Turk J Math
31 (2007) , 395 – 408.

The Generalized Hyers–Ulam–Rassias Stability of a Cubic Functional Equation

Abbas Najati

Abstract

In this paper, we obtain the general solution and the generalized Hyers–Ulam–Rassias stability for a cubic functional equation

$$f(mx+y)+f(mx-y)=mf(x+y)+mf(x-y)+2(m^3-m)f(x)$$

for a positive integer $m \geq 1$.

Key Words: Hyers–Ulam–Rassias stability, Quadratic function, Cubic function.

1. Introduction

In 1940, S. M. Ulam [18] gave a wide ranging talk before the mathematics club of the University of Wisconsin, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then does there exist a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: When the solutions of an equation differing slightly from a given one must be close to the true solution of the given equation. For

2000 *AMS Mathematics Subject Classification:* Primary 39B22, 39B52.

Banach spaces the Ulam problem was first solved by D. H. Hyers [7] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\Big\|f(x+y) - f(x) - f(y)\Big\|_Y \le \delta \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\Big\|f(x) - T(x)\Big\|_Y \le \delta$$

for all $x \in X$. Th. M. Rassias [15] succeeded in extending the result of Hyers by weakening the condition for the Cauchy difference to be unbounded. In recent decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 5, 9, 13]. The stability phenomenon that was introduced and proved by Th. M. Rassias in his 1978 paper is called the *Hyers–Ulam stability.* The terminology generalized Hyers–Ulam stability, originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we refer the reader to [8, 10, 16]. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

is related to a symmetric biadditive function [1, 14]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.2) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [1, 14]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}\Big(f(x+y) - f(x-y)\Big).$$

A Hyers–Ulam stability problem for the quadratic functional equation (1.2) was proved by F. Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [17]). P. W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. In the paper [4], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.2). A. Grabiec [6] has generalized these results mentioned above. K. W. Jun and Y. H. Lee [12] proved the Hyers–Ulam–Rassias stability of the pexiderized quadratic equation (1.2). In [11], K. W.

Jun and H. M. Kim introduced the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.3}$$

which are somewhat different from (1.2).

It is easy to see that the function $f(x)=cx^3$ is a solution of the above functional equation. Thus, it is natural that equation (1.3) is called a cubic functional equation and every solution of the cubic functional equation (1.3) is said to be a cubic function.

Now, we introduce the following functional equations, which are somewhat different from (1.3):

$$f(mx+y)+f(mx-y)=mf(x+y)+mf(x-y)+2(m^3-m)f(x), \tag{1.4}$$

where m is a positive integer and $m \geq 2$. For $m=2$, we obtain equation (1.3).

In this paper, we establish the general solution and the generalized Hyers–Ulam–Rassias stability problem for the equation (1.4), which are equivalent to (1.3).

2. Solution of Equation (1.4)

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be real vector spaces. We present the general solution of equation (1.4).

Theorem 2.1 *A function $f: E_1 \to E_2$ satisfies the functional equation* (1.3) *if and only if $f: E_1 \to E_2$ satisfies the functional equation* (1.4). *Therefore, every solution of functional equations* (1.4) *is also a cubic function.*

Proof. Let $f: E_1 \to E_2$ satisfy the functional equation (1.3). Putting $x=y=0$ in (1.3), we get $f(0)=0$. Set $x=0$ in (1.3) to get $f(-y)=-f(y)$. Letting $y=x$ and $y=2x$ in (1.3), respectively, we obtain that $f(2x)=8f(x)$ and $f(3x)=27f(x)$ for all $x \in E_1$. By induction, we lead to $f(kx)=k^3f(x)$ for all positive integer k. Replacing y by $x+y$ in (1.3), we have

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y)+12f(x) \tag{2.1}$$

for all $x, y \in E_1$. Once again Replacing y by $y-x$ in (1.3), we have

$$f(x+y)+f(3x-y)=2f(y)+2f(2x-y)+12f(x) \tag{2.2}$$

for all $x, y \in E_1$. Adding (2.1) to (2.2) and using (1.3), we obtain

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \tag{2.3}$$

for all $x, y \in E_1$. By using the above method, by induction, we infer that

$$f(mx+y)+f(mx-y)=mf(x+y)+mf(x-y)+2(m^3-m)f(x) \tag{2.4}$$

for all $x, y \in E_1$ and each positive integer $m \geq 3$.

Let $f : E_1 \to E_2$ satisfy the functional equation (1.4) with the positive integer $m \geq 3$. Putting $x = y = 0$ in (1.4), we get $f(0) = 0$. Set $x = 0$ in (1.4) to get $f(-y) = -f(y)$.

Let k be a positive integer. Replacing y by $kx + y$ in (1.4), we have

$$f((m+k)x+y)+f((m-k)x-y) \tag{2.5}$$

$$=mf((k+1)x+y)-mf((k-1)x+y)+2(m^3-m)f(x)$$

for all $x, y \in E_1$. Replacing y by $y - kx$ in (1.4), we have

$$f((m-k)x+y)+f((m+k)x-y) \tag{2.6}$$

$$=mf((k+1)x-y)-mf((k-1)x-y)+2(m^3-m)f(x)$$

for all $x, y \in E_1$. Adding (2.5) to (2.6), we obtain

$$\begin{aligned}&\Big[f((m+k)x+y)+f((m+k)x-y)\Big]\\&\quad+\Big[f((m-k)x+y)+f((m-k)x-y)\Big]\\&=m\Big[f((k+1)x+y)+f((k+1)x-y)\Big]\\&\quad-m\Big[f((k-1)x+y)+f((k-1)x-y)\Big]+4(m^3-m)f(x)\end{aligned} \tag{2.7}$$

for all $x, y \in E_1$ and for all integer $k \geq 1$. Let $\varphi_n(x, y) = f(nx + y) + f(nx - y)$ for each integer $n \geq 0$. Then (2.7) means that

$$\varphi_{m+k}(x,y)+\varphi_{m-k}(x,y)=m\varphi_{k+1}(x,y)-m\varphi_{k-1}(x,y)+4(m^3-m)f(x) \tag{2.8}$$

for all $x, y \in E_1$ and for all integer $k \geq 1$. For $k = 1$ and $k = m$ in (2.7), we obtain

$$\varphi_{m+1}+\varphi_{m-1}=m\varphi_2+4(m^3-m)f(x) \tag{2.9}$$

and

$$\varphi_{2m}=m\varphi_{m+1}-m\varphi_{m-1}+4(m^3-m)f(x) \tag{2.10}$$

for all $x, y \in E_1$. By the proof of the first part, since $f : E_1 \to E_2$ satisfies the functional equation (1.4) with the positive integer $m \geq 3$, then f satisfies the functional equation (1.4) with the positive integer $k \geq m$. It follows from (2.9) and (2.10) that f satisfies the functional equation (1.3) and

$$\begin{aligned} &f((m-1)x+y)+f((m-1)x-y) \\ &= (m-1)f(x+y)+(m-1)f(x-y)+2((m-1)^3-(m-1))f(x) \end{aligned} \tag{2.11}$$

for all $x, y \in E_1$. □

Remark 2.2 *If $f : E_1 \to E_2$ satisfies the functional the functional equation* (1.3)*, then for each rational number λ, we have*

$$f(\lambda x+y)+f(\lambda x-y)=\lambda f(x+y)+\lambda f(x-y)+2(\lambda^3-\lambda)f(x) \tag{2.12}$$

for all $x, y \in E_1$.

3. Hyers–Ulam–Rassias stability

In this section, let X be a real vector space and let Y be a Banach space unless we give any specific reference. We will investigate the Hyers–Ulam–Rassias stability problem for the functional equation (1.4). Thus we find the condition that there exists a true cubic function near an approximately cubic function.

Theorem 3.1 *Let δ be a real number and let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to [-\delta, +\infty)$ such that*

$$\widetilde{\varphi}(x) := \sum_{k=0}^{\infty} m^{-3k}\varphi(m^k x, 0) < \infty, \quad \lim_{n\to\infty} m^{-3n}\varphi(m^n x, m^n y) = 0, \tag{3.1}$$

and

$$\begin{aligned} &\Big\| f(mx+y)+f(mx-y)-mf(x+y) \\ &-mf(x-y)-2(m^3-m)f(x)\Big\|_Y \leq \delta+\varphi(x,y) \end{aligned} \tag{3.2}$$

for all $x, y \in X$, where m is a positive integer with $m > 1$. Then there exists a unique cubic mapping $T : X \to Y$ such that

$$\Big\|T(x) - f(x)\Big\|_Y \le \frac{\delta}{2(m^3-1)} + \frac{1}{2m^3}\widetilde{\varphi}(x) \tag{3.3}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (3.2), we obtain

$$\Big\|\frac{1}{m^3}f(mx) - f(x)\Big\|_Y \le \frac{1}{2m^3}\delta + \frac{1}{2m^3}\varphi(x,0), \quad (x \in X). \tag{3.4}$$

Replacing x by mx in (3.4), we get

$$\Big\|\frac{1}{m^6}f(m^2x) - \frac{1}{m^3}f(mx)\Big\|_Y \le \frac{1}{2m^6}\delta + \frac{1}{2m^6}\varphi(mx,0), \quad (x \in X). \tag{3.5}$$

Hence by using induction, we infer that

$$\Big\|m^{-3n}f(m^nx) - m^{-3(n-1)}f(m^{n-1}x)\Big\|_Y \le \frac{1}{2m^{3n}}\delta + \frac{1}{2m^{3n}}\varphi(m^{n-1}x,0) \tag{3.6}$$

for all $x \in X$ and integers $n \ge 1$. Therefore we have

$$\begin{aligned} &\Big\|\sum_{k=l+1}^{n}\Big(m^{-3k}f(m^kx) - m^{-3(k-1)}f(m^{k-1}x)\Big)\Big\|_Y \\ &\le \sum_{k=l+1}^{n}\Big\|m^{-3k}f(m^kx) - m^{-3(k-1)}f(m^{k-1}x)\Big\|_Y \\ &\le \tfrac{\delta}{2}\sum_{k=l+1}^{n} m^{-3k} + \tfrac{1}{2}\sum_{k=l+1}^{n} m^{-3k}\varphi(m^{k-1}x,0), \end{aligned} \tag{3.7}$$

for all $x \in X$ and integers $n > l \ge 0$. Hence we obtain from (3.7) that

$$\begin{aligned} &\Big\|m^{-3n}f(m^nx) - m^{-3l}f(m^lx)\Big\|_Y \\ &\le \tfrac{\delta}{2}\sum_{k=l+1}^{n} m^{-3k} + \tfrac{1}{2}\sum_{k=l+1}^{n} m^{-3k}\varphi(m^{k-1}x,0), \end{aligned} \tag{3.8}$$

and by letting $l = 0$ in (3.8), we obtain

$$\Big\|m^{-3n}f(m^nx) - f(x)\Big\|_Y \le \frac{\delta}{2}\sum_{k=1}^{n} m^{-3k} + \frac{1}{2}\sum_{k=1}^{n} m^{-3k}\varphi(m^{k-1}x,0) \tag{3.9}$$

for all $x \in X$ and integers $n > l \geq 0$. Thus (3.8) implies that $\{m^{-3n}f(m^n x)\}_n$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{n\to\infty} m^{-3n} f(m^n x)$$

for all $x \in X$. Letting $n \to \infty$ in (3.9), we get the inequality (3.3). It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\Big\|T(mx+y)+T(mx-y)-mT(x+y)-mT(x-y)-2(m^3-m)T(x)\Big\|_Y\\
&= \lim_{n\to\infty} m^{-3n}\Big\|f(m^n(mx+y))+f(m^n(mx-y))-mf(m^n(x+y))\\
&\quad -mf(m^n(x-y))-2(m^3-m)f(m^n x)\Big\|_Y\\
&\leq \lim_{n\to\infty} m^{-3n}\Big(\delta+\varphi(m^n x, m^n y)\Big)=0
\end{aligned}$$

for all $x, y \in X$. Hence, by Theorem 2.1, it proves that $T : X \to Y$ is a cubic mapping.

Let $Q : X \to Y$ be another cubic mapping satisfying (3.3). Then

$$\begin{aligned}
&\Big\|T(x)-Q(x)\Big\|_Y = \lim_{n\to\infty} m^{-3n}\Big\|T(m^n x)-Q(m^n x)\Big\|_Y\\
&\leq \lim_{n\to\infty} m^{-3n}\Big\|T(m^n x)-f(m^n x)\Big\|_Y\\
&\quad +\lim_{n\to\infty} m^{-3n}\Big\|f(m^n x)-Q(m^n x)\Big\|_Y\\
&\leq \lim_{n\to\infty} m^{-3n}\Big(\tfrac{\delta}{m^3-1}+\tfrac{1}{m^3}\widetilde{\varphi}(m^n x)\Big)\\
&= \tfrac{1}{m^3}\lim_{n\to\infty}\sum_{l=n}^{\infty} m^{-3l}\varphi(m^l x, 0)=0.
\end{aligned}$$

So we have $T(x) = Q(x)$ for all $x \in X$. This proves the uniqueness of T. □

Corollary 3.2 *Let $\delta, \epsilon, \theta, p$ and q be real numbers such that $\delta, \epsilon, \theta \geq 0, q > 0$ and $p, q < 3$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\begin{aligned}
&\Big\|f(mx+y)+f(mx-y)-mf(x+y)-mf(x-y)\\
&-2(m^3-m)f(x)\Big\|_Y \leq \delta+\epsilon\|x\|_X^p+\theta\|y\|_X^q
\end{aligned}\tag{3.10}$$

for all $x, y \in X$, where m is a positive integer with $m > 1$. Then there exists a unique cubic mapping $T : X \to Y$ such that

$$\Big\|T(x)-f(x)\Big\|_Y \leq \frac{\delta}{2(m^3-1)}+\frac{\epsilon}{2(m^3-m^p)}\|x\|_X^p \tag{3.11}$$

for all $x \in X$ *and for all* $x \in X \setminus \{0\}$ *if* $p < 0$.

Proof. In Theorem 3.1, let $\varphi(x, y) = \epsilon \|x\|_X^p + \theta \|y\|_X^q$. □

Theorem 3.3 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X \times X \to [0, +\infty)$ *such that*

$$\widetilde{\varphi}(x) := \sum_{k=0}^{\infty} m^{3k} \varphi(\frac{1}{m^k}x, 0) < \infty, \quad \lim_{n\to\infty} m^{3n} \varphi(\frac{1}{m^n}x, \frac{1}{m^n}y) = 0, \tag{3.12}$$

and

$$\begin{aligned} &\Big\| f(mx + y) + f(mx - y) - mf(x + y) \\ &- mf(x - y) - 2(m^3 - m)f(x) \Big\|_Y \le \varphi(x, y) \end{aligned} \tag{3.13}$$

for all $x, y \in X$*, where* m *is a positive integer with* $m > 1$*. Then there exists a unique cubic mapping* $T : X \to Y$ *such that*

$$\Big\| T(x) - f(x) \Big\|_Y \le \frac{1}{2} \widetilde{\varphi}(\frac{1}{m}x) \tag{3.14}$$

for all $x \in X$.

Proof. Putting $y = 0$ and replacing x by $\frac{1}{m}x$ in (3.13), we obtain

$$\Big\| m^3 f(\frac{1}{m}x) - f(x) \Big\|_Y \le \frac{1}{2} \varphi(\frac{1}{m}x, 0), \quad (x \in X). \tag{3.15}$$

Replacing x by $\frac{1}{m}x$ in (3.15), we get

$$\Big\| m^6 f(\frac{1}{m^2}x) - m^3 f(\frac{1}{m}x) \Big\|_Y \le \frac{1}{2} m^3 \varphi(\frac{1}{m^2}x, 0), \quad (x \in X). \tag{3.16}$$

Hence by using induction, we infer that

$$\Big\| m^{3n} f(\frac{1}{m^n}x) - m^{3(n-1)} f(\frac{1}{m^{n-1}}x) \Big\|_Y \le \frac{1}{2} m^{3(n-1)} \varphi(\frac{1}{m^n}x, 0) \tag{3.17}$$

for all $x \in X$ and integers $n \geq 1$. Therefore we have

$$\begin{aligned} &\Big\| \sum_{k=l+1}^{n} \Big(m^{3k} f(\tfrac{1}{m^k}x) - m^{3(k-1)} f(\tfrac{1}{m^{k-1}}x) \Big) \Big\|_Y \\ &\leq \sum_{k=l+1}^{n} \Big\| m^{3k} f(\tfrac{1}{m^k}x) - m^{3(k-1)} f(\tfrac{1}{m^{k-1}}x) \Big\|_Y \\ &\leq \tfrac{1}{2} \sum_{k=l+1}^{n} m^{3(k-1)} \varphi(\tfrac{1}{m^k}x, 0), \end{aligned} \tag{3.18}$$

for all $x \in X$ and integers $n > l \geq 0$. Hence we obtain from (3.18) that

$$\begin{aligned} &\Big\| m^{3n} f(\tfrac{1}{m^n}x) - m^{3l} f(\tfrac{1}{m^l}x) \Big\|_Y \\ &\leq \tfrac{1}{2} \sum_{k=l+1}^{n} m^{3(k-1)} \varphi(\tfrac{1}{m^k}x, 0), \end{aligned} \tag{3.19}$$

and by letting $l = 0$ in (3.19), we obtain

$$\Big\| m^{3n} f(\frac{1}{m^n}x) - f(x) \Big\|_Y \leq \frac{1}{2} \sum_{k=1}^{n} m^{3(k-1)} \varphi(\frac{1}{m^k}x, 0) \tag{3.20}$$

for all $x \in X$ and integers $n > l \geq 0$. Thus (3.19) implies that $\{m^{3n} f(\frac{1}{m^n}x)\}_n$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{n\to\infty} m^{3n} f(\frac{1}{m^n}x)$$

for all $x \in X$. Letting $n \to \infty$ in (3.20), we get the inequality (3.14). Similar to the proof of Theorem 3.1, it follows from (3.12) and (3.13) that

$$\begin{aligned} &\Big\| T(mx+y) + T(mx-y) - mT(x+y) - mT(x-y) - 2(m^3-m)T(x) \Big\|_Y \\ &= \lim_{n\to\infty} m^{3n} \Big\| f(\tfrac{mx+y}{m^n}) + f(\tfrac{mx-y}{m^n}) - mf(\tfrac{x+y}{m^n}) \\ &\quad - mf(\tfrac{x-y}{m^n}) - 2(m^3-m) f(\tfrac{x}{m^n}) \Big\|_Y \\ &\leq \lim_{n\to\infty} m^{3n} \varphi(\tfrac{x}{m^n}, \tfrac{y}{m^n}) = 0 \end{aligned}$$

for all $x, y \in X$. Therefore $T : X \to Y$ is a cubic mapping.

To prove the uniqueness of T, let $Q : X \to Y$ be another cubic mapping satisfying (3.14). Similar to the proof of Theorem 3.1, we have

$$\begin{aligned} \Big\| T(x) - Q(x) \Big\|_Y &= \lim_{n\to\infty} m^{3n} \Big\| f(\tfrac{1}{m^n}x) - Q(\tfrac{1}{m^n}x) \Big\|_Y \\ &\leq \tfrac{1}{2} \lim_{n\to\infty} m^{3n} \widetilde{\varphi}(\tfrac{1}{m^{n+1}}x) = 0 \end{aligned}$$

So we have $T(x) = Q(x)$ for all $x \in X$. This proves the uniqueness of T. $\square$

Corollary 3.4 *Let δ, ϵ, p and q be nonnegative real numbers such that $p, q > 3$. Suppose that $f : X \to Y$ is a mapping fulfilling*

$$\begin{aligned}&\Big\| f(mx+y) + f(mx-y) - mf(x+y) - mf(x-y) \\ &-2(m^3-m)f(x)\Big\|_Y \leq \epsilon\|x\|_X^p + \theta\|y\|_X^q\end{aligned} \tag{3.21}$$

for all $x, y \in X$, where m is a positive integer with $m > 1$. Then there exists a unique cubic mapping $T : X \to Y$ such that

$$\Big\| T(x) - f(x)\Big\|_Y \leq \frac{\epsilon}{2(m^p - m^3)}\|x\|_X^p \tag{3.22}$$

for all $x \in X$.

Proof. In Theorem 3.3, let $\varphi(x, y) = \epsilon\|x\|_X^p + \delta\|y\|_X^q$. $\square$

Let f be a mapping from X to Y. For each positive integer m, let $Df_m : X \times X \to Y$ be a mapping defined by

$$Df_m(x, y) = f(mx+y) + f(mx-y) - mf(x+y) - mf(x-y) - 2(m^3-m)f(x).$$

Proposition 3.5 *Let δ be a nonnegative real number and let $f : X \to Y$ be an odd mapping. Suppose that*

$$\Big\| Df_2(x, y)\Big\|_Y \leq \delta \tag{3.23}$$

for all x, y in X. Then there exists a sequence of nonnegative real numbers $\{\delta_n\}_{n=0}^{\infty}$ such that

$$\delta_0 = \delta, \quad \delta_1 = 4\delta, \quad \delta_2 = 10\delta, \quad \delta_m = 2\delta_{m-1} + (m+1)\delta + \delta_{m-2} \quad (m \geq 3)$$

and

$$\Big\| Df_m(x, y)\Big\|_Y \leq \delta_{m-2}, \quad (m \geq 2). \tag{3.24}$$

Proof. Replacing y by $y+x$ and $y-x$ in (3.23), respectively, we get from (3.23) that

$$\left\|Df_3(x,y)\right\|_Y \leq 4\delta = \delta_1 \tag{3.25}$$

for all x, y in X. Replacing y by $y+x$ and $y-x$ in (3.25), respectively, we get from (3.23) that

$$\left\|Df_4(x,y)\right\|_Y \leq 10\delta = \delta_2 \tag{3.26}$$

for all x, y in X. Replacing y by $y+x$ and $y-x$ in (3.26), respectively, we get from (3.23) and (3.25) that

$$\left\|Df_5(x,y)\right\|_Y \leq 28\delta = 2\delta_2 + 4\delta + \delta_1 = \delta_3 \tag{3.27}$$

for all x, y in X. Therefore by using this method, by induction we infer (3.24). □

Corollary 3.6 *Let $f : X \to Y$ be an odd mapping. Suppose that* (3.23) *holds. Then for each positive integer $m > 1$, there exists a unique cubic mapping $T_m : X \to Y$ such that*

$$\left\|T_m(x) - f(x)\right\|_Y \leq \frac{\delta_{m-2}}{2(m^3-1)}$$

for all x in X.

In the last part of this paper, let A be a unital Banach algebra with norm $\|.\|_A$, and let $\mathbb{X}$ and $\mathbb{Y}$ be left Banach A–modules with norms $\|.\|_{\mathbb{X}}$ and $\|.\|_{\mathbb{Y}}$, respectively. A cubic function $T : \mathbb{X} \to \mathbb{Y}$ is called A–cubic if $T(ax) = a^3T(x)$ for all $a \in A$, $x \in \mathbb{X}$.

The following corollary is a consequence of Theorem 3.1.

Corollary 3.7 *Let ϵ, p and q be real numbers such that $\epsilon \geq 0, q > 0$ and $p, q < 3$. Suppose that $f : \mathbb{X} \to \mathbb{Y}$ is a mapping fulfilling*

$$\begin{aligned} &\Big\|f(amx+ay) + f(amx-ay) - maf(x+y) - maf(x-y) \\ &-2(m^3-m)af(x)\Big\|_{\mathbb{Y}} \leq \epsilon(\|x\|_{\mathbb{X}}^p + \|y\|_{\mathbb{X}}^q) \end{aligned} \tag{3.28}$$

for all $a \in A$ ($\|a\|_A = 1$) and for all $x, y \in \mathbb{X}$ where m is a positive integer with $m > 1$. Also, if for each fixed $x \in \mathbb{X}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathbb{Y}$ is continuous, then *there exists a unique A–cubic function $T:\mathbb{X}\to\mathbb{Y}$ which satisfies equation* (1.4) *and the inequality*

$$\Big\|T(x)-f(x)\Big\|_{\mathbb{Y}}\le\frac{\epsilon}{2(m^3-m^p)}\|x\|_{\mathbb{X}}^p \tag{3.29}$$

for all $x\in\mathbb{X}$ and for all $x\in\mathbb{X}\setminus\{0\}$ if $p<0$.

Proof. In Theorem 3.1, let $\varphi(x,y)=\epsilon(\|x\|_{\mathbb{X}}^p+\|y\|_{\mathbb{X}}^q)$. By Theorem 3.1, it follows from the inequality of the statement for $a=1$ that there exists a unique cubic function $T:\mathbb{X}\to\mathbb{Y}$ satisfying the inequality (3.29). Under the assumption that $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in\mathbb{X}$, by the same reasoning as the proof of [4], the cubic function $T:\mathbb{X}\to\mathbb{Y}$ satisfies $T(tx)=t^3T(x)$ for all $t\in\mathbb{R}$, $x\in\mathbb{X}$. Also, T satisfies in the following equation

$$\begin{aligned}T(amx+ay)+T(amx-ay)=\ & maT(x+y)+maT(x-y)\\ & +2(m^3-m)aT(x)\end{aligned} \tag{3.30}$$

for all $a\in A$ ($\|a\|_A=1$) and for all $x,y\in\mathbb{X}$. For each fixed $a\in A$ ($\|a\|_A=1$), setting $y=0$ in (3.30), we have $T(ax)=a^3T(x)$ for all $x\in\mathbb{X}$. The last relation is also true for $a=0$. Let $a\in A$ ($a\ne0$) and $a_1=\frac{a}{\|a\|_A}$. Since T is $\mathbb{R}$–cubic and $T(bx)=b^3T(x)$ for all $x\in\mathbb{X}$ and $b\in A$ ($\|b\|_A=1$), then

$$T(ax)=T(\|a\|_A.a_1x)=\|a\|_A^3.a_1^3T(x)=a^3T(x)$$

for all $x\in\mathbb{X}$ and $a\in A$. So the unique $\mathbb{R}$–cubic function $T:\mathbb{X}\to\mathbb{Y}$ is also A–cubic, as desired. This completes the proof of the corollary. □

The following corollary is a consequence of Theorem 3.3.

Corollary 3.8 *Let ϵ,p and q be real numbers such that $\epsilon\ge0$ and $p,q>3$. Suppose that $f:\mathbb{X}\to\mathbb{Y}$ is a mapping fulfilling*

$$\begin{aligned}&\Big\|f(amx+ay)+f(amx-ay)-maf(x+y)-maf(x-y)\\ &-2(m^3-m)af(x)\Big\|_{\mathbb{Y}}\le\epsilon(\|x\|_{\mathbb{X}}^p+\|y\|_{\mathbb{X}}^q)\end{aligned} \tag{3.31}$$

for all $a\in A$ ($\|a\|_A=1$) and for all $x,y\in\mathbb{X}$, where m is a positive integer with $m>1$. Also, if for each fixed $x\in\mathbb{X}$ the mapping $t\mapsto f(tx)$ from $\mathbb{R}$ to $\mathbb{Y}$ is continuous, then there *exists a unique A–cubic function* $T : \mathbb{X} \to \mathbb{Y}$ *which satisfies Eq.* (1.4) *and the inequality*

$$\Big\|T(x) - f(x)\Big\|_{\mathbb{Y}} \le \frac{\epsilon}{2(m^p - m^3)}\|x\|_{\mathbb{X}}^p \tag{3.32}$$

for all $x \in \mathbb{X}$.

Proof. In Theorem 3.3, let $\varphi(x,y) = \epsilon(\|x\|_{\mathbb{X}}^p + \|y\|_{\mathbb{X}}^q)$. By Theorem 3.3, it follows from the inequality of the statement for $a = 1$ that there exists a unique cubic function $T : \mathbb{X} \to \mathbb{Y}$ satisfying the inequality (3.32). Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, by the same reasoning as the proof of [4], the cubic function $T : \mathbb{X} \to \mathbb{Y}$ satisfies $T(tx) = t^3T(x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{X}$. Also, T satisfies in (3.30) for all $a \in A$ ($\|a\|_A = 1$) and for all $x, y \in \mathbb{X}$. Therefore the result follows by using the same proof of Corollary 3.7. □

Acknowledgements

It is my pleasure to express my thanks to Professor Chun-Gil Park, who read a previous version of the manuscript, for his helpful comments. The author would like to thank the referee(s) for a number of valuable suggestions regarding a previous version of this paper.

References

[1] Aczél, J., Dhombres, J.: *Functional Equations in Several Variables.* Cambridge University Press, 1989.

[2] Baker, J.: *The stability of the cosine equation.* Proc. Amer. Math. Soc. **80**, 411–416 (1980).

[3] Cholewa, P. W.: *Remarks on the stability of functional equations.* Aequationes Math. **27**, 76–86 (1984).

[4] Czerwik, S.: *On the stability of the quadratic mapping in normed spaces.* Abh. Math. Sem. Univ. Hamburg **62**, 59–64 (1992).

[5] Forti, G. L.: *Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations.* J. Math. Anal. Appl. **295**, 127–133 (2004).

[6] Grabiec, A.: *The generalized Hyers–Ulam stability of a class of functional equations.* Publ. Math. Debrecen. **48**, 217–235 (1996).

[7] Hyers, D. H.: *On the stability of the linear functional equation.* Proc. Natl. Acad. Sci. U.S.A. **27**, 222–224 (1941).

[8] Hyers, D. H., Isac, G., and Rassias, Th. M.: *Stability of Functional Equations in Several Variables.* Birkhäuser, Basel, 1998.

[9] Hyers, D. H., Isac, G., and Rassias, Th. M.: *On the asymptoticity aspect of Hyers–Ulam stability of mappings.* Proc. Amer. Math. Soc. **126**, 425–430 (1998).

[10] Hyers, D. H., Rassias, Th. M.: *Approximate homomorphisms.* Aequationes Math. **44**, 125–153 (1992).

[11] Jun, K. W., Kim, H. M.: *The generalized Hyers–Ulam–Rassias stability of a cubic functional equation.* J. Math. Anal. Appl. **274**, 867–878 (2002).

[12] Jun, K. W., Lee, Y. H.: *On the Hyers–Ulam–Rassias stability of a pexiderized quadratic inequality.* Math. Ineq. Appl. **4**, 93–118 (2001).

[13] Jung, S. M.: *On the Hyers–Ulam–Rassias stability of a quadratic functional equation.* J. Math. Anal. Appl. **232**, 384–393 (1999).

[14] Kannappan, Pl.: *Quadratic functional equation and inner product spaces.* Results Math. **27**, 368–372 (1995).

[15] Rassias, Th. M.: *On the stability of the linear mapping in Banach spaces.* Proc. Amer. Math. Soc. **72**, 297–300 (1978).

[16] Rassias, Th. M.: *On the stability of functional equations in Banach spaces.* J. Math. Anal. Appl. **251**, 264–284 (2000).

[17] Skof, F.: *Proprietà̀ locali e approssimazione di operatori.* Rend. Sem. Mat. Fis. Milano. **53**, 113–129 (1983).

[18] Ulam, S. M.: *Problems in Modern Mathematics.* Chap. VI, Science Ed., Wiley, New York, 1960.

Abbas NAJATI
Faculty of Sciences, Department of Mathematics,
Mohaghegh Ardebili University
Ardebil, Islamic Republic of IRAN
e-mail: a.nejati@yahoo.com

Received 15.06.2006