Turk J Math
31 (2007) , Suppl, 25 – 43.

On Modular Lie Representations of Finite Groups

R. M. Bryant and Ralph Stöhr

Abstract

We give a new and substantially simplified proof of a key technical result in the theory of modular Lie representations of finite groups.

Key Words: free Lie algebras, modular Lie representations of groups, elimination

1. Introduction

Let G be a group and K a field. For any finite-dimensional KG-module V, let $L(V)$ be the free Lie algebra on V (that is, the free Lie algebra over K freely generated by any basis of V), and extend the action of G on V so that $L(V)$ is a KG-module on which each element of G acts as a Lie algebra automorphism. Each homogeneous component $L^n(V)$ is a finite-dimensional submodule of $L(V)$, called the nth Lie power of V.

The central problem on Lie powers is to describe the modules $L^n(V)$ up to isomorphism. In characteristic 0, the structure of $L^n(V)$ has been clarified in a number of papers, including those of Brandt [3], Klyachko [12] and Kraśkiewicz and Weyman [13]. In this paper we assume that $\mathrm{char}(K) = p > 0$, and we take G to be a finite group.

If the order of G is not divisible by p then the Lie powers $L^n(V)$ may be studied by methods similar to those in characteristic 0. Thus we assume that $|G|$ is divisible by p. The smallest such case, where $|G| = p$, turns out to be surprisingly difficult. A deep analysis of this case was conducted in [7], the main result being a recursive description of $L^n(V)$ for an arbitrary finite-dimensional KG-module V. This recursive description was used in [4] to obtain an explicit formula for $L^n(V)$ as an element of the Green ring of G over K.

2000 *AMS Mathematics Subject Classification:* 17B01.

Here we shall describe a proof of the main results of [7] incorporating a new and substantial simplification. However, we shall work in a slightly more general situation, as we now explain.

For certain applications of [7], it turns out to be necessary to generalise the main results from G to H, where H is the holomorph of G, a group of order $p(p-1)$. This generalisation is an important ingredient of the study, in [9], of Lie powers of modules for $GL(2,p)$ and was later used in [5] to describe the Lie powers of an arbitrary finite-dimensional module for a finite group with a Sylow p-subgroup of order p.

The required generalisation was stated in [9, Theorem 3.1 and Corollary 3.2]. However, [9] did not contain proofs of these two results. Instead, reference was made to another paper [14] intended to contain the proofs as an application of 'restricted Lazard elimination'. Unfortunately, some mistakes have been discovered in [14] that invalidate this application: the proof of Theorem 2 and the statements (ii) and (iii) of Lemma 4 in [14] are not correct. This means that there is no valid published proof of [9, Theorem 3.1 and Corollary 3.2].

In view of the importance of these results for [9] and subsequent work it is necessary to set the record straight. Thus we here provide proofs of these results. The main results of [7] for the cyclic group G can be obtained from the results for H by restriction from H to G, and, interpreted in this way, our proofs here give a major simplification of the original proofs in [7]. The key new ingredient is the use of restricted elimination, as in [14]. However, we cannot simplify all of [7]. Thus, for economy of space, reference will be made to [7] for a limited number of self-contained subsidiary results.

2. Preliminaries and statement of results

In the remainder of this paper, K is a field of prime characteristic p and H is the group defined by

$$H = \langle g, h : g^p = h^{p-1} = 1,\, h^{-1}gh = g^l \rangle,$$

where $l \in \{1, \dots, p-1\}$ and l has multiplicative order $p-1$ when considered as an element of K. Thus H has order $p(p-1)$. We wish to find a recursive method for describing $L^n(V)$ up to isomorphism, where n ranges over all positive integers and V ranges over all finite-dimensional KH-modules.

By [9, Theorem 4.1], if V is a finite-dimensional module for any group over any field then $L^n(V)$ is isomorphic to a direct sum of Lie powers of the form $L^m(W)$, where m is a divisor of n and W is an indecomposable direct summand of a tensor power of V. Thus it is enough to consider Lie powers $L^n(V)$ where V is indecomposable.

As is well known, there are, up to isomorphism, precisely $p(p-1)$ indecomposable KH-modules. Details are given in [6, §2], [9, §3] and [11, §1]. We follow the notation of [9] and denote the indecomposable KH-modules by $J_{i,r}$, for $i = 0, 1, \dots, p-2$ and $r = 1, 2, \dots, p$. Here r is the dimension of $J_{i,r}$. Furthermore, $J_{i,r}$ has a basis $Y^{(i,r)}$, where $Y^{(i,r)} = \{y_1^{(i,r)}, \dots, y_r^{(i,r)}\}$, such that the action of g is given by $y_j^{(i,r)}g = y_j^{(i,r)} + y_{j+1}^{(i,r)}$ for $j = 1, \dots, r-1$ and $y_r^{(i,r)}g = y_r^{(i,r)}$, and the action of h on $y_1^{(i,r)}$ is given by $y_1^{(i,r)}h = l^i y_1^{(i,r)}$, where l is regarded as an element of K. In particular, g acts trivially on $J_{i,1}$ and h acts on $J_{i,1}$ as the scalar l^i. For $s = 1, \dots, r$, the subspace $\langle y_s^{(i,r)}, y_{s+1}^{(i,r)}, \dots, y_r^{(i,r)} \rangle$ is a submodule of $J_{i,r}$. The submodules of this form are the only non-zero submodules of $J_{i,r}$ and they give a composition series with factors, from top to bottom, isomorphic to $J_{i,1}, J_{i+1,1}, \dots, J_{i+r-1,1}$. (In using the notation $J_{i,r}$ for $i > p-2$ we take the convention that i is reduced modulo $p-1$.) The unique one-dimensional submodule of $J_{i,r}$ is spanned by $y_r^{(i,r)}$ and h acts on this submodule as the scalar l^{i+r-1}. Furthermore, $J_{i,r}$ is projective if and only if $r = p$.

We take G to be the cyclic subgroup $\langle g \rangle$. On restriction from H to G, $J_{i,r}$ becomes a KG-module denoted by J_r. This is the same as the module J_r of [7], where the basis elements are denoted by $y_1^{(r)}, \dots, y_r^{(r)}$. The modules $J_1, \dots, J_p$ are the indecomposable KG-modules, up to isomorphism, and J_p is the regular KG-module, the unique projective indecomposable. We note that a finite-dimensional KH-module is projective if and only if its restriction to G is a free KG-module.

Since $J_{i,1}$ has dimension 1 we have $L^1(J_{i,1}) = J_{i,1}$ and $L^n(J_{i,1}) = 0$ for $n > 1$. Thus our main results will concern $L^n(J_{i,r})$ for $r \geqslant 2$.

For any set X we write $T(X)$ for the free associative K-algebra freely generated by X. As is well known, if $T(X)$ is regarded as a restricted Lie algebra under the operations given by $[a, b] = ab - ba$ and $a^{[p]} = a^p$, then the free restricted Lie algebra $R(X)$ and the free Lie algebra $L(X)$ may be identified, respectively, with the restricted Lie subalgebra and the Lie subalgebra generated by X in $T(X)$. Thus we take $L(X) \subseteq R(X) \subseteq T(X)$. Note also that $[R(X), R(X)] \subseteq L(X)$: see, for example, [7, §2].

For each non-negative integer n, let $T^n(X)$ denote the nth homogeneous component of $T(X)$ and write $R^n(X) = R(X) \cap T^n(X)$ and $L^n(X) = L(X) \cap T^n(X)$. Thus $T(X) = \bigoplus_{n \geqslant 0} T^n(X)$, $R(X) = \bigoplus_{n \geqslant 1} R^n(X)$ and $L(X) = \bigoplus_{n \geqslant 1} L^n(X)$.

The free metabelian Lie algebra $M(X)$ is defined to be the quotient $L(X)/L(X)''$, where $L(X)''$ is the second derived algebra of $L(X)$. We identify the elements of X, notationally, with their images in $M(X)$ under the natural homomorphism $L(X) \to M(X)$. Thus $M(X)$ is taken to be generated by X. For $n \geqslant 1$, we write $M^n(X)$ for the image of $L^n(X)$ in $M(X)$. Thus $M(X) = \bigoplus_{n \geqslant 1} M^n(X)$.

Let V be a vector space over K and let X be a basis of V. Then we identify V with the subspace of $T(X)$ spanned by X and write $T(V)$, $R(V)$, ... , to denote $T(X)$, $R(X)$, ... , respectively. (The notational ambiguity should cause no problems in practice.)

Suppose that V is a KH-module. Then the action of H on V extends to $T(V)$ so that each element of H acts as an algebra automorphism. Thus $T(V)$, $R(V)$, $L(V)$, $M(V)$ and their homogeneous components become KH-modules.

We shall apply the above notation mainly in the case where $V = J_{i,r}$, using the basis $Y^{(i,r)}$ described above. Thus $Y^{(i,r)}$ is a free generating set for $T(J_{i,r})$, $R(J_{i,r})$, $L(J_{i,r})$ and $M(J_{i,r})$.

For $r \geqslant s \geqslant 1$, there is a surjective homomorphism of KH-modules $J_{i,r} \to J_{i,s}$ given by $y_j^{(i,r)} \mapsto y_j^{(i,s)}$ for $j = 1, \ldots, s$ and $y_j^{(i,r)} \mapsto 0$ for $j = s+1, \ldots, r$. We call this map the *deletion map*. It induces algebra homomorphisms $T(J_{i,r}) \to T(J_{i,s})$, $R(J_{i,r}) \to R(J_{i,s})$, $L(J_{i,r}) \to L(J_{i,s})$ and $M(J_{i,r}) \to M(J_{i,s})$, which are also surjections of KH-modules.

When i and r are understood we write y_j to denote $y_j^{(i,r)}$ for $j = 1, \ldots, r$. In the remainder of this section we assume that $r \geqslant 2$. For $n \geqslant 2$, we write D_n for the subset of $L^n(J_{i,r})$ consisting of all left-normed Lie monomials of the form $[y_{j_1}, y_{j_2}, \ldots, y_{j_n}]$ with $j_1, j_2, \ldots, j_n \in \{1, \ldots, r\}$ and $j_1 < j_2 \geqslant j_3 \geqslant \cdots \geqslant j_n$. As is well known, the corresponding elements of $M^n(J_{i,r})$ form a basis of $M^n(J_{i,r})$. Let $D = \bigcup_{n \geqslant 2} D_n$. Thus the image of $\{y_1, \ldots, y_r\} \cup D$ in $M(J_{i,r})$ is a basis of $M(J_{i,r})$.

For $n \geqslant 2$, let E_n be the subset of D_n consisting of all those elements $[y_{j_1}, y_{j_2}, \ldots, y_{j_n}]$ satisfying $j_1 < j_2 \geqslant j_3 \geqslant \cdots \geqslant j_n$, as before, together with

$$\{m : j_m = 1\} \subseteq \{1, n-p+2, n-p+3, \ldots, n\}. \tag{2.1}$$

Condition (2.1) means that y_1 can occur only in the first and in the last $p-1$ positions. This condition is redundant when $n \leqslant p+1$, so $E_n = D_n$ for $n \leqslant p+1$. In §3 we shall consider the set E, where $E = \bigcup_{n\geqslant 2} E_n$.

For $s = 2, \ldots, r$ and $n \geqslant 2$, let $E_{n,s}$ denote the subset of E_n consisting of those elements $[y_{j_1}, y_{j_2}, \ldots, y_{j_n}]$ with $j_2 = s$. Thus E_n is a disjoint union, $E_n = E_{n,2} \cup \cdots \cup E_{n,r}$. Let $\widetilde{M}^n(J_{i,r})$ denote the subspace of $M^n(J_{i,r})$ with basis given by the images in $M^n(J_{i,r})$ of the elements of $E_{n,r}$. It is straightforward to check that $\widetilde{M}^n(J_{i,r})$ is a KH-submodule of $M^n(J_{i,r})$.

Define $\widehat{L}^p(J_{i,r})$ to be the kernel of the composite map

$$L^p(J_{i,r}) \to M^p(J_{i,r}) \to M^p(J_{i,2}),$$

where the first map is the canonical surjection and the second map is induced by the deletion map $J_{i,r} \to J_{i,2}$. Thus $\widehat{L}^p(J_{i,r})$ is a certain KH-submodule of $L^p(J_{i,r})$, and

$$L^p(J_{i,r}) \cap L(J_{i,r})'' \subseteq \widehat{L}^p(J_{i,r}). \tag{2.2}$$

Since $\dim M^p(J_{i,2}) = p - 1$ we have

$$\dim(L^p(J_{i,r})/\widehat{L}^p(J_{i,r})) = p - 1. \tag{2.3}$$

We set $x_k = y_1 g^{k-1}$ for $k = 1, \ldots, p$. Let $\widehat{L}(J_{i,r})$ and $L_S(J_{i,r})$ be the subspaces of $R(J_{i,r})$ defined by

$$\widehat{L}(J_{i,r}) = L^2(J_{i,r}) + \cdots + L^{p-1}(J_{i,r}) + \widehat{L}^p(J_{i,r}) + L^{p+1}(J_{i,r}) + \cdots$$

(where the right-hand side is, of course, a direct sum) and

$$L_S(J_{i,r}) = \widehat{L}(J_{i,r}) + \langle x_1^p, \ldots, x_p^p \rangle,$$

where $\langle x_1^p, \ldots, x_p^p \rangle$ denotes the subspace spanned by $x_1^p, \ldots, x_p^p$.

The notation $x_1, \ldots, x_p$ was also used in [7]. It is clear that $\langle x_1^p, \ldots, x_p^p \rangle$ is a KG-module, and an easy check shows that it is a KH-module. By [7, Lemma 3.2], $x_1^p, \ldots, x_p^p$ are linearly independent. Thus $\langle x_1^p, \ldots, x_p^p \rangle$ is a regular KG-module. By [6, Lemma 1], since $x_1^p h = (l^i x_1)^p = l^i x_1^p$, we have

$$\langle x_1^p, \ldots, x_p^p \rangle \cong J_{i,p}. \tag{2.4}$$

Also, by [7, Lemma 3.2], $\widehat{L}^p(J_{i,r})$ and $\langle x_1^p, \ldots, x_p^p \rangle$ span their direct sum:

$$\widehat{L}^p(J_{i,r}) + \langle x_1^p, \ldots, x_p^p \rangle = \widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \ldots, x_p^p \rangle. \tag{2.5}$$

Thus

$$L_S(J_{i,r}) = \widehat{L}(J_{i,r}) \oplus \langle x_1^p, \ldots, x_p^p \rangle. \tag{2.6}$$

It is easily seen that $\widehat{L}(J_{i,r})$ and $L_S(J_{i,r})$ are Lie subalgebras and KH-submodules of $R(J_{i,r})$. Following the terminology of [7], we call $L_S(J_{i,r})$ the *shifted Lie algebra* of $L(J_{i,r})$.

The two main results that we shall prove are the following: they are Theorem 3.1 and Corollary 3.2 of [9] or, equivalently, Theorem 2 and the Corollary in §5.2 of [14], except that we have not included statements (2.4) and (2.6) because these have already been established.

Theorem 2.1 *Suppose that $0 \leqslant i \leqslant p-2$ and $2 \leqslant r \leqslant p$. For each $n \geqslant 2$ there exists a KH-submodule U_n of $R^n(J_{i,r})$ such that*

(i) *the Lie subalgebra of $R(J_{i,r})$ generated by $U_2 \oplus U_3 \oplus \cdots$ is free, and $L_S(J_{i,r}) = L(U_2 \oplus U_3 \oplus \cdots)$,*

(ii) *for $n \neq p$, U_n is a direct summand of $L^n(J_{i,r})$,*

(iii) *U_p has the form $\langle x_1^p, \ldots, x_p^p \rangle \oplus V_p$, where V_p is a direct summand of $\widehat{L}^p(J_{i,r})$,*

(iv) *for $n < p$, $U_n \cong M^n(J_{i,r})$, and*

(v) *for $n \geqslant p$, U_n is a projective KH-module.*

Theorem 2.2 *There are KH-module isomorphisms $U_n \cong \bigoplus_{s=2}^{r} \widetilde{M}^n(J_{i,s})$, for $n > p$, and $V_p \cong \bigoplus_{s=3}^{r} \widetilde{M}^p(J_{i,s})$.*

As explained in [9, §3], the isomorphism types of the modules U_n and V_p in Theorems 2.1 and 2.2 can be completely identified, that is, it is possible to compute, recursively, the Krull-Schmidt multiplicities of the indecomposable KH-modules in each U_n and in V_p. Also, as explained in [9, §3 and §4], Theorems 2.1 and 2.2 provide the information necessary for a recursive description of the Lie powers $L^n(J_{i,r})$. Indeed, $L^n(J_{i,r})$ can be obtained from knowledge of $L^m(U_2 \oplus \cdots \oplus U_n)$ for $m < n$, and this allows $L^n(J_{i,r})$ to be obtained from knowledge of Lie powers $L^m(W)$ where W is indecomposable and $m < n$.

We conclude this section by stating without proof some further results about the modules U_n that can be deduced from Theorem 2.2. For $n > p$ these results give simpler descriptions of U_n than those of Theorem 2.2. As before we assume that $r \geqslant 2$. First, for

$n = p+1$, we have $U_{p+1} \cong M^{p+1}(J_{i,r})$. Now suppose that $n > p+1$. Then there is an injective linear map $\theta_n : M^{n-p}(J_{i,r}) \to M^n(J_{i,r})$ given by

$$[y_{j_1}, y_{j_2}, \ldots, y_{j_{n-p}}]\theta_n = [y_{j_1}, y_{j_2}, \ldots, y_{j_{n-p}}, x_1, x_2, \ldots, x_p]$$

for all $j_1, \ldots, j_{n-p} \in \{1, \ldots, r\}$. It is not hard to see that the image of θ_n is a KH-submodule of $M^n(J_{i,r})$ and that $\operatorname{im}\theta_n \cong J_{i,1} \otimes M^{n-p}(J_{i,r})$. The result that can be proved for U_n is

$$U_n \cong M^n(J_{i,r})/\operatorname{im}\theta_n.$$

Thus U_n is isomorphic to a certain factor module of $M^n(J_{i,r})$. Also, since U_n is projective, we have

$$U_n \oplus (J_{i,1} \otimes M^{n-p}(J_{i,r})) \cong M^n(J_{i,r}).$$

3. Elimination

We start with some general facts about free Lie algebras and free restricted Lie algebras.

Let $L(X)$ be the free Lie algebra on a free generating set X. If β is an invertible linear transformation of the subspace $\langle X \rangle$ of $L(X)$ then there is an automorphism of $L(X)$ in which $x \mapsto x\beta$ for all $x \in X$. Hence

$$\{x\beta : x \in X\} \text{ freely generates } L(X). \tag{3.1}$$

If X is a disjoint union, $X = X_1 \cup X_2$, and γ is any map from X_2 to the subalgebra $L(X_1)$ of $L(X)$ then there is an automorphism of $L(X)$ in which $x \mapsto x$ for all $x \in X_1$ and $x \mapsto x + x\gamma$ for all $x \in X_2$. Hence

$$X_1 \cup \{x + x\gamma : x \in X_2\} \text{ freely generates } L(X). \tag{3.2}$$

More generally, suppose that X is a countable disjoint union, $X = X_1 \cup X_2 \cup \cdots$. For each $n \geqslant 1$, let β_n be an invertible linear transformation of $\langle X_n \rangle$ and let γ_n be any map from X_n to $L(X_1 \cup \cdots \cup X_{n-1})$. Then there is a homomorphism $L(X) \to L(X)$ in which, for each n, we have $x \mapsto x\beta_n + x\gamma_n$ for all $x \in X_n$. By the proof of [8, Lemma 2.1], this homomorphism is an automorphism of $L(X)$. Hence

$$\bigcup_{n \geqslant 1} \{x\beta_n + x\gamma_n : x \in X_n\} \text{ freely generates } L(X). \tag{3.3}$$

We shall require 'Lazard elimination' with respect to an element z of X. This is a special case of [2, Chapter 2, §2.9, Proposition 10] and is cited as [14, (1.1)], namely,

$$L(X) = \langle z \rangle \oplus L(X|z), \tag{3.4}$$

where

$$X|z = \{[x, \underbrace{z, \dots, z}_{k}] : x \in X \setminus \{z\},\, k \geqslant 0\}$$

and $L(X|z)$ is the subalgebra of $L(X)$ generated by $X|z$, this subalgebra being freely generated by $X|z$. There is an analogue of (3.4) for the free restricted Lie algebra $R(X)$. This is stated as part of [14, (1.2)], namely,

$$R(X) = \langle z, z^p, z^{p^2}, \dots \rangle \oplus R(X|z). \tag{3.5}$$

The move from $L(X)$ to $L(X|z)$, or from $R(X)$ to $R(X|z)$, is called *full elimination* of z.

'Restricted elimination' is described in [14, Theorem B]. It gives

$$R(X) = \langle z \rangle \oplus R(X|_p z), \tag{3.6}$$

where

$$X|_p z = \{z^p\} \cup \{[x, \underbrace{z, \dots, z}_{k}] : x \in X \setminus \{z\},\, 0 \leqslant k \leqslant p-1\}.$$

The move from $R(X)$ to $R(X|_p z)$ is called *restricted elimination* of z.

We now consider $L(J_{i,r})$ where $r \geqslant 2$. We take i and r as fixed and write $Y = Y^{(i,r)}$ and $y_j = y_j^{(i,r)}$ for $j = 1, \dots, r$. Thus $Y = \{y_1, \dots, y_r\}$.

We use the sets $E_{p,s}$, for $s = 2, \dots, r$, as defined in §2. Let F be any basis of $L^p(J_{i,r}) \cap L(J_{i,r})''$. Since the image of $E_{p,2} \cup \dots \cup E_{p,r}$ in $M^p(J_{i,r})$ is a basis of $M^p(J_{i,r})$, it follows that $F \cup E_{p,2} \cup \dots \cup E_{p,r}$ is a basis of $L^p(J_{i,r})$. From the definition of $\widehat{L}^p(J_{i,r})$ in §2, it is easily verified that

$$F \cup E_{p,3} \cup \dots \cup E_{p,r} \text{ is a basis of } \widehat{L}^p(J_{i,r}). \tag{3.7}$$

For $k = 2, \dots, p$, let

$$d_k = [y_1, \underbrace{y_2, \dots, y_2}_{k-1}, \underbrace{y_1, \dots, y_1}_{p-k}].$$

Thus $E_{p,2} = \{d_2, \dots, d_p\}$. Hence

$$\{d_2, \dots, d_p\} \quad \text{is a basis for } L^p(J_{i,r}) \quad \text{modulo } \widehat{L}^p(J_{i,r}). \tag{3.8}$$

Recall that x_k denotes $y_1 g^{k-1}$ for $k = 1, \dots, p$, and that $\langle x_1^p, \dots, x_p^p\rangle$ has dimension p.

Lemma 3.1 *There exist elements $c_2, \dots, c_p$ of $\widehat{L}^p(J_{i,r})$ and elements $\alpha_{k,s}$ of K, for $k = 2, \dots, p$, $s = 1, \dots, r$, such that*

$$\{y_1^p,\ d_2 + c_2 + \alpha_{2,1}y_1^p + \cdots + \alpha_{2,r}y_r^p,\ \dots,\ d_p + c_p + \alpha_{p,1}y_1^p + \cdots + \alpha_{p,r}y_r^p\}$$

is a basis of $\langle x_1^p, \dots, x_p^p\rangle$.

Proof. By [7, (3.17)], there exist elements $l_2, \dots, l_p$ of $L^p(J_{i,r})$ satisfying

$$x_k^p = y_1^p + \binom{k-1}{1} y_2^p + \binom{k-1}{2} y_3^p + \cdots + \binom{k-1}{r-1} y_r^p + l_k, \tag{3.9}$$

for $k = 2, \dots, p$. Also, by [7, Corollary 3.3], $\{l_2, \dots, l_p\}$ is a basis for $L^p(J_{i,r})$ modulo $\widehat{L}^p(J_{i,r})$. Thus (3.8) yields that there are elements $c_2, \dots, c_p$ of $\widehat{L}^p(J_{i,r})$ such that $\{d_2 + c_2, \dots, d_p + c_p\}$ is a basis of $\langle l_2, \dots, l_p\rangle$. Hence there is an invertible $(p-1)\times(p-1)$ matrix M with entries from K such that

$$(l_2, \dots, l_p)M = (d_2 + c_2, \dots, d_p + c_p).$$

Therefore, by (3.9),

$$(x_2^p, \dots, x_p^p)M = (d_2 + c_2 + w_2, \dots, d_p + c_p + w_p),$$

where $w_2, \dots, w_p \in \langle y_1^p, \dots, y_r^p\rangle$. Since $x_1^p = y_1^p$, it follows that $\langle x_1^p, \dots, x_p^p\rangle$ is spanned by y_1^p, $d_2 + c_2 + w_2$, $\dots$, $d_p + c_p + w_p$. This gives the required result. □

We apply elimination to $L(Y)$ and $R(Y)$, where $Y = \{y_1, \dots, y_r\}$. We first apply (3.4) to $L(Y)$. Full elimination of $y_r, y_{r-1}, \dots, y_1$ (in that order) gives a direct decomposition

$$L(Y) = \langle y_1, \dots, y_r\rangle \oplus L(D),$$

where D is the set defined in §2. Since $L(J_{i,r}) = L(Y)$, it follows that

$$L(J_{i,r})' = L(D), \tag{3.10}$$

where $L(J_{i,r})'$ denotes the derived algebra of $L(J_{i,r})$. (This is a well-known result: see, for example, [1, Chapter 2, §2.4.2].)

We now apply (3.5) and (3.6) to $R(Y)$. Full elimination of $y_r, y_{r-1}, \ldots, y_2$ (in that order) followed by restricted elimination of y_1 gives a direct decomposition

$$R(Y) = \langle y_1, y_2, y_2^p, y_2^{p^2}, \ldots, y_3, y_3^p, y_3^{p^2}, \ldots, y_r, y_r^p, y_r^{p^2}, \ldots\rangle \oplus R(\{y_1^p\} \cup E), \tag{3.11}$$

where E is the set defined in §2.

It will be convenient to describe a subset A of $R(Y)$ as 'Lie-free' if A freely generates the Lie subalgebra of $R(Y)$ that it generates: this free Lie subalgebra is of course written as $L(A)$. Also, let us write

$$E' = E \setminus \{d_2, \ldots, d_p\} = E \setminus E_{p,2}. \tag{3.12}$$

Lemma 3.2 *Let*

$$T = \{y_1^p\} \cup E = \{y_1^p, d_2, \ldots, d_p\} \cup E'.$$

Then T is Lie-free.

Proof. As shown by (3.11), the restricted Lie subalgebra of $R(Y)$ generated by T is freely generated by T. A similar result therefore holds for the Lie subalgebra generated by T. □

It is easy to check that $L(J_{i,r})' \oplus \langle y_1^p\rangle$ is a Lie subalgebra of $R(Y)$. Hence $L(T) \subseteq L(J_{i,r})' \oplus \langle y_1^p\rangle$. Note that every element of D has the form

$$[y_{j_1}, y_{j_2}, \ldots, y_{j_m}, \underbrace{y_1, \ldots, y_1}_{s}, \underbrace{y_1^p, \ldots, y_1^p}_{t}],$$

where $j_1 < j_2 \geqslant j_3 \geqslant \cdots \geqslant j_m \geqslant 2$, $0 \leqslant s \leqslant p-1$ and $t \geqslant 0$. However,

$$[y_{j_1}, y_{j_2}, \ldots, y_{j_m}, \underbrace{y_1, \ldots, y_1}_{s}] \in E.$$

It follows that $D \subseteq L(T)$. Therefore, by (3.10), $L(J_{i,r})' \subseteq L(T)$ and we obtain

$$L(J_{i,r})' \oplus \langle y_1^p\rangle = L(T). \tag{3.13}$$

As follows from (3.13), every element of $L^p(J_{i,r}) \cap L(J_{i,r})''$ belongs to the Lie subalgebra generated by the elements of T of degree less than p (with respect to Y). Hence $L^p(J_{i,r}) \cap L(J_{i,r})'' \subseteq L(E')$, and so $\widehat{L}^p(J_{i,r}) \subseteq L(E')$, by (3.7). Thus, with $c_2, \ldots, c_p$ as in Lemma 3.1,

$$c_2, \ldots, c_p \in L(E'). \tag{3.14}$$

In the next result we use the scalars $\alpha_{k,s}$ of Lemma 3.1, but only those with $s > 1$.

Lemma 3.3 *Let*

$$T^* = \{y_1^p,\, d_2 + \alpha_{2,2}y_2^p + \cdots + \alpha_{2,r}y_r^p,\, \ldots,\, d_p + \alpha_{p,2}y_2^p + \cdots + \alpha_{p,r}y_r^p\} \cup E'.$$

Then T^* is Lie-free.

Proof. Let $\phi : T \to T^*$ be defined by $y_1^p\phi = y_1^p$, $d_k\phi = d_k + \alpha_{k,2}y_2^p + \cdots + \alpha_{k,r}y_r^p$ for $k = 2, \ldots, p$, and $t\phi = t$ for all $t \in E'$. Clearly, ϕ is surjective. Also, by Lemma 3.2, T is Lie-free. Thus ϕ extends to a Lie algebra homomorphism $\phi : L(T) \to R(Y)$. The image of this homomorphism is the Lie subalgebra of $R(Y)$ generated by T^*. Thus it suffices to show that $\ker\phi = 0$.

We may write $R(Y) = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ where, for each $m \geqslant 0$, every element of R_m is a linear combination of monomials of $T(Y)$ that have degree m in y_1. Let $u \in L(T)$, where u is a Lie monomial in the elements of T. For each $t \in T$, we have $t \in R_{m(t)}$ for some $m(t) \geqslant 0$. Thus $u \in R_{m(u)}$ for some $m(u) \geqslant 0$. For $t \in T \setminus \{d_2, \ldots, d_p\}$, we have $t\phi = t$ while, for $t \in \{d_2, \ldots, d_p\}$, we have $m(t) > 0$ and $t\phi = t + t'$ with $t' \in R_0$. Thus we may write $u\phi = u + u'$ where $u' \in R_0 \oplus \cdots \oplus R_{m(u)-1}$: this is interpreted as meaning that $u' = 0$ if $m(u) = 0$.

Now let u be any non-zero element of $L(T)$. For some $m \geqslant 0$, we may write $u = u_0 + u_1 + \cdots + u_m$, where $u_m \neq 0$ and where, for each j, u_j is a linear combination of Lie monomials of $L(T)$ belonging to R_j. Hence $u\phi = u_m + u'$ where $u' \in R_0 \oplus \cdots \oplus R_{m-1}$. Thus $u\phi \neq 0$. Hence $\ker\phi = 0$. □

Proposition 3.4 *Let*

$$S = \{x_1^p, x_2^p, \ldots, x_p^p\} \cup E'.$$

Then S is Lie-free.

Proof. By (3.14), $y_1^p, c_2, \dots, c_p \in L(\{y_1^p\} \cup E')$. Hence it follows from Lemma 3.3 and (3.2) that

$$\{y_1^p,\, d_2 + c_2 + \alpha_{2,1}y_1^p + \cdots + \alpha_{2,r}y_r^p,\, \dots,\, d_p + c_p + \alpha_{p,1}y_1^p + \cdots + \alpha_{p,r}y_r^p\} \cup E'$$

is Lie-free. Therefore, by Lemma 3.1 and (3.1), $\{x_1^p, x_2^p, \dots, x_p^p\} \cup E'$ is Lie-free. □

From the definitions of S and T we see that $|S \cap R^n(Y)| = |T \cap R^n(Y)|$ for all n. Thus there is an isomorphism from $L(S)$ to $L(T)$ induced by a degree-preserving bijection from S to T. It follows that

$$\dim(L(S) \cap R^n(Y)) = \dim(L(T) \cap R^n(Y)) \tag{3.15}$$

for all n. However, by (3.13), we have $L(T) \cap R^n(Y) = L^n(J_{i,r})$ for $n \neq 1, p$. Clearly $L(S) \cap R^n(Y) \subseteq L^n(J_{i,r})$ for $n \neq 1, p$. Thus, by (3.15),

$$L(S) \cap R^n(Y) = L^n(J_{i,r}) \quad \text{for } n \neq 1, p. \tag{3.16}$$

By (3.13),

$$L(T) \cap R^p(Y) = L^p(J_{i,r}) \oplus \langle y_1^p \rangle.$$

Thus, by (2.3),

$$\dim(L(T) \cap R^p(Y)) = p + \dim \widehat{L}^p(J_{i,r}).$$

By (3.7),

$$L(S) \cap R^p(Y) \subseteq \widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle,$$

where the right-hand side has dimension $p + \dim \widehat{L}^p(J_{i,r})$. Thus, by (3.15),

$$L(S) \cap R^p(Y) = \widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle. \tag{3.17}$$

Therefore, by (2.6), (3.16) and (3.17),

$$L(S) = L_S(J_{i,r}) = L^2(J_{i,r}) \oplus \cdots \oplus L^{p-1}(J_{i,r}) \oplus (\widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle) \oplus L^{p+1}(J_{i,r}) \oplus \cdots .$$

Thus S is a free generating set for the shifted Lie algebra $L_S(J_{i,r})$.

Write $Q = L_S(J_{i,r}) = L(S)$ and, for $n \geqslant 2$, $Q_n = Q \cap R^n(Y)$. Thus $Q = \bigoplus_{n\geqslant 2} Q_n$, where $Q_n = L^n(J_{i,r})$ for $n \neq p$ and $Q_p = \widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle$. Note that each Q_n is a KH-module. It will also be convenient to write S as the disjoint union $S = \bigcup_{n\geqslant 2} S_n$, where $S_n = S \cap R^n(Y)$. Thus $S_n = E_n$ for $n \neq p$ and $S_p = (E_p \setminus E_{p,2}) \cup \{x_1^p, \dots, x_p^p\}$.

4. Modules

As in §3, let $Q = L_S(J_{i,r})$, where $r \geqslant 2$. For $n \geqslant 1$, let $Q_{[<n]}$ denote the Lie subalgebra of Q generated by $Q_2, \dots, Q_{n-1}$, that is, the subalgebra of Q generated by all its elements of degree less than n, with the convention that $Q_{[<1]} = Q_{[<2]} = 0$. Of course, $Q_{[<n]} = L(S_2 \cup \cdots \cup S_{n-1})$. Also, for $n \geqslant 2$, let

$$I_n = Q_n \cap Q_{[<n]} = R^n(Y) \cap Q_{[<n]}.$$

Thus I_n is a KH-submodule of Q_n. Since $Q_{[<n]}$ has no elements of degree 1 we have $I_n = Q_n \cap Q_{[<n-1]}$. For $n \leqslant p+1$, $Q_{[<n-1]}$ is generated by $L^2(J_{i,r}), \dots, L^{n-2}(J_{i,r})$, and so $I_n = Q_n \cap L(J_{i,r})''$. Hence, for $n \leqslant p+1$ with $n \neq p$, we have $I_n = L^n(J_{i,r}) \cap L(J_{i,r})''$. Also, by (2.2),

$$I_p = Q_p \cap L(J_{i,r})'' = \widehat{L}^p(J_{i,r}) \cap L(J_{i,r})'' = L^p(J_{i,r}) \cap L(J_{i,r})''.$$

Hence

$$I_n = L^n(J_{i,r}) \cap L(J_{i,r})'' \quad \text{for} \quad n \leqslant p+1, \tag{4.1}$$

and $I_p \subseteq \widehat{L}^p(J_{i,r})$.

Since $Q = L(S)$, we have that

$$Q_n = I_n \oplus \langle S_n \rangle \quad \text{for all} \;\; n. \tag{4.2}$$

Lemma 4.1 *For $n \geqslant 2$ with $n \neq p$, $L^n(J_{i,r})/I_n$ has basis*

$$\{e + I_n : e \in E_{n,2} \cup \cdots \cup E_{n,r}\}.$$

Also, $\widehat{L}^p(J_{i,r})/I_p$ has basis

$$\{e + I_p : e \in E_{p,3} \cup \cdots \cup E_{p,r}\}.$$

Proof. The result for $n \neq p$ is an immediate consequence of (4.2), while the result for $n = p$ follows from (4.1) and (3.7). □

Lemma 4.2 *For each $n \geqslant 2$, let U_n be a subspace of Q_n such that $Q_n = I_n \oplus U_n$, and let B_n be a basis of U_n. Then Q is freely generated by $B_2 \cup B_3 \cup \cdots$, that is, $Q = L(U_2 \oplus U_3 \oplus \cdots)$.*

Proof. In view of (4.2), for each $n \geqslant 2$ there exists an invertible linear transformation β_n of $\langle S_n \rangle$ such that

$$\{b + I_n : b \in B_n\} = \{x\beta_n + I_n : x \in S_n\}.$$

Hence there is a map $\gamma_n : S_n \to I_n$ such that

$$B_n = \{x\beta_n + x\gamma_n : x \in S_n\}.$$

Therefore, by (3.3), Q is freely generated by $B_2 \cup B_3 \cup \cdots$. □

We shall aim to find subspaces U_n satisfying the hypothesis of Lemma 4.2 such that U_n is a KH-submodule of Q_n.

Suppose first that n satisfies $2 \leqslant n \leqslant p-1$. Then $Q_n = L^n(J_{i,r})$, and, by (4.1), $I_n = Q_n \cap L(J_{i,r})''$. By [7, §2], since we have $n < p$, $L^n(J_{i,r})$ splits over $L^n(J_{i,r}) \cap L(J_{i,r})''$ as a KH-module. Thus there is a KH-submodule U_n of Q_n such that $Q_n = I_n \oplus U_n$ and

$$U_n \cong L^n(J_{i,r})/(L^n(J_{i,r}) \cap L(J_{i,r})'') \cong M^n(J_{i,r}).$$

In order to deal with the case $n \geqslant p$ we shall use the modules $\widetilde{M}^n(J_{i,r})$ defined in §2. By [7, (3.20)], the restriction of $\widetilde{M}^n(J_{i,r})$ to G is free as a KG-module for $r \geqslant 2$ and $n \geqslant p+2$. Exactly the same argument shows that this holds, more generally, when $r \geqslant 2$ and $n + r \geqslant p + 3$. Hence, as a KH-module,

$$\widetilde{M}^n(J_{i,r}) \text{ is projective when } r \geqslant 2 \text{ and } n + r \geqslant p + 3. \tag{4.3}$$

Recall that $E_{n,r}$ consists of all Lie monomials $[y_{j_1}, y_r, y_{j_3}, \ldots, y_{j_n}]$ satisfying $j_1 < r$,

$$j_3 \geqslant \cdots \geqslant j_n, \tag{4.4}$$

and

$$\{m : j_m = 1\} \subseteq \{1, n-p+2, n-p+3, \dots, n\}. \tag{4.5}$$

We shall require the following technical lemma which gives a simplified treatment of the essential content of [7, Lemma 3.5] and an argument on page 361 of [7].

Lemma 4.3 *Suppose that $n \geqslant p$ and $r \geqslant 3$. Then $(I_n \cap L(J_{i,r})'') \oplus \langle E_{n,r}\rangle$ is a KH-submodule of $L^n(J_{i,r})$.*

Proof. Let $F_{n,r}$ be the set of all Lie monomials $[y_{j_1}, y_r, y_{j_3}, \dots, y_{j_n}]$ satisfying $j_1 < r$ and (4.5) (but not necessarily (4.4)). For each $f \in F_{n,r}$ there is a unique element f^* of $E_{n,r}$ obtained from f by re-arranging the entries $y_{j_3}, \dots, y_{j_n}$ to satisfy (4.4). Clearly

$$f - f^* \in L(J_{i,r})''. \tag{4.6}$$

We shall show that we also have

$$f - f^* \in I_n. \tag{4.7}$$

For $e \in E_{n,r}$ and $x \in H$, it is easy to see that ex may be written as a linear combination of elements of $F_{n,r}$. Thus, given (4.7), we obtain

$$ex \in (I_n \cap L(J_{i,r})'') \oplus \langle E_{n,r}\rangle,$$

and Lemma 4.3 follows. It remains to prove (4.7).

For $n = p$ and $n = p+1$, (4.7) follows from (4.6) and (4.1). Thus we may assume that $n \geqslant p+2$. Let

$$f = [y_{j_1}, y_r, y_{j_3}, \dots, y_{j_n}] \in F_{n,r}.$$

In order to re-arrange the entries $y_{j_3}, \dots, y_{j_n}$ we need to be able to interchange y_{j_t} and $y_{j_{t+1}}$ when $j_t < j_{t+1}$. However, by (4.5), we do not need an interchange with $y_{j_t} = y_1$ unless $t \geqslant n-p+2$. By the Jacobi identity,

$$[y_{j_1}, y_r, \dots, y_{j_t}, y_{j_{t+1}}, \dots, y_{j_n}] - [y_{j_1}, y_r, \dots, y_{j_{t+1}}, y_{j_t}, \dots, y_{j_n}] = u,$$

where

$$u = [y_{j_1}, y_r, y_{j_3}, \dots, y_{j_{t-1}}, [y_{j_t}, y_{j_{t+1}}], y_{j_{t+2}}, \dots, y_{j_n}].$$

Therefore it suffices to verify that $u \in I_n$ in each of the two cases (i) $1 < j_t < j_{t+1}$ and (ii) $j_t = 1$ with $t \geqslant n - p + 2$. Let $a = [y_{j_1}, y_r, y_{j_3}, \dots, y_{j_{t-1}}]$ and $b = [y_{j_t}, y_{j_{t+1}}]$. Thus we have $u = [a, b, y_{j_{t+2}}, \dots, y_{j_n}]$. By repeated use of the Jacobi identity (see [7, (2.1)]), we may write u as a sum of elements of the form

$$[[a, y_{k_1}, \dots, y_{k_s}], [b, y_{k_{s+1}}, \dots, y_{k_{n-t-1}}]], \tag{4.8}$$

where the list $y_{k_1}, \dots, y_{k_{n-t-1}}$ is a re-arrangement of the list $y_{j_{t+2}}, \dots, y_{j_n}$. It suffices to show that each element (4.8) belongs to I_n. Thus it suffices to show that (4.8) belongs to $Q_{[<n]}$. Write $a^* = [a, y_{k_1}, \dots, y_{k_s}]$ and $b^* = [b, y_{k_{s+1}}, \dots, y_{k_{n-t-1}}]$. Thus $a^* \in L^{t+s-1}(J_{i,r})$ and $b^* \in L^{n-t-s+1}(J_{i,r})$. It suffices to show that $a^* \in Q_{t+s-1}$ and $b^* \in Q_{n-t-s+1}$. The result for a^* is clear if $t + s - 1 \neq p$ and the result for b^* is clear if $n - t - s + 1 \neq p$. However, if $t + s - 1 = p$ then $a^* \in \widehat{L}^p(J_{i,r}) \subseteq Q_p$ because a^* involves y_r and $r \geqslant 3$. Suppose finally that $n - t - s + 1 = p$. Then $t = n - p - s + 1 < n - p + 2$. Hence case (ii) cannot arise here: we must be in case (i). Thus $j_{t+1} \geqslant 3$. Since b^* involves $y_{j_{t+1}}$ we have $b^* \in \widehat{L}^p(J_{i,r}) \subseteq Q_p$, as required. □

We can now prove the key result of this section.

Proposition 4.4 *For $r = 2$, $I_p = \widehat{L}^p(J_{i,r})$, and, for $r \geqslant 3$,*

$$\widehat{L}^p(J_{i,r})/I_p \cong \widetilde{M}^p(J_{i,3}) \oplus \cdots \oplus \widetilde{M}^p(J_{i,r}).$$

For $n \geqslant p + 1$ and $r \geqslant 2$,

$$L^n(J_{i,r})/I_n \cong \widetilde{M}^n(J_{i,2}) \oplus \cdots \oplus \widetilde{M}^n(J_{i,r}).$$

Proof. We first consider the case where $r = 2$. Clearly, $\widehat{L}^p(J_{i,2}) = L^p(J_{i,2}) \cap L(J_{i,2})''$. Thus, for $r = 2$, (4.1) gives $I_p = \widehat{L}^p(J_{i,r})$.

Now suppose that $r = 2$ and $n \geqslant p + 1$. Let $v, w \in E_{n,2}$, where

$$v = [y_1, \underbrace{y_2, \dots, y_2}_{n-p}, \underbrace{y_1, \dots, y_1}_{p-1}], \quad w = [y_1, \underbrace{y_2, \dots, y_2}_{n-1}].$$

Let V be the KG-submodule of $L^n(J_{i,r})$ generated by v. By [7, Lemma 3.4], V is a regular KG-module, and its unique one-dimensional submodule is spanned by w. However, both

y_1 and y_2 are eigenvectors for h, and hence V is a KH-submodule. Since $w \in E_{n,2}$, we have $w \notin I_n$, by Lemma 4.1. It follows that $V \cap I_n = 0$. However,

$$\dim(L^n(J_{i,2})/I_n) = |E_{n,2}| = p = \dim V.$$

Thus $L^n(J_{i,2})/I_n \cong V$. Since the image of w in $M^n(J_{i,2})$ is non-zero, V is isomorphic to its image in $M^n(J_{i,2})$. However, this image is clearly contained in $\widetilde{M}^n(J_{i,2})$, and both V and $\widetilde{M}^n(J_{i,2})$ have dimension p. Thus $V \cong \widetilde{M}^n(J_{i,2})$ and we have $L^n(J_{i,2})/I_n \cong \widetilde{M}^n(J_{i,2})$.

We now suppose that $r \geqslant 3$ and use induction. Thus we may assume that Proposition 4.4 holds with $r-1$ in place of r. We shall need to consider the shifted Lie algebra $L_S(J_{i,r-1})$. For this we use notation similar to that for $L_S(J_{i,r})$, but with a superscript $(i, r-1)$. Thus we write $L_S(J_{i,r-1}) = Q^{(i,r-1)} = \bigoplus_{n \geqslant 2} Q_n^{(i,r-1)}$, where $Q_n^{(i,r-1)} = Q^{(i,r-1)} \cap R^n(J_{i,r-1})$. Furthermore, $I_n^{(i,r-1)} = Q_n^{(i,r-1)} \cap Q_{[<n]}^{(i,r-1)}$, and so on.

Let $\delta : R(J_{i,r}) \to R(J_{i,r-1})$ be the homomorphism induced by the deletion map $J_{i,r} \to J_{i,r-1}$. Since δ is a module homomorphism,

$$x_k\delta = (y_1 g^{k-1})\delta = (y_1\delta)g^{k-1} = y_1^{(i,r-1)} g^{k-1} = x_k^{(i,r-1)},$$

for $k = 1, \ldots, p$. Also, it is easily verified that $\widehat{L}^p(J_{i,r})\delta = \widehat{L}^p(J_{i,r-1})$. Thus, by (2.6), $Q\delta = Q^{(i,r-1)}$, and it follows that $Q_n\delta = Q_n^{(i,r-1)}$ for all n. Therefore $I_n\delta \subseteq I_n^{(i,r-1)}$, and so δ induces surjective homomorphisms of KH-modules

$$\bar{\delta}_n : L^n(J_{i,r})/I_n \to L^n(J_{i,r-1})/I_n^{(i,r-1)}, \quad \text{for } n \geqslant p+1,$$

and

$$\bar{\delta}_p : \widehat{L}^p(J_{i,r})/I_p \to \widehat{L}^p(J_{i,r-1})/I_p^{(i,r-1)}.$$

For each n, let W_n be the KH-submodule of $L^n(J_{i,r})$, or $\widehat{L}^p(J_{i,r})$ when $n = p$, such that $W_n \supseteq I_n$ and $W_n/I_n = \ker \bar{\delta}_n$. By the inductive hypothesis, the image of $\bar{\delta}_n$ is isomorphic to $\widetilde{M}^n(J_{i,2}) \oplus \cdots \oplus \widetilde{M}^n(J_{i,r-1})$, or $\widetilde{M}^p(J_{i,3}) \oplus \cdots \oplus \widetilde{M}^p(J_{i,r-1})$ when $n = p$. By (4.3), this image is projective, and hence $\bar{\delta}_n$ splits over its kernel W_n/I_n. Thus it suffices to show that $W_n/I_n \cong \widetilde{M}^n(J_{i,r})$ for all $n \geqslant p$.

For $n \geqslant p+1$, Lemma 4.1 shows that $L^n(J_{i,r})/I_n$ has basis

$$\{e + I_n : e \in E_{n,2} \cup \cdots \cup E_{n,r}\},$$

while $L^n(J_{i,r-1})/I_n^{(i,r-1)}$ has a similar basis corresponding to $E_{n,2}^{(i,r-1)} \cup \cdots \cup E_{n,r-1}^{(i,r-1)}$. It is easy to see that δ maps $E_{n,2} \cup \cdots \cup E_{n,r-1}$ bijectively to $E_{n,2}^{(i,r-1)} \cup \cdots \cup E_{n,r-1}^{(i,r-1)}$, whereas $E_{n,r}\delta = 0$. It follows that W_n/I_n has basis $\{e + I_n : e \in E_{n,r}\}$. Similarly, $\widehat{L}^p(J_{i,r})/I_p$ has a basis corresponding to $E_{p,3} \cup \cdots \cup E_{p,r}$ while $\widehat{L}^p(J_{i,r-1})/I_p^{(i,r-1)}$ has a basis corresponding to $E_{p,3}^{(i,r-1)} \cup \cdots \cup E_{p,r-1}^{(i,r-1)}$. Again we find that W_p/I_p has basis $\{e + I_p : e \in E_{p,r}\}$.

Let $Z = (I_n \cap L(J_{i,r})'') \oplus \langle E_{n,r}\rangle$, the module given by Lemma 4.3. We know that I_n and $\langle E_{n,r}\rangle$ span their direct sum in $L^n(J_{i,r})$. So, too, do $L^n(J_{i,r}) \cap L(J_{i,r})''$ and $\langle E_{n,r}\rangle$, since the image of $E_{n,r}$ in $M^n(J_{i,r})$ is a basis of $\widetilde{M}^n(J_{i,r})$. Hence

$$Z \cap I_n = I_n \cap L(J_{i,r})'' = Z \cap L(J_{i,r})''.$$

However, $I_n + Z = W_n$ and the image of Z in $M^n(J_{i,r})$ is $\widetilde{M}^n(J_{i,r})$. Thus

$$W_n/I_n \cong Z/Z \cap I_n = Z/Z \cap L(J_{i,r})'' \cong \widetilde{M}^n(J_{i,r}),$$

as required. This completes the proof of Proposition 4.4. □

We now apply Proposition 4.4 to obtain, for $n \geqslant p$, a KH-submodule U_n of Q_n satisfying $Q_n = I_n \oplus U_n$. We have $Q_p = \widehat{L}^p(J_{i,r}) \oplus \langle x_1^p, \ldots, x_p^p\rangle$. By Proposition 4.4 and (4.3), we may write $\widehat{L}^p(J_{i,r}) = I_p \oplus V_p$ where

$$V_p \cong \widetilde{M}^p(J_{i,3}) \oplus \cdots \oplus \widetilde{M}^p(J_{i,r}).$$

Thus we may take $U_p = \langle x_1^p, \ldots, x_p^p\rangle \oplus V_p$. For $n \geqslant p+1$, $Q_n = L^n(J_{i,r})$. Thus, by Proposition 4.4 and (4.3), we may take $Q_n = I_n \oplus U_n$ where

$$U_n \cong \widetilde{M}^n(J_{i,2}) \oplus \cdots \oplus \widetilde{M}^n(J_{i,r}).$$

It now follows that the modules $U_2, U_3, \ldots$ have all the properties required for Theorems 2.1 and 2.2.

References

[1] Bakhturin, Yu. A.: *Identical relations in Lie algebras,* Nauka, Moscow, 1985 (Russian). English translation: VNU Science Press, Utrecht, 1987.

[2] Bourbaki, N.: *Lie groups and Lie algebras*, Part I: Chapters 1-3 (Hermann, Paris, 1975).

[3] Brandt, A. J.: The free Lie ring and Lie representations of the full linear group, *Trans. Amer. Math. Soc.* **56** (1944), 528–536.

[4] Bryant, R. M.: Modular Lie representations of groups of prime order, *Math. Z.* **246** (2004), 603–617.

[5] Bryant, R. M.: Modular Lie representations of finite groups, *J. Austral. Math. Soc.* **77** (2004), 401–423.

[6] Bryant, R. M., Kovács, L. G. and Stöhr, R.: Free Lie algebras as modules for symmetric groups, *J. Austral. Math. Soc. (Ser. A)* **67** (1999), 143–156.

[7] R. M. Bryant, L. G. Kovács and Stöhr, R.: Lie powers of modules for groups of prime order, *Proc. London Math. Soc. (3)* **84** (2002), 343–374.

[8] Bryant, R. M., Kovács, L. G. and Stöhr, R.: Invariant bases for free Lie rings, *Q. J. Math.* **53** (2002), 1–17.

[9] Bryant, R. M., Kovács, L. G. and Stöhr, R.: Lie powers of modules for GL(2,p), *J. Algebra* **260** (2003), 617–630.

[10] Bryant R. M. and Stöhr, R.: Lie powers in prime degree, *Q. J. Math.* **56** (2005), 473–489.

[11] Hughes, I. and Kemper, G.: Symmetric powers of modular representations for groups with a Sylow subgroup of prime order, *J. Algebra* **241** (2001), 759–788.

[12] Klyachko, A. A.: Lie elements in the tensor algebra, *Sibirsk Mat. Ž.* **15** (1974), 1296–1304, 1430 (Russian). English translation: *Siberian Math. J.* **15** (1974), 914–921 (1975).

[13] Kraśkiewicz, W. and Weyman, J.: Algebra of coinvariants and the action of a Coxeter element, *Bayreuth. Math. Schr.* **63** (2001), 265–284.

[14] Stöhr, R.: Restricted Lazard elimination and modular Lie powers, *J. Austral. Math. Soc.* **71** (2001), 259–277.

R. M. BRYANT and Ralph STÖHR
School of Mathematics
University of Manchester
Manchester
M13 9PL
United Kingdom
e-mail: Roger.Bryant@manchester.ac.uk
e-mail: Ralph.Stohr@manchester.ac.uk

Received 13.09.2007