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Noetherian Hopf Algebras

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To the memory of my teacher and friend Brian Hartley

Abstract

This short survey article reviews our current state of understanding of the structure of noetherian Hopf algebras. The focus is on homological properties. A number of open problems are listed.

Key Words: Hopf algebra, noetherian, homological integral, AS-Gorenstien, dualizing complex.

1. Introduction

For the first 30 years after Hopf algebras were defined by H. Hopf around 1940 the theory developed quite slowly. The publication of Sweedler's monograph [30] in 1969 quickened the pace, so that understanding of the finite dimensional case in particular grew considerably in the 1970s. But the tectonic plates really shifted with the discovery of quantum groups [7], [12] in the early 1980s, and the years since then have witnessed a massive expansion in both the range of known examples and of our understanding of them.

Many of these new examples of the past 25 years have been noetherian algebras, so it makes sense to ask what features noetherian Hopf algebras hold in common, and which aspects of the finite dimensional theory extend to infinite dimensional noetherian

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Hopf algebras. (We remark in passing that artinian Hopf algebras give us nothing new, since every artinian Hopf algebra is finite dimensional [18].) Such an investigation was proposed in the survey article [2], presented at an AMS meeting in Seattle in 1997. The purpose of the present article is to review what has happened since then: there have indeed been some interesting and beautiful developments. As well as describing some of these, I will list a number of questions which may help to stimulate research on noetherian Hopf algebras over the next decade.

2. Definition and examples

All the algebras in this paper will be defined over a field k which for convenience we shall always assume to be algebraically closed. To say that an algebra A is *affine* means that A is finitely generated as an algebra. A Hopf algebra H is an associative $k-$algebra with a unit element, which is also equipped with

(a) a *counit*; that is, an algebra homomorphism $\varepsilon : H \longrightarrow k$;

(b) a *comultiplication*; that is, an algebra homomorphism $\Delta : H \longrightarrow H \otimes_k H$, which we write using the Sweedler notation: $\Delta(h) = \sum h_1 \otimes h_2$ for $h \in H$;

(c) an *antipode*; that is, an algebra antihomomorphism[1] $S : H \longrightarrow H$.

This apparatus is required to satisfy a number of axioms (essentially the duals of the axioms for a group). We won't list these here as they can be found in all the standard references, for example in [23], [26], [30]. In addition, we'll assume[2] throughout that

the antipode S is bijective. (2.0.1)

This hypothesis may in fact be vacuous - see (7.2) for a discussion. We'll usually assume also that our Hopf algebras H are *left noetherian* - that is, all their left ideals are finitely generated. Thanks to the antiautomorphism of H gauranteed by (2.0.1), this is equivalent to H being right noetherian.

Recall that H is said to be *cocommutative* if $\Delta^{\mathrm{op}}(h) := \sum h_2 \otimes h_1 = \Delta(h)$ for all $h \in H$. In the list of examples below we shall first review the most important classes of

[1]An antihomomorphism is an algebra homomorphism from H to H^{op}.

[2]By no means all results stated here require this hypothesis, but we won't complicate matters by discussing details.

cocommutative Hopf algebras (Exs. 1 and 2); then discuss the noetherian *commutative* Hopf algebras (Exs. 3) ; and then consider some classes of noetherian Hopf algebras which may be neither cocommutative nor commutative (Exs. 4-6).

2.1. Examples

1. Group algebras. For any group G, the group algebra $H = kG$ is a Hopf algebra, with $\varepsilon(g) = 1$, $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$ for $g \in G$. By a variant of Hilbert's basis theorem due to Philip Hall [24, Corollary 10.2.8], kG is noetherian if kG is polycyclic-by-finite, (where this means that G has a finite series $1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$ of subgroups, with $G_i \lhd G_{i+1}$ and G_{i+1}/G_i cyclic or finite for $i = 0, \ldots, n-1$.) It's easy to see that if kT is any noetherian group algebra then T satisfies the ascending chain condition on subgroups, but more than 50 years after Hall proved his theorem it's still not known if T has to be polycyclic-by-finite. So we ask:

Question A: Let kG be a noetherian group algebra. Is G polycyclic-by-finite?

2. Enveloping algebras. Let $\mathfrak{g}$ be a $k-$Lie algebra. Then the enveloping algebra $H = \mathcal{U}(\mathfrak{g})$ is a Hopf algebra with $\varepsilon(x) = 0$, $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $S(x) = -x$ for $x \in \mathfrak{g}$. By (the proof of) the Poincaré-Birkhoff-Witt theorem, $\mathcal{U}(\mathfrak{g})$ is a filtered algebra whose associated graded algebra is a commutative polynomial algebra in $\dim_k(\mathfrak{g})$ indeterminates. Thus, if $\dim_k(\mathfrak{g}) < \infty, \mathcal{U}(\mathfrak{g})$ is a noetherian domain. It doesn't seem to be known whether there are any other examples:

Question B: Suppose that $\mathcal{U}(\mathfrak{g})$ is noetherian. Is $\dim_k(\mathfrak{g}) < \infty$?

Over characteristic 0 Examples 1 and 2 are not far from the complete story for cocommutative Hopf algebras - by theorems of Cartier, Gabriel and Kostant [23, Corollary 5.6.4(3) and Theorem 5.6.5], if k has charactersitic 0 and H is *any* cocommutative Hopf $k-$algebra (not necessarily noetherian), then H is a skew group algebra over the enveloping algebra of the Lie algebra of primitive elements of H.[3] However other examples can occur in positive characteristic - see [23, pages 82-83].

3. Commutative Hopf algebras. The category of commutative affine Hopf $k-$algebras is equivalent to the category of algebraic groups over k [26, Corollary 1.7]: if G is such a group then its coordinate ring $\mathcal{O}(G)$ is a Hopf algebra, with $\varepsilon(f) = f(1_G)$, $\Delta(f)$ the

[3]This needs the algebraic closure of k.

function in $\mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$ defined by $\Delta(f)((x,y)) := f(xy)$ for $x, y \in G$, and $S(f)(x) := f(x^{-1})$ for $x \in G$. And by a theorem of Molnar [22], a commutative Hopf algebra is affine if and only if it is noetherian.

In contrast to the above examples, quantum groups are neither commutative nor cocommutative. Speaking crudely, these split into two families, 4(i) and 4(ii) below, which are, respectively, deformations of some of the algebras in Examples 2 and Examples 3.

4. Quantum groups. (i) **Quantized enveloping algebras** The key examples of these are deformations of the enveloping algebras of semisimple Lie algebras. For each finite dimensional semisimple Lie algebra $\mathfrak{g}$ and each non-zero scalar $q \in k$, (avoiding a few "bad" values), $H = \mathcal{U}_q(\mathfrak{g})$ is a noncommutative noncocommutative noetherian Hopf $k-$algebra.

(ii) **Quantized coordinate rings** For each semisimple algebraic $k-$group G and non-zero scalar q (again avoiding a few values), the quantized coordinate ring $H := \mathcal{O}_q(G)$ is a deformation of the classical coordinate ring of G. It is a noncommutative noncocommutative noetherian Hopf algebra.

There are many references where details of the definitions and basic properties of these algebras in Examples 4 can be found - see, for example, [11], [13], [4]. For H in either of the above classes, there is a fundamental dichotomy determined by the value of the deformation parameter q: namely,

$$H \text{ is a finite module over its centre if and only if } q \text{ is a root of 1 in } k. \tag{2.1.1}$$

5. Hopf algebras satisfying a polynomial identity. For the definition of *a ring satisfying a polynomial identity*, see for example [21]. The dichotomy (2.1.1) just identified for quantum groups can be examined for the other example classes listed above. Thus a group algebra kG is a noetherian polynomial identity algebra if and only if G is a finitely generated abelian-by-finite group [24, Corollaries 5.3.8, 5.3.10]. And the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ satisfies a polynomial identity if and only if $\mathfrak{g}$ is abelian or k has positive characteristic [16], [41]. Prompted by this rather weak evidence, we ask (i) below:

Question C: (i) Suppose that H is a semiprime noetherian Hopf algebra satisfying a polynomial identity. Is H a finite module over its centre?[4]

[4] I'm grateful to Ed Letzter for pointing out to me that [9, Remark 3.9] gives a counterexample if the

(ii) (Wu, Zhang, [36]) Let H be a noetherian Hopf algebra satisfying a polynomial identity. Is H affine?

(iii) Let H be an affine Hopf algebra satisfying a polynomial identity. Is H noetherian?

Molnar's characterisation [22] of commutative noetherian Hopf algebras gives some support to (ii) and (iii). In [2, Question B] I asked whether every affine noetherian PI Hopf algebra was a finite module over a commutative normal sub-Hopf algebra. (For the meaning of *normal* here, see [23, Definition 3.4.1].) It was noted by Gelaki and Letzter in [9] that this is not the case, but their example does not rule out the following refinement:

Question D: Suppose that H is an affine noetherian Hopf algebra satisfying a polynomial identity. Is H a finite module over a commutative normal right co-ideal subalgebra?

This is true for all the PI algebras in the classes 1, 2 and 4.

We introduce the following class primarily so as to include factor Hopf algebras of Examples 4(ii):

6. Filtered algebras. Let H be a Hopf $k-$algebra. We'll say that H is *normally* $\mathbb{N}-$*filtered* if $H = \cup_{i \geq 0} H_i$, with $H_0 = k$ and each H_i a finite dimensional $k-$vector space with $H_i H_j \subseteq H_{i+j}$ for all i, j, such that the associated graded algebra $\mathrm{gr}(H)$ is connected graded noetherian, and so that every graded prime factor ring of $\mathrm{gr}(H)$ is either k, or contains a homogeneous normal element of positive degree.

3. Motivation: finite dimensional Hopf algebras

In the subsequent sections we'll consider generalisations of the classical facts about finite dimensional Hopf algebras which we recall in (3.1) and (3.2).

3.1. Frobenius algebras

Recall that a finite dimensional algebra A is a *Frobenius algebra* if it admits a bilinear form $\phi : A \times A \longrightarrow k$ which is non-degenerate (meaning that $\phi(x, A) \neq 0 \neq \phi(A, x)$ for all $x \in A \setminus \{0\}$), and associative (meaning that $\phi(xh, y) = \phi(x, hy)$ for all $x, y, h \in A$). Notice that this makes A isomorphic to its vector space dual A^* as left and right $A-$module, so that in particular A is an injective left and right $A-$module - in other words, A is *quasi-Frobenius*.

semiprime condition is left out.

In 1969 Larson and Sweedler proved the following fundamental theorem:

Theorem. (Larson, Sweedler, [15]) *Let H be a finite dimensional Hopf algebra. Then H is a Frobenius algebra.*

3.2. Integrals

The left $H-$module isomorphism of H and H^* implies that H contains a unique ideal $\int_H^l$ with $\dim_k(\int_H^l) = 1$ and $hx = \varepsilon(h)x$ for all $x \in \int_H^l$. The ideal $\int_H^l$ is called the *left integral* of H. In a similar way H has a *right integral* $\int_H^r$. If $\int_H^l = \int_H^r$, H is called *unimodular*. For example if G is a finite group then $H = kG$ is unimodular with $\int_H^l = \int_H^r = k\sum_{g\in G} g$.

4. Injective dimension

Self-injective algebras are artinian [28, Proposition XIV.3.1], so it's clear that Theorem 3.1 doesn't generalise directly to infinite dimensional algebras. On the other hand, it's easy to see that, when k has characteristic 0, commutative affine Hopf $k-$algebras are *regular* - that is, they have finite global (projective) dimension. (This is essentially because in characteristic 0 commutative Hopf algebras are semiprime by a theorem of Cartier [34, Theorem 11.4], and the regular action of the group G defines automorphisms mapping any given maximal ideal of $\mathcal{O}(G)$ to any other.) More generally, over any field, commutative noetherian Hopf algebras are *Gorenstein* - that is, they have finite injective dimension, [2, Proposition 2.3]. Now *any* commutative affine Gorenstein (or, *a fortiori*, regular) algebra has injective dimension equal to the "size" d of the algebra, [8, Theorem 21.8]. Here, "size" means the Gelfand-Kirillov dimension, GKdim(−), or Krull dimension (which are equal for a commutative affine algebra). The *Gelfand-Kirillov dimension* of an affine algebra A is a measure of its rate of growth; it has many attractive properties, [14], but unfortunately is often infinite. *Krull dimension*, on the other hand, is *always* defined for a noetherian algebra, but its use often involves difficult technical problems. In any case, it seems that the correct way to impose the relevant "size" constraints in a noncommutative setting may be to demand more stringent homological conditions than simply having finite injective dimension. The relevant definitions are introduced in the next paragraph.

4.1. Homological definitions

Useful sources for the basic facts concerning the following ideas are [1], [3], [17]. A simple but key point to appreciate when considering (b) and (f) is that, for say a *left* $A-$module M, $\mathrm{Ext}^i_A(M, A)$ is a *right* $A-$module via the right action on A.

Definition. Let A be a ring.

(a) The *grade* of the left $A-$module M is

$$j(M) := \inf\{j : \mathrm{Ext}^j_A(M, A) \neq 0\}.$$

(b) A satisfies the *Auslander condition* if, for every noetherian left or right $A-$module M and for all $i \geq 0$, $j(N) \geq i$ for all submodules N of $\mathrm{Ext}^i_A(M, A)$.

(c) The ring A is *Auslander-Gorenstein* if it is noetherian, satisfies the Auslander condition, and has finite right and left injective dimensions (which are then equal by a theorem of Zaks [40]).

(d) If A is Auslander-Gorenstein and has finite global dimension then it is called *Auslander-regular.*

(e) The ring A is *Cohen-Macaulay* (with respect to GKdim) if, for all non-zero noetherian $A-$modules M,

$$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A).$$

(f) Suppose that A is a noetherian Hopf $k-$algebra. Then A is *AS−Gorenstein* if it has right and left injective dimension $d < \infty$, and $\mathrm{Ext}^i_A(k, A) = \delta_{id}k$, where the module k is as usual the trivial (right or left) $A-$module, with A acting through ε.

(g) The Hopf algebra A is *AS−regular* if it is $AS-$Gorenstein and has finite global dimension.

These definitions are closely connected, at least for noetherian Hopf algebras:

Lemma. *([5, Lemma 6.1]) Let H be a noetherian Hopf $k-$algebra. If H is Auslander-Gorenstein and Cohen-Macaulay, then H is $AS-$Gorenstein.*

4.2. Injective dimension of Hopf algebras

In [2, 3.1] and also in [3, 1.15] we asked whether every noetherian Hopf algebra has finite injective dimension. This question remains open, so we restate it here, taking the opportunity to refine it in the light of evidence gathered in the last decade:

Question E: Is every noetherian Hopf algebra $AS-$Gorenstein?

At the time of writing, the answer is “yes” for all known noetherian Hopf algebras. In particular, the algebras listed in (2.1) are all $AS-$Gorenstein. Detailed proofs for classes 1, 2 and 4(i) can be found in [5, §6]; see [10] for class 4(ii). The proof for class (2.1)6 given in [19] is different in flavour; we discuss it briefly in Remark 5(b). The most striking of these positive cases for Question E is class (2.1)5, affine noetherian Hopf algebras satisfying a polynomial identity - the result is a theorem of Wu and Zhang which is both beautiful and technical. In fact, at least formally, they prove a bit more:

Theorem. (Wu, Zhang [36]) *Every affine noetherian Hopf algebra satisfying a polynomial identity is Auslander-Gorenstein and Cohen-Macaulay.*

To illustrate the power of these homological properties we state a sample non-homological corollary, the second (much deeper) part of which follows from the theorem together with results of Stafford and Zhang [29]:

Corollary. *Let H be as in the theorem.*

- *(a) [36, Theorem 0.2(2)] H has a quasi-Frobenius (artinian) ring of fractions.*
- *(b) Suppose that H has finite global dimension. Then H is a finite direct sum of prime rings, and is a finite module over its centre.*

4.3. Integrals of Hopf algebras

While it is perhaps not so surprising that finiteness of the injective dimension should generalise from artinian to noetherian Hopf algebras, it was very surprising - to me at least - that the idea of the integral should do so also. Let ${}_{\varepsilon}k$ denote the trivial left $H-$module. The key points are first, to think of $\int_H^l$ in the artinian case as $\mathrm{Hom}_H({}_{\varepsilon}k, H)$; second, to regard $\mathrm{Hom}_H({}_{\varepsilon}k, H)$ as the case $i = 0$ of $\mathrm{Ext}^i_H({}_{\varepsilon}k, H)$; and third, to recall that these Ext$-$groups are $H-$bimodules, with *left* $H-$action induced by the right action on

${}_{\varepsilon}k$ (and so trivial), and *right* action induced from the right action on H. The definition is due to Lu, Wu and Zhang:

Definition. [20, Definition 1.1] Let H be an $AS-$Gorenstein Hopf algebra of injective dimension d.

(a) The one-dimensional $k-$vector space and $H-$bimodule $\mathrm{Ext}^d_H({}_{\varepsilon}k, {}_{|}H)$ is called the *left integral* of H, denoted $\int^l_H$.

(b) The one-dimensional $k-$vector space and $H-$bimodule $\mathrm{Ext}^d_H(k_{\varepsilon}, H_{|})$ is called the *right integral* of H, denoted $\int^r_H$.

(c) H is *unimodular* if $\int^l_H$ is right trivial as well as left trivial.

One can show quite easily [20, Lemma 1.3] that H is unimodular if and only if $\int^r_H$ is left trivial.

4.4. The Nakayama automorphism

As we saw in (3.1), if A is any Frobenius algebra (for example a finite dimensional Hopf algebra) then A^* is isomorphic to A as left and as right $A-$module. But in general this is *not* an isomorphism of bimodules: in fact the correction is provided by twisting the module on one side by a suitable algebra automorphism

$$A^* \cong {}^{\nu}A^1,$$

[38, Theorem 2.4.1]. Here, ${}^{\nu}A^1$ is the $A-A-$bimodule which is left and right isomorphic to A, with $a.b.c := \nu(a)bc$ for all $a, c \in A$, for all $b \in {}^{\nu}A^1$. In the theory of Frobenius algebras, ν is called the *Nakayama automorphism* of A, well-defined up to an inner automorphism of A. For many purposes - for instance, in representation theory - it's useful to know ν explicitly. When $\nu = \mathrm{Id}$, A is called a *symmetric algebra.*

Recall that if H is *any* Hopf algebra (not necessarily finite dimensional) and $\pi : H \longrightarrow k$ is an algebra epimorphism, the *left winding automorphism* τ^l_{π} is the algebra automorphism

$$\tau^l_{\pi} : H \longrightarrow H : h \mapsto \sum \pi(h_1)h_2.$$

The *right winding automorphism* τ^r_{π} is defined by $\tau^r_{\pi}(h) = \sum h_1\pi(h_2)$ for $h \in H$. The Nakayama automorphism of a finite dimensional Hopf algebra has the following description:

Theorem. (Schneider, [26, Proposition 3.6]) *Let H be a finite dimensional Hopf algebra and let $\pi : H \longrightarrow k$ be the algebra epimorphism whose kernel is the right annihilator of $\int_H^l$. Then the Nakayama automorphism ν of H is $S^2 \circ \tau_\pi^l$.*

5. Dualizing complexes

Theorem 4.4 generalises in a natural way to AS-Gorenstein Hopf algebras, provided we work in the derived category, in particular using concepts developed by Yekutieli [39] and Van den Bergh [33]. Recall that if A is a noetherian algebra, a bounded complex R_A of $A - A-$bimodules (viewed as an object of the bounded derived category $\mathsf{D}(A^e\text{-}\mathsf{Mod})$ of $A - A-$bimodules) is a *rigid dualizing complex* over A if

(a) R has finite injective dimension over A and over A^{op} respectively.

(b) R is homologically finite over A and over A^{op} respectively.

(c) The canonical morphisms $A \to \mathrm{RHom}_A(R, R)$ and $A \to \mathrm{RHom}_{A^{\mathsf{op}}}(R, R)$ are isomorphisms in $\mathsf{D}(A^e\text{-}\mathsf{Mod})$.

(d) A dualising complex R over A is called *rigid* if there is an isomorphism

$$R \cong \mathrm{RHom}_{A^e}(A, R \otimes R^{\mathsf{op}})$$

in $\mathsf{D}(A^e\text{-}\mathsf{Mod})$. (Here the $A - A-$bimodule structure of $R \otimes R^{\mathsf{op}}$ comes from the left A-module structure of R and the left A^{op}-module structure of R^{op}.)

When such a complex R exists it is unique, and $\mathrm{RHom}_A(-, R)$ defines a duality - that is, a contravariant equivalence - between the bounded derived categories of left and right $A-$modules. For example, if A is any finite dimensional algebra then R_A exists and is A^*. So if A is a Frobenius algebra,

$$R_A = A^* \cong {}^{\nu}A^1.$$

If M is an $A-$module and $d \in \mathbb{Z}$, we write $M[d]$ for the complex which has M moved d places to the left (from the 0th place) and 0 elsewhere. We can now state a result generalising this left-right duality from the finite-dimensional case to noetherian AS-Gorenstein Hopf algebras:

Theorem. *[5, Proposition 4.5] Let H be an AS-Gorenstein Hopf algebra of injective dimension d.*

(a) H has rigid dualizing complex ${}^{\nu}H^{1}[d]$, for a certain algebra automorphism ν of H.

(b) The automorphism ν, which we call the Nakayama automorphism of H, is $S^2 \circ \tau^l_\pi$, where π is the epimorphism from H to $H/(\mathrm{r}-\mathrm{ann}(\int^l_H))$.

Naturally, we should ask the following question, which is probably closely related to Question E:
Question F: Does every noetherian Hopf algebra have a rigid dualizing complex?

Remarks. (a) It follows from the above that the Nakayama automorphism and the integrals are crucial to the two-sided structure of AS-Gorenstein Hopf algebras. The calculation of these entities for classes (2.1)1, 2 and 4 is not difficult and has been carried out in [5, §6].

(b) The treatment [19] of the normally $\mathbb{N}-$filtered Hopf algebras of (2.1)6 is the reverse of that given here. Namely, one shows first that such an H *has* a rigid dualizing complex satisfying a rather natural additional property, and then deduces *from this* that H is AS-Gorenstein. As this indicates, it seems that Questions E and F are closely related.

6. Applications of the dualizing complex

6.1. Poincaré duality

For the definition of the Hochschild homology groups $H_i(A, M)$ and cohomology groups $H^i(A, M)$ of an $A-$bimodule M we refer to [35, Chapter 9]. Although classical Poincaré duality fails for noncommutative noetherian Hopf algebras, it seems that it may be valid if we allow twisting by the Nakayama automorphism. Combining Theorem 5 with a result of Van den Bergh [32] we obtain

Theorem. *Let H be a noetherian AS-regular Hopf algebra of global dimension d with Nakayama automorphism ν.*

(a) For every $A-$bimodule M and all $i = 0, \dots, d$

$$H^i(H,\, M) \cong H_{d-i}(H,\, {}^{1}M^{\nu}).$$

(b) *In particular,*

$$H^d(H, {}^{\nu}H^1) \cong H/[H,H] \neq 0,$$

and

$$H_d(H, {}^1H^{\nu}) \cong Z(A) \neq 0.$$

6.2. The antipode

If H is a Hopf algebra (with bijective antipode as usual) then so is $H' := (H, \Delta^{op}, S^{-1}, \varepsilon)$, where $\Delta^{op}(h) := \sum h_2 \otimes h_1$ [23, Lemma 1.5.11]. If H is noetherian and AS-Gorenstein we can apply Theorem 5(b) to it *and* to H'. The latter case yields the Nakayama automorphism

$$\nu' = \tau^r_\pi \circ S^{-2},$$

where τ^r_π is the *right* winding automorphism associated to the epimorphism $\pi : H \longrightarrow H/\mathrm{r}-\mathrm{ann}(\int^l_H)$; see (4.4). However, the Nakayama automorphism of H is unique up to an inner automorphism, by the uniqueness property of rigid dualizing complexes. Since the underlying algebra for H' is the same as for H, ν and ν' differ only by an inner automorphism, proving:

Theorem. *[5, Corollary 4.6] Let H be a noetherian AS-Gorenstein Hopf algebra. Then there exists an inner automorphism γ such that*

$$S^4 = \gamma \circ \tau^r_\pi \circ (\tau^l_\pi)^{-1},$$

where τ^l_π and τ^r_π are the left and right winding automorphisms given by the left integral of H.

Of course we immediately ask:
Question G: What is the inner automorphism γ?

When H is finite dimensional γ is conjugation by the group-like element which is the character of the right structure on $\int^l_{H^*}$, by a 1976 paper of Radford [25]. This suggests that the Hopf dual H° may feature in the answer to Question G.

It's not hard to see that the maps γ, τ^l_π and τ^r_π in the theorem commute with each other [5, Proposition 4.6]. Moreover, when H is a finite module over its centre, τ^l_π and τ^r_π have finite order [6, Theorem 2.3(b)]. It follows that:

Corollary. *If H is a noetherian Hopf algebra which is a finite module over its centre, then some power of the antipode of H is inner.*

Question H: Is the corollary true for an affine noetherian Hopf algebra satisfying a polynomial identity?

The antipode of a finite dimensional Hopf algebra has finite order [25], and $S^2 = \mathrm{Id}$ for a commutative Hopf algebra, [23, Corollary 1.5.12], so it's natural to ask:
Question I: If H is as in the corollary, does S have finite order?

7. Further questions

7.1. Finite global dimension

The possibility that *all* noetherian Hopf algebras have finite injective dimension, together with the motivating commutative, cocommutative and finite dimensional cases, combine to suggest that there may be natural structural conditions on a noetherian Hopf algebra sufficient to guarantee other homological properties such as finite global dimension. If we examine our favourite classes of examples, at least three structural "indicators" of infinite global dimension for a noetherian Hopf algebra H become apparent:

(a) H is not semiprime;

(b) H has a finite dimensional Hopf subalgebra which is not semisimple;

(c) H has a finite dimensional irreducible module of dimension divisible by the characteristic of k.

Of course, more than one of these features can occur in the same example; and easy group algebra examples show that (c) can happen in a regular Hopf algebra. Nevertheless, in part to stimulate the creation of more esoteric examples, we ask:

Question J: Suppose a noetherian Hopf algebra H is not regular. Must at least one of (a), (b), (c) occur?

If this seems too optimistic or too difficult, one might try:

Question K: Let H be a noetherian domain and a Hopf $k-$algebra, and suppose that k has characteristic 0. Is H regular?

Some (slight) positive evidence for Question J is given by

Theorem. (Wu, Zhang [37]) *Let H be a noetherian affine Hopf algebra satisfying a polynomial identity. Suppose that H is involutary - that is, $S^2 = \mathrm{Id}$. If neither (a) nor (c) occurs for H, then H is regular.*

7.2. Bijectivity of the antipode

Recall that we've assumed throughout that our Hopf algebras have a bijective antipode (2.0.1). Examples of Takeuchi [31] show that this hypothesis fails in general. However no example is known of a *noetherian* Hopf algebra whose antipode is *not* bijective, and we have the following theorem and final question:

Theorem. (Skryabin, [27]) *If H is a noetherian Hopf algebra which is either semiprime or affine with a polynomial identity, then its antipode is bijective.*

Question L: (Skryabin) Let H be a noetherian Hopf algebra. Is the antipode S bijective?

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