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Centralizers in Locally Finite Groups

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Abstract

The topic of the present paper is the following question.

Let G be a locally finite group admitting an automorphism ϕ of finite order such that the centralizer $C_G(\phi)$ satisfies certain finiteness conditions. What impact does this have on the structure of the group G?

Equivalently, one can ask the same question when ϕ is an element of G. Sometimes the impact is quite strong and the paper is a survey of results illustrating this phenomenon. In particular, we concentrate on results where G is shown to have a large nilpotent or soluble subgroup. Naturally, in each case the result depends on the order of the automorphism ϕ and kind of conditions imposed on $C_G(\phi)$. We shall be considering mostly the classical finiteness conditions such as $C_G(\phi)$ being finite, Chernikov, and of finite rank, respectively. It is not a purpose of the paper to survey numerous results on automorphisms of finite groups. In particular, among important topics that are left out of the present discussion are "p-automorphisms of p-groups" (see [36]) and "length problems" (see [73]). However, in some situations (like, for example, when $C_G(\phi)$ is finite) problems on infinite groups quickly reduce to finite groups and in those cases working with finite groups is very natural.

A separate section of the paper is devoted to the case when ϕ is of order two, a prime, four, or other, respectively. However, before anything else we address in Section 1 the following related question.

Given a periodic group G with an automorphism ϕ, what additional assumptions on G and $C_G(\phi)$ ensure that G is locally finite?

Key Words: Locally finite groups, centralizer.

1. On local finiteness of periodic groups with automorphisms

An automorphim of order two is called involutory. An automorphism ϕ is called fixed-point-free if $C_G(\phi) = 1$ (sometimes such an automorphism is also called regular). W. Burnside knew that any finite group admitting a fixed-point-free involutory automorphism is abelian [10]. Of course, this is a well-known elementary result. Perhaps it is less known that even any periodic group admitting a fixed-point-free involutory automorphism is abelian (B. H. Neumann [52]). It seems at present Neumann's result is not as well-known as it should be and so we give below a proof.

Theorem 1.1 *Let G be a periodic group admitting a fixed-point-free involutory automorphism ϕ. Then $x^\phi = x^{-1}$ for every $x \in G$. Hence G is abelian without elements of order two.*

Proof. If G has an element of order two, say x, then the subgroup $K = \langle x, \phi \rangle$ in the semidirect product $G\langle \phi \rangle$ is a finite dihedral group. One can see easily that ϕ centralizes an involution in $K \cap G$. This leads to a contradiction and shows that G has no involution.

Now consider the mapping $\tau : G \to G$ such that $x^\tau = x^{-\phi}x$ for every $x \in G$. Since ϕ is fixed-point-free, it is clear that τ is injective. Let $I = \{x^\tau ; x \in G\}$ be the image of G under τ. It is straightforward to check that I is precisely the set of elements sent by ϕ to their inverses. Therefore if $y \in I$, then $y^\tau = y^2$. Clearly, I is closed under taking powers of its elements. Since every element of G has odd order and since τ acts on I by the rule $y^\tau = y^2$, it follows that τ is surjective on I. Thus, the restriction of τ to I is a bijection. Combining this with the fact that τ is injective, it follows that $I = G$. So $x^\phi = x^{-1}$ for every $x \in G$ and the result follows. $\square$

It is noteworthy that the above theorem provides a criterion for a periodic group to be locally finite. As a topic closely related to our main theme let us consider the following general question.

Problem 1.2 *Let G be a periodic group acted on by a finite group A. Under what assumptions about $C_G(A)$ does it follow that G is locally finite?*

It is now well-known that periodic groups need not be locally finite [53]. The following important theorem is due to Shunkov [67].

Theorem 1.3 *Let G be a periodic group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is finite. Then G is locally finite.*

Once Theorem 1.3 is proved a lot more about the structure of G can be said. In fact in his paper [67] Shunkov showed that G has a soluble subgroup of finite index. Later, using methods of finite group theory Hartley and Meixner showed that under the hypothesis of Theorem 1.3 G has a nilpotent subgroup of finite index and of class at most two (see Section 2 for details). A shorter proof of Shunkov's result can be found in Belyaev [5]. It was shown in Deryabina and Ol'shanskii [12] that for any integer n having at least one odd divisor there exists a periodic non-(locally finite) group G admitting an automorphism ϕ of order n whose centralizer in G is finite. Therefore, if it is possible to generalize Theorem 1.3 without imposing additional conditions on the structure of G, it is only for the case that ϕ is of 2-power order. However even this case is very hard to handle. In [46, 12.100] the author proposed the question whether a periodic group admitting a fixed-point-free automorphism of order four is necessarily locally finite and in the fifteen years no progress with respect to that question has been made. We mention however that in [57] we proved that if a group G admits a fixed-point-free four-group of automorphisms, and if every two elements of G generate a finite subgroup, then G is locally finite. Another relevant result is that any group of exponent five that admits a fixed-point-free four-group of automorphisms is locally finite [58].

Well-known examples constructed in [16, 18, 19, 69] show that there exist residually finite periodic groups that are not locally finite. In [55] we proposed a method to handle residually finite periodic groups with automorphisms.

Theorem 1.4 *Let G be a residually finite periodic group admitting an automorphism ϕ of 2-power order such that $C_G(\phi)$ is finite. Then G is locally finite.*

The proof of the above theorem runs as follows. It is well-known that if a finite group G is acted on by a finite group A of coprime order, then for any A-invariant normal subgroup N of G we have $C_{G/N}(A) = C_G(A)N/N$. This easily extends to the case where G is a locally finite group. Suppose that A is a finite group acting on a locally finite group G that has no $|A|$-torsion. If N is an A-invariant normal subgroup of G, then $C_{G/N}(A) = C_G(A)N/N$. The following lemma shows that in the case that A is a 2-group this remains true even if G is merely periodic rather than locally finite.

Lemma 1.5 *Let G be a periodic $2'$-group acted on by a finite 2-group A. If N is an A-invariant normal subgroup of G, then $C_{G/N}(A) = C_G(A)N/N$.*

Now suppose that G and ϕ are as in Theorem 1.4 and assume that ϕ has order 2^n. Since G is residually finite, we can choose a ϕ-invariant subgroup $H \leq G$ of finite index such that $C_H(\phi) = 1$. It is sufficient to show that H is locally finite. Thus, without loss of generality we can assume that $C_G(\phi) = 1$. With this assumption it is easy to show that G has no involutions (because otherwise some of them would be contained in $C_G(\phi)$). Now Lemma 1.5 shows that $C_Q(\phi) = 1$ for every ϕ-invariant section Q of G. Therefore we can use results on finite groups admitting a fixed-point-free automorphism. According to a theorem of Berger [7] a finite group admitting a fixed-point-free automorphism of order 2^n has Fitting height at most n. Therefore our group G has a normal series of length at most n all of whose quotients are residually nilpotent. It is sufficient to show that each of the quotients is locally finite. Hence, without loss of generality we can assume that G is a finitely generated p-group for some prime p.

Write $D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k}$. The subgroups D_i form a central series of G known as the Zassenhaus-Jennings-Lazard series (see [31, Chapter 8]). Set $L(G) = \oplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field with p elements. We denote by L the subalgebra of $L(G)$ generated by D_1/D_2. One can show that G is finite if and only if L is nilpotent. For any $x \in G$ we let $i = i(x)$ be the largest integer such that $x \in D_i$ and denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$. A lemma of Lazard now allows us to deduce that $\tilde{x}$ is ad-nilpotent for every $x \in G$ [50, p. 131].

The automorphism ϕ acts on every quotient D_i/D_{i+1}. This action extends by linearity to the whole $L(G)$ and it is clear that L is ϕ-invariant. Lemma 1.5 shows that $C_L(\phi) = 0$. A well-known theorem of Kreknin [44] says that a Lie algebra admitting an automorphism of finite order without non-trivial fixed points is soluble. Thus, L is a finitely generated soluble algebra spanned by ad-nilpotent elements. It follows that L is nilpotent, as required. □

Shortly after Theorem 1.4 was proved a number of more general results have been published. Proofs of the results quoted below mostly follow the same general scheme as in Theorem 1.4. Improvements were achieved mainly due to using more sophisticated Lie-theoretic tools. In particular, instead of the Kreknin theorem a combination of a theorem of Zelmanov on nilpotency of finitely generated PI Lie algebras [74, III(0.4)] with a theorem of Bahturin and Zaicev on Lie algebras admitting an automorphism whose fixed point subalgebra is PI [2] was used. It is unlikely that Lemma 1.5 remains true for odd primes p so when dealing with centralizers of elements of odd order imposing some 2-generated conditions like in Theorems 1.7 and 1.8 below seems inevitable.

Theorem 1.6 *(Shalev [56]) Let G be a residually finite periodic group having a finite 2-subgroup whose centralizer is finite. Then G is locally finite.*

Theorem 1.7 *(Shumyatsky [62]) Let G be a residually finite group in which every two-generated subgroup is finite. Suppose G contains a finite p-subgroup with finite centralizer. Then G is locally finite.*

Theorem 1.8 *(Kuzucuoğlu and Shumyatsky [47]) Let G be a periodic residually finite group containing a nilpotent subgroup A such that $C_G(A)$ is finite. Assume that $\langle A, A^g\rangle$ is finite for any $g \in G$. Then G is locally finite.*

The reader can find in the papers [56, 62] a number of other results related to the problem on local finiteness of periodic groups with automorphisms.

2. Involutory automorphisms

In this section we discuss results that show that if G is a locally finite group with an involutory automorphism ϕ, and if $C_G(\phi)$ is small in some sence, then the structure of G is close to that of an abelian group. We say that a group almost has certain property if it has a subgroup of finite index with that property. We use the term "$\{a, b, c\dots\}$-bounded" to mean "bounded from above by some function depending only on the parameters $a, b, c\dots$". As usual, if ϕ is an automorphism of G we denote by $[G, \phi]$ the subgroup of G generated by the set $\{x^{-1}x^{\phi} | x \in G\}$. It is easy to see that $[G, \phi]$ is normal in $G\langle\phi\rangle$.

2.1. Finite Centralizers

Centralizers of involutions have played a central rôle in the theory of finite groups. In particular, the famous Brauer-Fowler Theorem that for any integer m there exist only finitely many finite simple groups containing an involution whose centralizer has order m has served as one of the key tools in the classification of finite simple groups. Fong showed in [14] that if a finite group G contains an element ϕ of prime order p such that $|C_G(\phi)| \le m$, then G contains a normal soluble subgroup S such that the index $|G : S|$

is m-bounded. For odd primes p this result depends on the classification of finite simple groups. However in the case $p = 2$ the result is in fact independent of the classification and uses only the Brauer-Fowler Theorem. A simple inverse limit argument along the lines of Kegel and Wehrfritz [33, p. 54] shows that this also holds for locally finite groups. Therefore we have

Theorem 2.1 *Let G be a locally finite group containing an element ϕ of prime order such that $C_G(\phi)$ is finite. Then G is almost locally soluble.*

The study of locally finite groups G admitting an involutory automorphism ϕ showed that quite often $[G, \phi]'$ and $G/[G, \phi]$ have properties similar to those of the centralizer $C_G(\phi)$. In particular, Belyaev and Sesekin proved the following theorem [6].

Theorem 2.2 *Let G be a locally finite group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is finite. Then $[G, \phi]'$ and $G/[G, \phi]$ are finite, too.*

One immediate corollary of the above theorem is that G has a subgroup N of finite index which is nilpotent of class at most two (for example, one may take $N = C_G([G, \phi]') \cap [G, \phi]$. Theorem 2.2 was proved in [6] using methods typical for infinite groups. The proof provides no information on what the index of the subgroup N might be. In [26] Hartley and Meixner studied the situation using mostly methods of finite group theory. Their results can be summarized as follows.

Theorem 2.3 *Let G be a locally finite group admitting an involutory automorphism ϕ such that $|C_G(\phi)| = n$ is finite. Then*

1. *G has a subgroup of finite n-bounded index which is nilpotent of class at most two.*
2. *If G is a q-group and $n = q^m$, then G has a subgroup of nilpotency class at most two and index at most $q^{1^2+2^2+\cdots+m^2}$.*
3. *If G is a 2-group, then G has an abelian subgroup of n-bounded index.*

2.2. Chernikov Centralizers

A group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasicyclic p-groups). By a deep result obtained independently by Shunkov [66] and Kegel and Wehrfritz [32]

Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup.

Kegel and Wehrfritz raised in [33] the question whether any locally finite group admitting an involutory automorphism ϕ with $C_G(\phi)$ Chernikov is almost locally soluble. This was confirmed by Asar in [1]. Asar's work does not use the full classification of finite simple groups but, still, it is much more involved than, say, the involutory case of Theorem 2.1. A more precise information about the structure of G is given in the following theorem, due to Hartley [20].

Theorem 2.4 *If G is a locally finite group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is Chernikov, then both $[G,\phi]'$ and $G/[G,\phi]$ are Chernikov.*

It is easy to deduce from the above theorem that G has a normal ϕ-invariant subgroup N such that N is nilpotent of class at most two and G/N is Chernikov. It is sufficient for example to take $N = C_{[G,\phi]}([G,\phi]')$.

2.3. Centralizers of Finite Rank

A group is said to have finite rank r if any of its finitely generated subgroups can be generated by at most r elements. Locally finite groups of finite rank have been a subject of a study for many years. By a result of Shunkov [68] any locally finite group of finite rank is almost locally soluble. A theorem of Kargapolov says that a periodic locally soluble group of finite rank G has a normal locally nilpotent subgroup N such that the quotient G/N is almost abelian with finite Sylow p-subgroups for all primes p (see [13, 3.2.3]. Finally a result of Blackburn guarantees that a locally finite p-group G has finite rank if and only if G is Chernikov [8]. Thus, it is safe to say that up to certain questions on finite groups of bounded rank r locally finite groups of finite rank are fairly well understood. It is natural to consider locally finite groups admitting an involutory automorphism ϕ such that $C_G(\phi)$ has finite rank. The situation here is more complicated than in the case where $C_G(\phi)$ is Chernikov. In particular an infinite simple locally finite group can have an involution with centralizer of finite rank. Such an example is provided by the group $PSL(2,K)$, where K is an infinite locally finite field of odd characteristic. Besides, even the case that G is almost locally soluble is not easy to handle. In [61] we proved the following result that plays a crucial rôle in the study of locally finite groups admitting an involutory automorphism with centralizer of finite rank.

Theorem 2.5 *Let G be a finite group of odd order admitting an involutory automorphism ϕ such that $C_G(\phi)$ is of rank r. Then the ranks of both $[G,\phi]'$ and $G/[G,\phi]$ are r-bounded.*

A traditional and very effective tool for study of finite groups of given rank is the powerful p-groups introduced by Lubotzky and Mann in [51]. The use of powerful p-groups constitutes an important step in the proof of Theorem 2.5. With Theorem 2.5 at hand it is not too difficult to treat periodic almost locally soluble groups. The next result, obtained in [63], is very natural.

Theorem 2.6 *If G is a periodic almost locally soluble group admitting an involutory automorphism ϕ such that $C_G(\phi)$ has finite rank, then both $[G,\phi]'$ and $G/[G,\phi]$ are of finite rank.*

We have already mentioned that an infinite simple locally finite group can have an involution with centralizer of finite rank. This rules out any hope of extending Theorem 2.6 to arbitrary locally finite groups. Yet, somewhat surprisingly, in a joint work of Kuzucuoğlu and the author [48] a very detailed description of the general case was given.

Let G be an infinite locally finite group admitting an involutory automorphism ϕ such that $C_G(\phi)$ has finite rank. Suppose first that G is simple. By the well-known theorem classifying simple periodic linear groups [4, 9, 28, 70], G is of Lie type over some locally finite field of odd characteristic. Combining this with the fact that $C_G(\phi)$ is almost locally soluble (because it has finite rank!) and with results of Hartley and Kuzucuoğlu on centralizers in locally finite simple groups [25] it follows that G is isomorphic to the group $PSL(2,K)$, for some infinite locally finite field K of odd characteristic.

Armed with the knowledge of the simple groups admitting an involutory automorphism with centralizer of finite rank, Kuzucuoğlu and the author obtained the following description of the general case.

Theorem 2.7 *Let G be a locally finite group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is of finite rank. Then $G/[G,\phi]$ has finite rank. Furthermore, $[G,\phi]'$ contains a characteristic subgroup B such that*

1. *B is a product of finitely many normal in $[G,\phi]$ subgroups isomorphic to either $PSL(2,K)$ or $SL(2,K)$ for some infinite locally finite fields K of odd characteristic, and*

2. *$[G,\phi]'/B$ has finite rank.*

We just note that Theorem 2.6 follows from Theorem 2.7 as a particular case.

2.4. Centralizers with the SF-property

A group is said to satisfy min-π, where π is a set of primes, if it satisfies the minimal condition on π-subgroups. A locally finite group satisfying min-p for every prime $p \in \pi(G)$ is called an SF-group. Belyaev proved in [3] that an SF-group is necessarily almost locally soluble.

The next theorem was proved in [59].

Theorem 2.8 *If G is a periodic almost locally soluble group admitting an involutory automorphism ϕ such that $C_G(\phi)$ satisfies min-π for a set of primes π containing 2, then both $[G,\phi]'$ and $G/[G,\phi]$ satisfy min-π.*

The following corollary is straightforward from the above theorem.

Corollary 2.9 *If G is a periodic almost locally soluble group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is an SF-group, then both $[G,\phi]'$ and $G/[G,\phi]$ are SF-groups.*

Handling arbitrary locally finite groups admitting an involutory automorphism whose centralizer is an SF-group seemed hard back in 1993. Eventually our experience in dealing with centralizers of finite rank proved to be very helpful here. In fact, slightly modifying the proof of Theorem 2.7 Kuzucuoğlu and the author obtained the following theorem [49].

Theorem 2.10 *Let G be a locally finite group admitting an involutory automorphism ϕ such that $C_G(\phi)$ is an SF-group. Then $G/[G,\phi]$ is an SF-group. Furthermore, $[G,\phi]'$ contains a characteristic subgroup B such that*

1. *B is a product of finitely many subgroups, normal in $[G,\phi]$, each isomorphic to either $PSL(2,K)$ or $SL(2,K)$ for some infinite locally finite fields K of odd characteristic, and*

2. *$[G,\phi]'/B$ is an SF-group.*

3. Automorphisms of prime order

3.1. Fixed-point-free automorphisms of prime order

The main facts concerning groups admitting a fixed-point-free automorphism of prime order are the following famous results of Higman and Thompson.

Theorem 3.1 *(Higman [29]). There exists a function $h(p)$ depending only on p such that every nilpotent group admitting a fixed-point-free automorphism of prime order p is nilpotent of class at most $h(p)$.*

Theorem 3.2 *(Thompson [71]). Every finite group admitting a fixed-point-free automorphism of prime order is nilpotent.*

These two results perfectly complement each other. Using the inverse limit argument we obtain the following theorem.

Theorem 3.3 *There exists a function $h(p)$ such that every locally finite group admitting a fixed-point-free automorphism of prime order p is nilpotent of class at most $h(p)$.*

Let $h(p)$ denote the minimal function satisfying the above theorem. The value of $h(p)$ is still not known. Higman showed that $h(p) \geq \frac{p^2-1}{4}$ for $p > 2$. He also showed that $h(5) = 6$. Scimemi showed that $h(7) = 12$ (see Hughes [30] for the proof). The upper bound for $h(p)$ was given by Kreknin and Kostrikin in [45]. They showed that

$$h(p) \leq \frac{(p-1)^{2^{p-1}-1} - 1}{p-2}.$$

Recently in [65] this was “improved” to

$$h(p) \leq \frac{(p-2)^s - 1}{p-3}, \text{ where } s = 2^{p-5} + [log_2(p-3)] + 2$$

for $p \geq 11$.

Still, the latter bound does not seem to be in the right order of magnitude.

3.2. Finite Centralizers

In most cases problems on locally finite group admitting an automorphism of finite order with finite centralizer easily reduce to finite groups. In view of Theorem 3.3 it is natural to expect that the structure of a locally finite group admitting an automorphism of prime order with finite centralizer is close to that of a nilpotent group. In 1982 Hartley posed in the Kourovka Notebook [46, 8.81] a question equivalent to the following one.

Let p be a prime and m an integer. Do there exist functions $c = c(p)$ of p and $i = i(m, p)$ of m and p such that any finite group admitting an automorphism of prime order p with centralizer of order m contains a nilpotent subgroup of index at most i and nilpotency class at most c?

Of course, Theorem 2.3 shows that if $p = 2$, the answer to the above problem is positive. Fong's Theorem 2.1 reduces the problem to the soluble case. This was dealt with in the work of Hartley and Meixner [27], where the following result was obtained.

Theorem 3.4 *Let G be a finite group admitting an automorphism of prime order p with centralizer of order m. Then G has a nilpotent subgroup of $\{m, p\}$-bounded index.*

Independently this was proved in Pettet [54]. With the above result Hartley's question was essentially reduced to the case that G is nilpotent. The nilpotent case was successfully handled by Khukhro in [34] using Lie methods. Thus, the problem has been solved in the affirmative. This yields the following theorem which is rightfully considered one of the main achievments of the theory of locally finite groups.

Theorem 3.5 *Let G be a locally finite group admitting an automorphism of prime order p whose centralizer is finite of order m. Then G contains a nilpotent subgroup of finite $\{m, p\}$-bounded index and p-bounded nilpotency class.*

3.3. Chernikov Centralizers

Unlike the situation with finite centralizers there is no obvious way to reduce problems on locally finite groups with Chernikov centralizer of an automorphism to finite groups. Yet, experience shows that often results in the Chernikov case resemble those in the finite one. First of all we note that the result of Blackburn [8] implies that if a locally finite p-group G admits an automorphism of order p whose centralizer is Chernikov, then G is Chernikov itself. More generally, the following proposition is noteworthy (see [33, 3.2]).

Proposition 3.6 *Let G be a locally finite group acted on by a finite p-group A in such a way that $C_G(A)$ satisfies min-p. Then G satisfies min-p, too.*

The following theorem was proved by Hartley in [22].

Theorem 3.7 *Let G be a locally finite group admitting an automorphism ϕ of prime-power order such that $C_G(\phi)$ is Chernikov. Then G is almost locally soluble.*

As was already mentioned, the case of the theorem where ϕ has order two was handled by Asar in [1]. For automorphisms of arbitrary prime order the theorem was independently proved by Turau in [72]. Both works – Hartley's and Turau's – depend on the classification of finite simple groups. Proposition 3.6 is important in reducing the theorem to the case that G is an infinite simple group of Lie type.

Earlier Hartley studied locally soluble groups admitting an automorphism of prime order with Chernikov centralizer [20]. Let $F(G)$ denote the Hirsch-Plotkin radical of a group G, that is, $F(G)$ is the product of all normal locally nilpotent subgroups of G. Hartley's main result is as follows.

Theorem 3.8 *Let G be a periodic locally soluble group admitting an automorphism ϕ of prime order such that $C_G(\phi)$ is Chernikov, and let $F = F(G)$. Then G/F is Chernikov.*

Together Theorems 3.7 and 3.8 reduce further study to the case that G is locally nilpotent. Specifically, in [21] Hartley raised the following question.

Let G be a locally finite q-group admitting an automorphism ϕ of prime order $p \neq q$ such that $C_G(\phi)$ is Chernikov. Does it follow that $[G, \phi]$ is hypercentral?

Recall that a group G is called hypercentral if every quotient of G has non-trivial center. The above question was addressed in [60], where it was answered in the affirmative. The main result of [60] can be stated as follows.

Theorem 3.9 *Let G be a locally finite group admitting an automorphism ϕ of prime order such that $C_G(\phi)$ is Chernikov. Then G is nilpotent-by-Chernikov.*

3.4. Centralizers of Finite Rank

Until very recently our knowledge of locally finite groups admitting an automorphism whose centralizer has finite rank was somewhat vague. However in the past year a real breakthrough has occured. Of course, if the automorphism has order two, a detailed description of the structure of the group is given in Theorem 2.7. The case that the automorphism has odd prime order was studied by Khukhro and Mazurov. In [41] they proved the following theorem.

Theorem 3.10 *Let G be a periodic locally soluble group admitting an automorphism ϕ of prime order p such that $C_G(\phi)$ is of finite rank r. Then G has normal subgroups $G \geq N \geq R$ such that N/R is locally nilpotent and both G/N and R have finite $\{p, r\}$-bounded ranks.*

This theorem reduces all further questions on locally soluble groups admitting an automorphism of prime order with centralizer of finite rank to the (locally) nilpotent case. However we must remember that the general case does not reduce to the locally soluble one as examples of the groups $PSL(2, K)$, where K is an infinite locally finite field of odd characteristic, show. In view of this we should mention another relevant theorem of Khukhro and Mazurov [42].

Theorem 3.11 *Let G be a locally finite p'-group admitting an automorphism ϕ of order p such that $C_G(\phi)$ is of finite rank r. Then G has a normal locally soluble subgroup of $\{p, r\}$-bounded index.*

Thus, at least in the coprime case, the study of locally finite groups with an automorphism of prime order whose centralizer is of finite rank is reduced to (locally) nilpotent groups. Back in the nineties Khukhro raised the following problem (see [46, 13.58]).

Problem 3.12 *Let G be a finite nilpotent group admitting an automorphism ϕ of prime order p. Does G possess a normal subgroup N such that the rank of G/N is $\{p, r\}$-bounded and N is of p-bounded nilpotency class?*

In the case that $p = 2$ this question has been answered positively in [61]. For arbitrary prime p some partial results have been obtained in Khukhro [37]. In particular he proved that if G is a finite soluble group with derived length d admitting an automorphism of prime order p with centralizer of rank r, then G has a subnormal nilpotent subgroup of p-bounded class connected to the group by a subnormal series of $\{d, p, r\}$-bounded length with quotients of $\{d, p, r\}$-bounded ranks. Combining this with the recent results on characteristic subgroups (see Theorem 4.2 in Section 4) Khukhro deduced that if G is a finite nilpotent group of derived length d, then G has a characteristic subgroup C of

p-bounded class such that G/C has $\{d, p, r\}$-bounded rank. Finally, using a very clever argument he was able to show that C can be chosen in such a way that the rank of G/C would be independent of d and thus be $\{p, r\}$-bounded [38]. This gives a positive answer to Problem 3.12. Thus, we have the following theorem.

Theorem 3.13 *(Khukhro) Let G be a locally finite p'-group admitting an automorphism ϕ of order p such that $C_G(\phi)$ is of finite rank r. Then G has normal subgroups $G \geq N \geq R$ such that N/R is nilpotent of r-bounded class and both G/N and R have finite $\{p, r\}$-bounded ranks.*

4. Automorphisms of order four

Kovács proved that if a finite group G admits a fixed-point-free automorphism of order four, then $G/Z(G)$ is metabelian [43]. Kovács' proof uses the famous Feit-Thompson Theorem that any finite group of odd order is soluble [15]. A proof of solubility of a finite group admitting a fixed-point-free automorphism of order four that does not use the Feit-Thompson Theorem can be found in Gorenstein [17, Theorem 10.4.2].

A question about the structure of a finite group G admitting an automorphism ϕ of order four such that $|C_G(\phi)| \leq m$ was suggested by the author in [46, 11.126]. The question was studied in a series of papers by Khukhro and Makarenko. Recently their study was completed (see [40] and references therein), main result being the following theorem.

Theorem 4.1 *There exist an m-bounded number $i = i(m)$ and a constant c such that if a (locally) finite group G admits an automorphism of order four whose centralizer is of finite order m, then G possesses a characteristic subgroup K with the properties that the index $[G : K]$ is at most i and $\gamma_3(K)$ is nilpotent of class at most c.*

One technical result that enabled Khukhro and Makarenko to prove the above theorem deserves a special mentioning. It is well-known that if a group G has an abelian subgroup of finite index n then it also has a characteristic abelian subgroup of n-bounded index. The following nice generalization of this fact is an important step in the proof of Theorem 4.1.

Theorem 4.2 *(Khukhro and Makarenko) If a group G has a subgroup H of finite index n satisfying an identity $\kappa(H) \equiv 1$, where κ is a multilinear commutator of weight w, then G also has a characteristic subgroup C of finite $\{n, w\}$-bounded index satisfying the same identity $\kappa(C) \equiv 1$.*

It is natural to ask if the theorem remains true with κ an arbitrary identity. In particular, the question is interesting if the identity is $x^e \equiv 1$.

With Theorem 4.1 at hand the author was able to prove almost solubility of G in the case that $C_G(\phi)$ is Chernikov [64].

Theorem 4.3 *Let G be a locally finite group admitting an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov. Then G is almost soluble.*

Modulo Theorem 4.1 the proof of the above result is very short and elementary. Comparing this with the results on automorphisms of prime order one comes to the following questions.

Problem 4.4 *Let G be a locally finite group admitting an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov. Is it true that G is metanilpotent-by-Chernikov?*

Problem 4.5 *What is the structure of a (locally) finite group admitting an automorphism of order four whose centralizer has finite rank at most r?*

5. Automorphisms of composite order

Recall Hartley's Theorem 3.7.

Let G be a locally finite group admitting an automorphism ϕ of prime-power order such that $C_G(\phi)$ is Chernikov. Then G is almost locally soluble.

The theorem was proved in [22], where Hartley makes the following comment.

"It seems likely that much more remains to be said. Possibly the theorem remains true even if 'locally soluble' is replaced by 'soluble' and 'prime power order' is replaced by 'finite order', and if that is too much to hope for, then at least some progress in that direction might be feasible".

Of course, one can find many implicit questions in the above paragraph. The most obvious is the following one.

Problem 5.1 *Let G be a locally finite group admitting an automorphism ϕ of finite order such that $C_G(\phi)$ is Chernikov. Is G necessarily almost soluble?*

It is natural to look first at the above problem under the assumption that $C_G(\phi) = 1$.

Assume the hypothesis of Problem 5.1 with $C_G(\phi) = 1$. It is a well-known corollary of the classification of finite simple groups that a finite group with a fixed-point-free automorphism is soluble. Thus, it follows that G is locally soluble. The classical theorem of Dade says that the Fitting height of a finite soluble group is bounded by a function that depends only on the composition length of the Carter subgroup [11]. Applying this to our situation and using the routine inverse limit argument, it follows that G possesses a characteristic series of finite length all of whose quotients are locally nilpotent. Recall that if α is an automorphism of a finite group K and N is a normal α-invariant subgroup of K, then $|C_{K/N}(\alpha)| \leq |C_K(\alpha)|$ (see [35, 1.6.1]). Therefore ϕ induces a fixed-point-free automorphism on every locally nilpotent quotient of the aforementioned series. We conclude that G is soluble if and only if so is every ϕ-invariant locally nilpotent section of G. Thus, we can assume from the outset that G is a q-group for a prime q and we arrive at the the following well-known problem.

Problem 5.2 *Suppose a finite q-group G admits a fixed-point-free automorphism ϕ of order n. Is then G soluble with derived length bounded by a function depending only on n?*

The above problem has been open for many years. So far the existence of a bound on the derived length of G was established only in the cases that n is a prime (Theorem 3.3) or n=4 (Kovács' theorem quoted in the last section). By contrast, the similar question on Lie rings was answered long ago: the earlier mentioned Kreknin theorem says that if a Lie algebra L admits a regular automorphism of order n, then L is soluble with derived length at most $2^n - 2$. Khukhro and Makarenko even managed to prove an "almost regular" analog of this theorem [39]:

Theorem 5.3 *If a Lie algebra L admits an automorphism of finite order n with finite-dimensional fixed-point subalgebra of dimension m, then L has a soluble ideal of derived length bounded by a function of n whose codimension is bounded by a function of m and n.*

Unfortunately it is not clear how the results on Lie algebras can be used in the context of Problem 5.2. Hence, both – Problem 5.2 and Problem 5.1 – at the moment seem impregnable unless n is a prime or $n = 4$. We finish the paper quoting two important results of Hartley that are directly related to Problem 5.1. The first, obtained in [23], generalizes Fong's Theorem 2.1.

Theorem 5.4 *If a locally finite group G contains an element ϕ of order n such that $|C_G(\phi)| \leq m$, then G contains a normal locally soluble subgroup S such that the index $|G : S|$ is m-bounded.*

The next theorem, obtained in a joint work with Belyaev is discussed in [24, Theorem 3.2].

Theorem 5.5 *Let G be a simple locally finite group containing an element ϕ whose centralizer is a Chernikov group. Then G is finite.*

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