Lengths of Subsets in Coxeter Groups

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1. Introduction

A Coxeter group W is a group which possesses a presentation of the form

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R \rangle$$

where $m_{rs} = m_{sr} \in \mathbb{N} \cup \{\infty\}$, $m_{rr} = 1$ and $m_{rs} \geq 2$ for $r, s \in R, r \neq s$. We shall only deal with finite rank Coxeter groups, so R is assumed to be a finite set. The length of an element w of W, denoted by $l(w)$, is defined to be

$$l(w) = \begin{cases} \min\{l \mid w = r_1 \cdots r_l, r_i \in R\} & \text{if } w \neq 1 \\ 0 & \text{if } w = 1. \end{cases}$$

The notion of the length of an element is a frequent player in proofs of results about Coxeter groups. It is usually to be observed in inductive arguments and its importance can be gauged by seeing how many results would remain if we were to deny ourselves the use of this concept. Undoubtedly its importance has much to do with its interpretation in terms of the root system of W. Almost all significant results about Coxeter groups involve some use of the root system which we now prepare the ground for. Let V be a real vector space with basis $\Pi = \{\alpha_r \mid r \in R\}$. Define a symmetric bilinear form $\langle \, , \, \rangle$ on V by

$$\langle \alpha_r, \alpha_s \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{rs}}\right) & \text{if } m_{rs} < \infty \\ -1 & \text{if } m_{rs} = \infty, \end{cases}$$

where $r, s \in R$ and the m_{rs} are as in the above presentation of W. (By $m_{rs} = \infty$ we mean that rs has infinite order.) Now for $r, s \in R$ if we define

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r$$

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this then extends to an action of W on V which is both faithful and respects the bilinear form $\langle \, , \, \rangle$ (see [12]). The module V is sometimes called a reflection module for W. The following subset of V

$$\Phi = \{w \cdot \alpha_r \mid r \in R, w \in W\}$$

is the root system of W. Setting $\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all R}\}$ and $\Phi^- = -\Phi^+$ we have the fundamental fact that Φ is the disjoint union $\Phi^+ \dot\cup \Phi^-$ (see [12] again). The sets Φ^+ and Φ^- are referred to, respectively, as the positive and negative roots of Φ.

For X a subset of W we define

$$N(X) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in X\}.$$

If $X = \{w\}$, we write $N(w)$ instead of $N(\{w\})$. Clearly $N(X) = \cup_{w \in X} N(w)$. The connection between $l(w)$ and the root system of W mentioned above is that $l(w) = |N(w)|$ (see Section 5.6 of [12]). For a subset X of W we define its Coxeter length, or C-length, to be the cardinality of $N(X)$ and denote it by $l(X)$. It is to be hoped that this more general idea of length will prove to be of some value in investigating Coxeter groups. At the moment – putting our cards on the table – we have yet to see any results which do not explicitly or implicitly mention length in their statements. Nevertheless this general length function has many more properties than one might initially suppose and in this article we will give a sample of some of these properties. Many of these results are drawn from [13], [9], [10] and [11], but we also include new results such as Propositions 3.5, 3.6 and 3.10. The main aim of this survey is to bring these ideas and results to a wider audience.

Our main attention will be focussed upon subsets of W which have a "group theoretic" flavour. So in Section 2 we look at conjugacy classes, followed by subgroups in Section 3. In Section 4 we examine cosets and are led to discuss X-posets which are a vast generalization of the Bruhat order of a Coxeter group. Our final section considers some open questions.

We begin by answering an obvious question – when is $l(X)$ finite?

Lemma 1.1 *[Lemma 2.1; [13]] Let $X \subseteq W$. Then X has infinite Coxeter length if and only if X is infinite.*

To get some inkling of what this general length function looks like, we look at some small examples. Suppose $\mathcal{W}$ is a collection of subsets of W and a_r is the number of subsets in

$\mathcal{W}$ whose C-length is r. Then set

$$\Lambda_{\mathcal{W}}(t) = \sum_{r=0}^{\infty} a_r t^r.$$

So $\Lambda_{\mathcal{W}}(t)$ is analogous to the Poincaré series defined in [Section 1.11; [12]]. Choosing $\mathcal{W}$ to be the collection of all subgroups of W we have

$$
\begin{aligned}
\Lambda_{\mathcal{W}}(t) &= 1 + 2t + 3t^2 \text{ when } W \text{ is of type } A_2; \\
\Lambda_{\mathcal{W}}(t) &= 1 + 3t + 2t^2 + 6t^3 + t^4 + t^5 + 16t^6 \text{ when } W \text{ is of type } A_3; \text{ and} \\
\Lambda_{\mathcal{W}}(t) &= 1 + 4t + 6t^2 + 9t^3 + 12t^4 + 2t^5 + 34t^6 \\
&\quad + t^7 + 7t^8 + 6t^9 + 74t^{10} \text{ when } W \text{ is of type } A_4.
\end{aligned}
$$

(Recall that W being of type A_n means that W is isomorphic to $\mathrm{Sym}(n+1)$.) Not much of a pattern is apparent in the above examples. In Section 3, though, we will give formulae for $\Lambda_{\mathcal{W}}(t)$ when $\mathcal{W}$ consists of all conjugates of certain standard parabolic subgroups of W (with W finite). A standard parabolic subgroup of W is a subgroup of the form $\langle I \rangle$ where I is a subset of R – any subgroup of W conjugate to a standard parabolic subgroup is called a parabolic subgroup of W.

One noticeable feature of the above three polynomials is that the largest coefficient occurs for the largest power of t. Part of the explanation for this is contained in

Proposition 1.2 *(see Proposition 4.1; [10]) Suppose X is a finite subgroup of W which is not contained in any proper parabolic subgroup of W. Then $N(X) = \Phi^+$.*

Proposition 1.2 follows immediately from the more general Proposition 3.5, which is proved in Section 3.

2.   Conjugacy Classes

A conjugacy class X of W is called a flat class if all its elements have the same length. Flat classes in finite irreducible Coxeter groups have been classified in [Theorem 1.3; [14]]. We remark that flat classes are to be found in infinite Coxeter groups. For example, let $W = \langle r, s, t : m_{rr} = m_{ss} = m_{tt} = 1, m_{rs} = m_{rt} = m_{st} = 3 \rangle$, the Coxeter group of type $\tilde{A}_2$. Then the element $rstrst$ lies in a flat conjugacy class (the other members of which are $strstr$ and $trstrs$).

Theorem 2.1 *Let W be an infinite Coxeter group and X a conjugacy class of W. Then $N(X) = \Phi^+$ if and only if X is not a flat class.*

Broadly speaking the situation is the same for finite Coxeter groups as we see with our next result.

Theorem 2.2 *Suppose that W is a finite irreducible crystallographic Coxeter group, and that X is a conjugacy class of W. Then $N(X) = \Phi^+$ if and only if X is neither the identity class or of type $Cl(6)$.*

We shall not give the definition of a type $Cl(6)$ class here, referring the reader to [Definition 1.5 [13]], but remark that such classes are flat. So, from the above mentioned classification, it can be seen that the only type $Cl(6)$ classes in Theorem 2.2 are:-

(i) $X = (x^2)^W$, $|X| = 16$, $l(X) = 22$ with W of type F_4 and x any Coxeter element of W.

(ii) $X = (x^5)^W$, $|X| = 4480$, $l(X) = 119$ with W of type E_8 and x any Coxeter element of W.

This gives us pause for thought – among all the irreducible crystallographic Coxeter groups and all their non-trivial conjugacy classes X, $N(X) \neq \Phi^+$ in only two instances. For W of type H_3 and H_4, see [Table 2; [13]] for a list of those non-trivial classes X with $l(X) < |\Phi^+|$ (H_3 has one and H_4 has four).

3. Subgroups

For $w \in W$ and a fundamental reflection r (that is $r \in R$), it is well known that $l(rwr) = l(w) \pm 2$ or $l(w)$. Now if X is a finite subgroup of W and r a fundamental reflection there is, in general, not such a close relationship between $l(X)$ and $l(X^r)$. However we do have the following result.

Lemma 3.1 *Suppose X is a finite subgroup of W, $r \in R$ and let $\mathcal{O}_r$ be the X-orbit of α_r. Then*

$l(X^r) < l(X)$ *if $\mathcal{O}_r \subseteq \{\alpha_r\} \cup (\Phi^-\backslash\{-\alpha_r\})$.*

$l(X^r) > l(X)$ *if $\mathcal{O}_r \subseteq \Phi^+$ and $\mathcal{O}_r \neq \{\alpha_r\}$.*

$l(X^r) = l(X)$ *if either $\mathcal{O}_r \subseteq \{\pm\alpha_r\}$ or both $\mathcal{O}_r \cap \Phi^- \neq \emptyset$ and $\mathcal{O}_r \cap \Phi^+ \neq \{\alpha_r\}$.*

Amongst the subgroups of a Coxeter group, the parabolic subgroups and particularly the standard parabolic subgroups occupy a premier position. So it is to these subgroups that we turn next.

Theorem 3.2 *[Proposition 1.1; [13]] Let X be a finite standard parabolic subgroup of W, and Y be conjugate to X. Then $l(X) \leq l(Y)$, with equality if and only if Y is also a standard parabolic subgroup of W.*

A great deal more specific information about parabolic subgroups has been obtained. For X any subgroup of W we use $\mathbb{X}$ to denote the conjugacy class of X in W and let $\mathbb{X}_{\min}$ be the set consisting of all Y of minimal Coxeter length in $\mathbb{X}$, and, when it is defined, $\mathbb{X}_{\max}$ denotes those conjugates of X of maximal length in $\mathbb{X}$.

Theorem 3.3 *[Theorem 1.2; [9]]*

(i) If $W \cong A_n$ and $X \cong A_i$, then

$$\Lambda_{\mathbb{X}}(t) = t^{i(i+1)/2} \sum_{r=0}^{n-i} (n+1-i-r) \binom{r+i-1}{i-1} t^{(i+1)r};$$

(ii) If $W \cong B_n$ and $X \cong B_i$, then

$$\Lambda_{\mathbb{X}}(t) = t^{i^2} \sum_{r=0}^{n-i} \binom{r+i-1}{i-1} t^{2ir};$$

(iii) If $W \cong D_n$ and $X \cong D_i$, then

$$\Lambda_{\mathbb{X}}(t) = t^{i(i-1)} \sum_{r=0}^{n-i} \binom{r+i-1}{i-1} t^{2ir}.$$

Theorem 3.3 does not cover the connected parabolic subgroups of type A_m when W is of type either B_n or D_n. For such subgroups see Lemma 3.2 and Theorems 3.10 and 4.5 of [9]. When X is not connected, the situation appears to be very complicated and there are virtually no results here – see Section 5 of [9] for an indication of the quagmire waiting there. When it comes to the finite exceptional Coxeter groups, the connected parabolic subgroups are dealt with in [9]. To whet the reader's appetite we give the details for type E_6 in Table 1 (see Table 1 in [9]).

Lengths of parabolic subgroups are particularly important in the light of Proposition 3.5. In order to state Propositions 3.5 and 3.10, we require the following definition.

Table 1. Connected parabolic subgroups of E_6

Type of X	$\Lambda_{\mathbb{X}}(t)$
A_1	$6 + 5t^3 + 5t^5 + 5t^7 + 4t^9 + 3t^{11} + 3t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21}$
A_2	$5t^3 + 10t^6 + 15t^9 + 16t^{12} + 15t^{15} + 18t^{18} + 14t^{21} + 8t^{24} + 9t^{27} + 10t^{30}$
A_3	$5t^6 + 12t^{10} + 9t^{12} + 8t^{14} + 16t^{16} + 14t^{18} + 38t^{20} + 14t^{22} + 16t^{24}$
	$+23t^{26} + 2t^{28} + 46t^{30} + 67t^{34}$
A_4	$4t^{10} + 4t^{15} + 28t^{20} + 10t^{25} + 40t^{30} + 130t^{35}$
A_5	$t + 35t^{35}$
D_4	$t^{12} + 8t^{20} + 8t^{28} + 28t^{36}$
D_5	$2t^{20} + 25t^{36}$

Definition 3.4 *Let $X \leq W$. Define $\langle X \rangle_p$ to be the intersection of all parabolic subgroups containing X. Similarly, define $\langle X \rangle_{sp}$ to be the intersection of all standard parabolic subgroups containing X.*

Proposition 3.5 *Let $X \leq W$ be a finite subgroup of W with $P = \langle X \rangle_p$. Then $N(X) = N(P)$.*

Note that by Exercise 3.3 of [8], $\langle X \rangle_p$ is a parabolic subgroup of W.

Proof. By a result of Tits, any finite subgroup of a Coxeter group W is contained in a finite parabolic subgroup of W (see [3], Chapter 5, Section 4). Since X is finite by hypothesis, P must also be finite. Now $X \leq P$, so clearly $N(X) \subseteq N(P)$. Suppose, for a contradiction, that there exists $\alpha \in N(P) \setminus N(X)$. Define $v \in V$ to be

$$v = \sum_{x \in X} x \cdot \alpha.$$

Now $v \neq 0$ because $x \cdot \alpha \in \Phi^+$ for each $x \in X$, by assumption. In addition P is finite, which implies that the form $\langle \, , \, \rangle$ is positive definite on the linear span of $N(P)$ (which contains v). Thus the radical is trivial and therefore the stabilizer of v cannot be the whole of P. It is well known that the stabilizer of any non-zero vector in V is a parabolic subgroup, say Q, of W. Clearly X is contained in Q. But then $X \subseteq Q \cap P \lneq P$, which contradicts the minimality of P. Hence $N(X) = N(P)$, as stated. $\qquad\square$

Proposition 3.6 *Let X be a finite subgroup of W and assume that $X \in \mathbb{X}_{\min}$. Then there is a standard parabolic subgroup W_I of W such that $X \leq W_I$ with $N(X) = N(W_I)$.*

Proof. By Proposition 3.5, $N(X) = N(\langle X \rangle_p)$. By Theorem 3.2, $\langle X \rangle_p$ has minimal length if and only if it is a standard parabolic subgroup. In order for X to have minimal length in $\mathbb{X}$ therefore, we must have $\langle X \rangle_p = W_I$ for some $I \subseteq R$, giving $N(X) = N(W_I)$.
□

Let X and Y be conjugate subgroups of W. We say that $X \searrow Y$ if there exists a sequence $r_1, r_2, \ldots, r_n$ of fundamental reflections such that $Y = X^{r_1 \cdots r_n}$, and, for $i \geq 1$, $l(X^{r_1 \cdots r_{i-1} r_i}) \leq l(X^{r_1 \cdots r_{i-2} r_{i-1}})$. We can make an analogous definition, that $X \nearrow Y$ if there exists a sequence $r_1, r_2, \ldots, r_n$ of fundamental reflections such that $Y = X^{r_1 \cdots r_n}$, and, for $i \geq 1$, $l(X^{r_1 \cdots r_{i-1} r_i}) \geq l(X^{r_1 \cdots r_{i-2} r_{i-1}})$. Supposing C is a conjugacy class of elements of W, we let

$$C_{\min} = \{w \in C \mid w \text{ has minimal length in } C\}; \text{ and}$$
$$C_{\max} = \{w \in C \mid w \text{ has maximal length in } C\}.$$

If X is a cyclic subgroup of W, we have

Theorem 3.7 *[Theorem 1.2; [13]] Suppose that W is a finite irreducible Coxeter group and that $X = \langle x \rangle$ is a cyclic subgroup of W. Let C denote the W-conjugacy class of x.*

(i) For each $Y \in \mathbb{X}$ there exists $Y' \in \mathbb{X}_{\min}$ such that $Y \searrow Y'$.

(ii) If $w \in C_{\min}$, then $\langle w \rangle \in \mathbb{X}_{\min}$.

Theorem 3.8 *[Theorem 1.3; [13]] Suppose that W is a finite irreducible Coxeter group and that $X = \langle x \rangle$ is a cyclic subgroup of W. Let C denote the W-conjugacy class of x.*

(i) For each $Y \in \mathbb{X}$ there exists $Y' \in \mathbb{X}_{\max}$ such that $Y \nearrow Y'$.

(ii) If $w \in C_{\max}$, then $\langle w \rangle \in \mathbb{X}_{\max}$.

These results are obtained with the aid of analogous results about elements $w \in W$ in [7] and [6]. Our next Proposition may be viewed as a generalization of Theorem 3.2.9 (ii) of [8].

Proposition 3.9 *Let X be a finite subgroup of W and suppose $Y_1, Y_2 \in \mathbb{X}_{\min}$. Then there exist $g \in W$ and $r_1, \ldots, r_n \in R$ such that, writing $Z = Y_1^g$, the following hold:*

(i) $N(Z) = N(Y_2)$ and $l(Y_1 g) = l(Y_1) + l(g)$; and

(ii) $Y_2 = r_n \cdots r_1 Z r_1 \cdots r_n$, and for $1 \le i \le n$, $N(Z^{r_1 \cdots r_i}) = N(Y_2)$, and hence, in particular, $l(Z^{r_1 \cdots r_i}) = l(Y_2)$.

Proof. Let $Y_1, Y_2 \in \mathbb{X}_{\min}$. Then by Corollary 3.6, there exist $I, J \subseteq R$ such that $N(Y_1) = N(W_I)$ and $N(Y_2) = N(W_J)$. In particular W_I and W_J are conjugate, which implies that there exists $g \in W$, a minimal double coset representative of W_I in W, such that $J = I^g$. Then $N(g^{-1}) \cap N(Y_1) = N(g^{-1}) \cap N(W_I) = \emptyset$, and so, using Lemma 4.3 (see Section 4),

$$N(Xg) = g^{-1}N(Y_1) \dot\cup N(g).$$

Thus $l(Xg) = l(Y_1) + l(g)$, and setting $Z = Y_1^g$ we see that $\langle Z \rangle_p = W_I^g = W_J$, so giving $N(Z) = N(Y_2)$. We have now established part (i).

For part (ii), note that Z and Y_2 are both contained in the standard parabolic subgroup W_J. They are therefore conjugate in W_J. So there exist $r_1, \ldots, r_n \in J \subseteq R$ with $Y_2 = r_n \cdots r_1 Z r_1 \cdots r_n$. It is easy to see that for any set H with $N(H) = N(W_J)$, and any $r \in J$, we have $N(rHr) = N(H) = \Phi_J^+$, and (ii) immediately follows. $\square$

If $X \le W$, it is possible that for each $Y \in \mathbb{X}$, $l(Y) = l(X)$. If this holds, we say that X is a *flat* subgroup of W. The following result gives a condition which is necessary for this to occur.

Proposition 3.10 *Let $X \ne 1$ be a finite subgroup of W, and let W_I denote the standard parabolic subgroup $\langle \mathbb{X} \rangle_{sp}$ of W. If X is flat, then $N(X) = \Phi_{W_I}^+$.*

Proof. By a result of Tits (see [3], Chapter 5, Section 4), X is contained in a finite parabolic subgroup of W. This implies that $\langle X \rangle_p$ is finite. Suppose now that X is flat. Then all conjugates of X have the same length. We may therefore assume, without loss of generality, that $\langle X \rangle_p$ is a finite standard parabolic subgroup W_K for some $K \subseteq R$. Hence, by Proposition 3.5, $N(X) = N(\Phi_{W_K}^+)$. Clearly W_K is a standard parabolic subgroup of W_I. Suppose, for a contradiction, that $K \ne I$. By definition of W_I, we see that there must exist $r \in K$ and $s \in I \setminus K$ such that $m_{rs} > 2$ (if this were not the case, then we would have $X^W \subseteq W_K \subsetneq W_I$). Letting $Y = sXs$, we will show that $l(Y) > l(X)$.

Let $\alpha \in \Phi_{W_K}^+ = N(X)$. Then $s \cdot \alpha \in \Phi^+$, because $\alpha_s \notin \Phi_{W_K}^+$. Choose $x \in X$ with $\alpha \in N(x)$. Then we have

$$(sxs) \cdot (s \cdot \alpha) = sx \cdot \alpha \in s \cdot (\Phi_{W_K}^-) \subseteq \Phi^-.$$

Therefore $s \cdot \Phi^+_{W_K} \subseteq N(Y)$. Now the stabilizer of α_s is a parabolic subgroup of W which is not equal to W_K. Since W_K is the intersection of all parabolic subgroups containing X, we deduce that X is not contained in the stabilizer of α_s. Hence there exists $x \in X$ with $x \cdot \alpha_s \in \Phi^+ \setminus \{\alpha_s\}$ and hence

$$(sxs) \cdot \alpha_s = -s(x \cdot \alpha_s) \in \Phi^-.$$

We have now shown that $s \cdot N(X) \dot\cup \{\alpha_s\} \subseteq N(Y)$. But this means that $l(Y) \geq l(X) + 1$, contradicting our assumption that X is flat. Therefore $I = K$, which gives $N(X) = N(W_I) = \Phi^+_{W_I}$, so proving Proposition 3.10. $\qquad\square$

4. Cosets

We begin by examining the length of Xr where r is a fundamental reflection and X is a subset of W. We draw the reader's attention to the fact that our group elements are written on the right of X – so when X is a subgroup we are looking at right cosets of X. We return to discuss what is behind this choice shortly.

Proposition 4.1 *[Proposition 1.6; [10]] If X is a finite subset of W and $r \in R$, then*

$$l(Xr) = \begin{cases} l(X) + 1 & \text{if } \alpha_r \notin N(X); \\ l(X) - 1 & \text{if } \alpha_r \in N(x) \text{ for all } x \in X; \\ l(X) & \text{otherwise.} \end{cases}$$

Setting $|X| = 1$, we retrieve the familiar fact that for each $w \in W$, $r \in R$, we have

$$l(wr) = \begin{cases} l(w) + 1 & \text{if } w \cdot \alpha_r \in \Phi^+; \\ l(w) - 1 & \text{if } w \cdot \alpha_r \in \Phi^-. \end{cases}$$

If we demand that X be a subgroup of W then we have

Proposition 4.2 *[Proposition 1.7; [10]]Let $g = r_1 \cdots r_k \in W$ be a reduced expression for g and let X be a finite subgroup of W. Then*

$$l(X) \leq l(Xr_1) \leq \cdots \leq l(Xr_1 \cdots r_k) = l(Xg).$$

In particular, for all $g \in G$, $l(Xg) \geq l(X)$.

The 'left handed' versions of Propositions 4.1 and 4.2, comparing the length of gX with the length of X, do not hold. This lack of symmetry is due to the fact that we are acting on the left of Φ. For example let W be the Coxeter group of type A_4. Then $W \cong \mathrm{Sym}(5)$ and if we take $r = (12)$ and $X = \{(132), (12)(34)\}$ we have $N(X) = \{\alpha_{(12)}, \alpha_{(23)}, \alpha_{(34)}, \alpha_{(13)}\}$. Thus $l(X) = 4$ but $l(rX) = 2 < l(X) - 1$. If we take X to be the parabolic subgroup generated by (13) and (45), then $l(X) = 4$ but $l((12)X) = 3 < l(X)$.

It is sometimes useful to know $N(Xg)$ explicitly. To this end, we have the following

Lemma 4.3 *[Lemma 2.2; [10]] Suppose that X is a subgroup of W. Then for $g \in W$,*

$$N(Xg) = N(g) \dot\cup g^{-1}(N(X) \backslash N(g^{-1}))$$

and hence, when X is finite,

$$l(Xg) = l(g) + l(X) - |N(X) \cap N(g^{-1})|.$$

The following is Proposition 3.3 of [10] – we denote the set of reflections of W by $\mathrm{Ref}(W)$.

Proposition 4.4 *Let X be a finite subset of W and let $s \in \mathrm{Ref}(W)$. If $l(Xs) \leq l(X)$, then there exists $x \in X$ such that $l(xs) < l(x)$.*

Proposition 4.4 implies that for the specified $x \in X$, xs precedes x in the well-known Bruhat order on W (see Section 5.9 of [12]). In [10] and [11] a generalized notion of the Bruhat order on cosets of subgroups in a finite Coxeter groups is introduced and investigated. In [5], Deodhar extended the usual Bruhat order on finite Coxeter groups to a partial order on the (right) cosets of a standard parabolic subgroup X of W. The fact that every right coset of X has a unique element whose length is minimal in that coset is crucial in this construction. For cosets Xg, Xh, this partial order is defined by $Xg < Xh$ if $g < h$ (where g and h are the elements of minimal length in their cosets). We shall write $\mathcal{B}(X)$ for this partial order on the (right) cosets of X in W. A further generalization of $\mathcal{B}(X)$ (where X is a standard parabolic subgroup of W), called generalized quotients, was given in [2] by Björner and Wachs. This poset has many properties in common with $\mathcal{B}(X)$. For example the Möbius function of generalized quotients (and $\mathcal{B}(X)$) takes values $-1, 0$ or 1 [Corollary 3.6; [2]]. However the Möbius function for the posets to be introduced in Definition 4.5 does not necessarily take values in $\{-1, 0, 1\}$ (see [11]) and so these posets do indeed differ from generalized quotients.

Definition 4.5 *Suppose that $X \leq W$ where W is a finite Coxeter group.*

(i) For right cosets Xg and Xh of X we write $Xg \sim Xh$ whenever $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and $l(Xg) = l(Xh)$. Let $\approx$ be the equivalence relation generated by $\sim$ on the set of right cosets of X in W and let $\mathfrak{X}$ be the set of $\approx$ equivalence classes.

(ii) Let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$. We write $\mathbf{x} \rightsquigarrow \mathbf{x}'$ if there is a right coset Xg in $\mathbf{x}$ and a right coset Xh in $\mathbf{x}'$ such that $Xgt = Xh$ for some $t \in \mathrm{Ref}(W)$ and $l(Xg) < l(Xh)$. The partial order $\preccurlyeq$ on $\mathfrak{X}$ is defined by $\mathbf{x} \preccurlyeq \mathbf{x}'$ if and only if there exist $\mathbf{x_1}, \dots, \mathbf{x_m} \in \mathfrak{X}$ such that $\mathbf{x} \rightsquigarrow \mathbf{x_1} \rightsquigarrow \dots \rightsquigarrow \mathbf{x_m} \rightsquigarrow \mathbf{x}'$. We shall call $\mathfrak{X}$ the X-poset (of W).

(iii) If, in (i) and (ii), we use the set of fundamental reflections instead of $\mathrm{Ref}(W)$ we may define, analogously, the weak X-poset (of W) denoted by $\mathfrak{X}_w$ with ordering $\preccurlyeq_w$.

For the coset Xg we use $[Xg]$, respectively $[Xg]_w$, to denote the $\approx$ equivalence class, respectively the $\approx_w$ equivalence class containing Xg. Of many results about X-posets to be found in [10] we mention the following.

Theorem 4.6 (Theorems 1.2 and Theorem 1.3(iii); [10]) *Suppose W is a finite Coxeter group, $X \leq W$ and let $\mathfrak{X}$, respectively $\mathfrak{X}_w$, denote the X-poset, respectively weak X-poset. Then both $\mathfrak{X}$ and $\mathfrak{X}_w$ have a unique minimal element, namely the $\approx$ (respectively $\approx_w$) equivalence class containing X. In addition, both the weak X-poset and the X-poset are symmetric. That is if $[Xg] \preccurlyeq [Xh]$, respectively $[Xg]_w \preccurlyeq_w [Xh]_w$, then $[Xhw_0] \preccurlyeq [Xgw_0]$, respectively $[Xhw_0]_w \preccurlyeq_w [Xgw_0]_w$ (where w_0 is the unique longest element of W).*

For $\mathbf{x} \in \mathfrak{X}$, $l(\mathbf{x})$ is defined to be $l(Xg)$ for any $Xg \in \mathbf{x}$. As an example of an X-poset, Figure 1 gives the X-poset in the case where W is of type F_4, with Coxeter graph
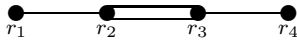
 and $X = \langle [r_3r_2r_3r_4r_3r_2r_3r_4] \rangle \cong \mathbb{Z}_2$. Elements on the same horizontal level have their length indicated on the right. In fact this is the Hasse diagram of the poset, which is a ranked poset as labelled by the length of the elements. For a collection of Hasse diagrams of X-posets of all shapes and sizes we direct the reader to [11].
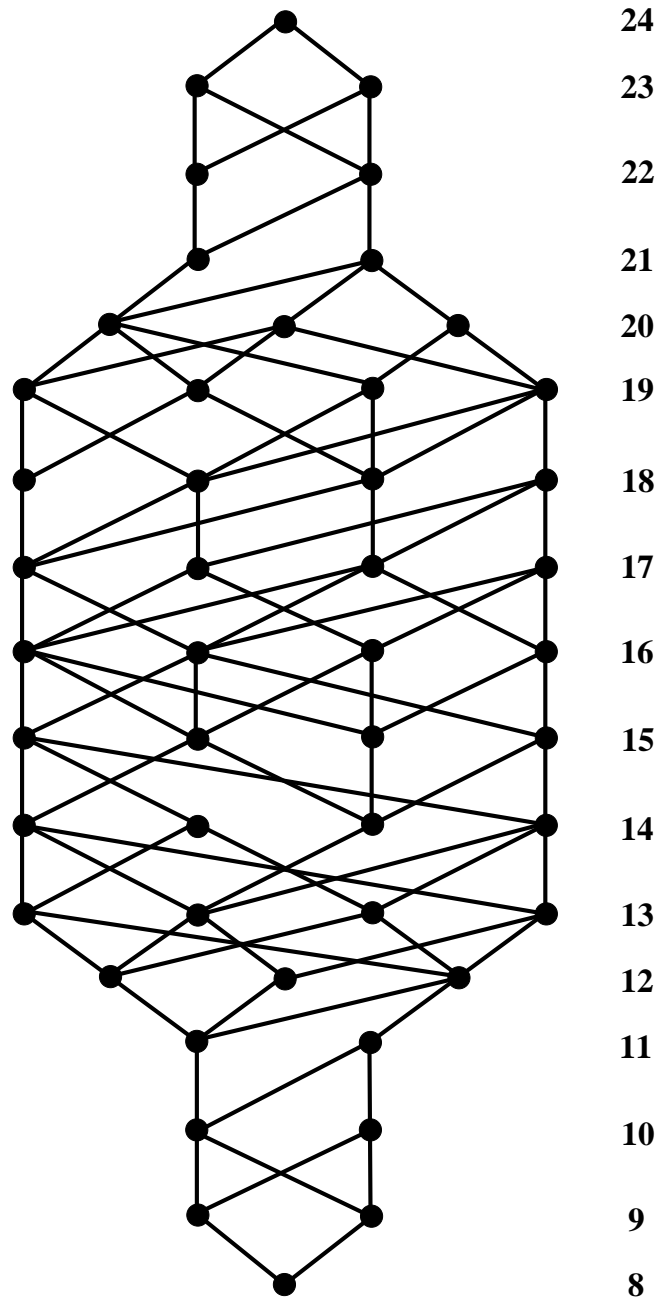
Figure 1. $W \cong F_4, X = \langle [r_3 r_2 r_3 r_4 r_3 r_2 r_3 r_4] \rangle \cong \mathbb{Z}_2$

5. Some Open Questions

Theorems 3.7 and 3.8 may be seen as generalizing the results of Geck, Pfeiffer [7] and Geck, Kim, Pfeiffer [6] to cyclic groups. So we ask the following.

Question 5.1 Suppose that X is a subgroup of the finite irreducible Coxeter group W.

(i) For each $Y \in \mathbb{X}$, does there exist $Y' \in \mathbb{X}_{\min}$ such that $Y \searrow Y'$?

(ii) For each $Y \in \mathbb{X}$, does there exist $Y' \in \mathbb{X}_{\max}$ such that $Y \nearrow Y'$?

Enlisting the aid of MAGMA [4], Question 5.1 part (i) has been answered in the affirmative for all finite irreducible Coxeter groups of rank less than or equal to 6 (much to our surprise, as we expected to find a counterexample). We have yet to do comparable calculations for Question 5.1 part (ii).

Another obvious question is whether there is a maximal analogue of Proposition 3.9.

Question 5.2 Suppose X is a finite subgroup of W and that $Y_1, Y_2 \in \mathbb{X}_{\max}$. Do there exist $g \in W$ and $r_1, \ldots, r_n \in R$ such that, writing $Z = Y_1^g$, the following hold:

(i) $N(Z) = N(Y_2)$ and $l(Y_1 g) = l(Y_1) - l(g)$;

(ii) $Y_2 = r_n \cdots r_1 Z r_1 \cdots r_n$, and for $1 \leq i \leq n$, $N(Z^{r_1 \cdots r_i}) = N(Y_2)$, and hence, in particular, $l(Z^{r_1 \cdots r_i}) = l(Y_2)$.

One difficulty in establishing Question 5.2 is that we do not understand where subgroups of maximal length in a conjugacy class may be found. Because of Proposition 3.5, this reduces to the problem of parabolic subgroups of maximal length, about which we have limited knowledge. Thus we ask

Question 5.3 For $X \leq W$, with X finite and of maximal length in its conjugacy class, what can be said about the location of X in W?

As was proved in 3.10, if X is a finite subgroup, W is irreducible and $\mathbb{X}$ is a flat class, then $N(X) = \Phi^+$. However, having $N(X) = \Phi^+$ does not guarantee that $\mathbb{X}$ is flat. A trivial illustration this occurs in W of type A_n where X is generated by the longest element of W.

Question 5.4 Find conditions that characterize flat classes of subgroups.

We recall that a *ranked* poset is a poset $\mathcal{P}$ such that for each $p \in \mathcal{P}$ all maximal chains in $\{q \in \mathcal{P} | q \leq p\}$ have the same finite length, called the rank of p. A *graded* poset is a ranked poset with a minimum and a maximum element (see [1]). The following, which we believe to be true, has caused the present authors much anguish to date.

Question 5.5 Suppose W is a finite irreducible Coxeter group, and X a subgroup of W. Is the X-poset $\mathfrak{X}$ graded?

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