Turk J Math
31 (2007) , Suppl, 113 – 130.

Finitary Actions and Invariant Ideals

D. S. Passman

Abstract

Let K be a field and let G be a group. If G acts on an abelian group V, then it acts naturally on any group algebra $K[V]$, and we are concerned with classifying the G-stable ideals of $K[V]$. In this paper, we consider a rather concrete situation. We take G to be an infinite locally finite simple group acting in a finitary manner on V. When G is a finitary version of a classical linear group, then we show that the augmentation ideal $\omega K[G]$ is the unique proper G-stable ideal of $K[V]$. On the other hand, if G is a finitary alternating group acting on a suitable permutation module V, then there is a rich family of G-stable ideals of $K[V]$, and we show that these behave like certain graded ideals in a polynomial ring.

Key Words: group algebra, invariant ideal, locally finite simple group, finitary permutation group, permutation module, finitary linear group.

1. Introduction

If H is a nonidentity group, then the group algebra $K[H]$ always has at least three distinct ideals, namely 0, the augmentation ideal $\omega K[H]$, and $K[H]$ itself. Thus it is natural to ask, as I. Kaplansky did, if groups exist for which the augmentation ideal is the unique nontrivial ideal. In such cases, we say that $\omega K[H]$ is simple. Certainly H must be a simple group for this to occur and, since the finite situation is easy enough to describe, we might as well assume that H is infinite simple. The first examples discovered were the algebraically closed groups and the universal groups. From this, it appeared that such groups would be quite rare. But F. Leinen, A. E. Zalesskiĭ, and others have shown

2000 *AMS Mathematics Subject Classification:* 16S34, 20B07, 20G40, 20H20.

that, for locally finite groups, this phenomenon is really the norm. Indeed, for all locally finite infinite simple groups, the characteristic 0 group algebras $K[H]$ tend to have very few ideals.

As suggested by B. Hartley and A. E. Zalesskiĭ, the next stage of this program should be concerned with certain abelian-by-(quasi-simple) groups. Specifically, these are the locally finite groups H having a minimal normal abelian subgroup V with H/V being infinite simple (or perhaps just close to being simple). Note that $G = H/V$ acts as automorphisms on V, and hence on the group algebra $K[V]$. Furthermore, if I is any nonzero ideal of $K[H]$, then it is easy to see that $I \cap K[V]$ is a nonzero G-stable ideal of $K[V]$. Thus, for the most part, this second stage is concerned with classifying the G-stable ideals of $K[V]$. Even in concrete cases, this turns out to be a surprisingly difficult task. Fortunately, there has been some recent progress on this problem, and a survey of this material can be found in [7]. For the most part, the methods used in these classifications are quite different from the usual group ring techniques.

In this paper, we consider a rather concrete situation. We take G to be an infinite locally finite simple group acting in a finitary manner on V. When G is a finitary version of a classical linear group, then we show that the augmentation ideal is the only proper G-stable ideal of $K[V]$. On the other hand, if G is a finitary alternating group acting on a suitable permutation module V, then there is a rich family of G-stable ideals of $K[V]$, and we show that these behave like certain graded ideals in a polynomial ring. This problem was suggested by F. Leinen at the recent Antalya Algebra Days conference in memory of Brian Hartley. The author is pleased to thank J. Hall, F. Leinen, U. Meierfrankenfeld and A. E. Zalesskiĭ for helpful conversations.

We close this section with a fairly standard observation on group actions. Let G act as automorphisms on a ring S. Then S is said to be G-prime if, for all G-stable ideals A and B of S, $AB = 0$ implies that $A = 0$ or $B = 0$. The following result is based on I. Connell's well-known criterion for a group algebra to be prime.

Lemma 1.1 *Let G act on the abelian group V and hence on the group algebra $K[V]$. If V contains no nonidentity finite G-stable subgroup, then $K[V]$ is G-prime. Indeed, if G_0 is any subgroup of G of finite index, then $K[V]$ is G_0-prime.*

Proof. We can assume that G acts faithfully on V, and we form the semidirect product $\overline{G} = V \rtimes G$. If N is a finite normal subgroup of $\overline{G}$, then $N \cap V$ is a finite G-stable subgroup of V and hence, by assumption, $N \cap V = 1$. Thus N acts trivially on V and, since $\overline{G}/V \cong G$ acts faithfully, we conclude that $N \subseteq V$ and hence $N = 1$. It now follows from [5, Theorem 4.2.10] that $K[\overline{G}]$ is prime. Suppose A and B are G-stable ideals of $K[V]$ with $AB = 0$. Then $\overline{A} = A \cdot K[\overline{G}] = K[\overline{G}] \cdot A$ and $\overline{B} = B \cdot K[\overline{G}] = K[\overline{G}] \cdot B$ are ideals of $K[\overline{G}]$ with $\overline{A}\,\overline{B} = 0$. Thus, by primeness, either $\overline{A}$ or $\overline{B}$ is zero, and hence either A or B is zero. In other words, $S = K[V]$ is G-prime.

Finally, if G_0 is a subgroup of G of finite index and if M is a finite G_0-stable subgroup of V, then M has only finitely many G-conjugates $\{M^x \mid x \in G\}$, and these finitely many finite subgroups of V generate M^G, a finite G-stable subgroup of V. It follows by assumption that $M^G = 1$ and hence that $M = 1$. Thus, by the result of the previous paragraph, $K[V]$ is also G_0-prime. □

2. Permutation Groups

Let Ω be an infinite set and let $G \subseteq \mathrm{Sym}(\Omega)$ be a group of permutations on Ω. Assume that G is n-transitive for all integers n. Thus, for example, we could take G to be $\mathrm{Sym}(\Omega)$, the full symmetric group, or $\mathrm{FSym}(\Omega)$, the group of finitary permutations on Ω, or $\mathrm{FAlt}(\Omega)$, the group of even finitary permutations. Let F be a field of characteristic $p > 0$, and let $V = F\Omega$ denote the corresponding permutation module for G. Then G acts on V, so it acts on any group algebra $K[V]$, and the goal of this section is to understand the G-stable ideals of $K[V]$ when $\operatorname{char} K \neq p$.

We start with a special case having two additional assumptions, namely that K contains a primitive pth root of unity and that $F = \mathrm{GF}(p)$. We then drop each of these assumptions in turn. While the general case could be handled at once, this approach will hopefully make the arguments somewhat more transparent. Note that the coefficient ring of the polynomial ring given below is irrelevant so, for simplicity, we just take it to be the ring of integers $\mathbb{Z}$. Specifically, we are concerned with ideals of $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ that are generated by monomials. We say that such an ideal I is cancellative provided that, for any monomial α, we have $\alpha \in I$ if and only if $\alpha\zeta_i \in I$ for all $i = 1, 2, \ldots, p$.

Proposition 2.1 *Let Ω be an infinite set and let G be a subgroup of* $\mathrm{Sym}(\Omega)$ *that acts n-transitively for all integers n. Let $V = \mathrm{GF}(p)\Omega$ be the permutation module for G over* $\mathrm{GF}(p)$*, and let K be a field of characteristic $\neq p$ that contains a primitive pth root of unity. Then there is a one-to-one order-preserving correspondence between the G-stable ideals of the group algebra $K[V]$ and the ideals of the polynomial ring $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ that are* *generated by monomials and are cancellative. In particular, the lattice of G-stable ideals of $K[V]$ is Noetherian, and any G-stable ideal is generated by the G-conjugates of finitely many of its elements.*

Proof. We write V multiplicatively as the weak direct product $V = \prod_{x\in\Omega} V_x$, where $V_x = \langle x\rangle$ corresponds to $\mathrm{GF}(p)x$ and is cyclic of order p. Since K contains ε, a primitive pth root of unity, it follows that $K[V_x]$ is a direct sum of p copies of K. Indeed, $K[V_x] = \oplus\sum_{i=1}^{p} Ke(i,x)$, where $e(i,x)$ is the primitive idempotent of $K[V_x]$ corresponding to the linear character given by $x \mapsto \varepsilon^i$. We say that $e(i,x)$ is of type i and we assign the symbol $[i]$ to this type. It is clear that the action of G on such idempotents preserves type, namely $e(i,x)^g = e(i,x^g)$ for all $g \in G$.

Next, let $x_1, x_2, \ldots, x_n$ be n distinct elements of Ω and consider the finite direct product $W = V_{x_1} \times V_{x_2} \times \cdots \times V_{x_n}$. Then $K[W]$ is a direct sum of $|W| = p^n$ copies of K with corresponding primitive idempotents given by the products

$$e = e(j_1,x_1)e(j_2,x_2)\cdots e(j_n,x_n) = \prod_{k=1}^{n} e(j_k,x_k).$$

For convenience, we say that such idempotents are locally primitive in $K[V]$. Now, for each $1 \le i \le p$, let n_i denote the number of subscripts k with $j_k = i$, and then we let the symbol $\prod_{i=1}^{p}[i]^{n_i}$ denote the type of this locally primitive idempotent e. Notice that the type of e is the formal product of the types of the individual idempotents $e(j_k,x_k)$. Since G is n-transitive on Ω for all n, it is clear that two locally primitive idempotents are G-conjugate if and only if they have the same type.

Now let I be a G-stable ideal of $K[V]$. If $\alpha \in I$, then $\alpha \in I \cap K[W]$ for some $W = V_{x_1} \times V_{x_2} \times \cdots \times V_{x_n}$, as above. Hence $\alpha = \sum_e a_e e$, where $a_e \in K$ and where the sum is over the p^n primitive idempotents e of $K[W]$. Since these primitive idempotents are orthogonal, it follows that $a_e e = e\alpha \in I$, and hence I is the K-linear span of all the locally primitive idempotents that it contains. Since I is G-stable, we know that $e \in I$ implies that $e^g \in I$ for all $g \in G$. But these G-classes of locally primitive idempotents correspond precisely to the type of the idempotent. Thus, I is uniquely determined by the types of the locally primitive idempotents it contains. We encapsulate this information into the polynomial ring $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ as follows.

If I is as above, define $\mathcal{T}(I) \subseteq \mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ to be the $\mathbb{Z}$-linear span of all monomials of the form $\zeta_1^{n_1}\zeta_2^{n_2}\cdots\zeta_p^{n_p}$ when $\prod_{i=1}^{p}[i]^{n_i}$ is the type of a locally primitive idempotent contained in I. Then certainly I determines $\mathcal{T}(I)$ and, by the observations of the preceding paragraph, this map is one-to-one. It still remains to understand the possible images of $\mathcal{T}$. We know at least that each $\mathcal{T}(I)$ is a $\mathbb{Z}$-submodule of the polynomial ring and it is generated by monomials.

To continue, suppose $e = \prod_{k=1}^{n} e(j_k, x_k)$ is a locally primitive idempotent contained in I. Then $I \triangleleft K[V]$ implies that $eK[V] \subseteq I$, and certainly $eK[V]$ is the ideal of $K[V]$ consisting of all elements β with $\beta = e\beta$. In particular, $eK[V]$ is the K-linear span of all locally primitive idempotents e' with $e' = ee'$, and it follows easily that any such e' must be of the form $e' = e\prod_{k=n+1}^{m} e(j_k, x_k)$, where $x_{n+1}, x_{n+2}, \ldots .x_m$ are finitely many elements of Ω distinct from $x_1, x_2, \ldots, x_n$. With this, we see that if $\prod_{i=1}^{p}[i]^{n'_i}$ is the type of e' and if $\prod_{i=1}^{p}[i]^{n_i}$ is the type of e, then we must have $n'_i \geq n_i$ for all i. In other words, in the polynomial ring, the monomial $\prod_{i=1}^{p} \zeta_i^{n'_i}$ is a multiple of $\prod_{i=1}^{p} \zeta_i^{n_i}$. Furthermore, it is clear that any such multiple can occur by taking a suitable locally primitive idempotent $e' \in eK[V] \subseteq I$, and it follows that $\mathcal{T}(I)$ is indeed an ideal of $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ generated by monomials.

Now suppose that α is a monomial in the polynomial ring with $\alpha\zeta_i \in \mathcal{T}(I)$ for all i, and let f be a locally primitive idempotent of $K[V]$ having type corresponding to α. If V_y is disjoint from the support of f, then $f{\cdot}e(i, y)$ is a locally primitive idempotent of $K[V]$ of type $\alpha\zeta_i \in \mathcal{T}(I)$, and hence $f{\cdot}e(i, y) \in I$. But $\sum_{i=1}^{p} e(i, y) = 1$, so $f \in I$ and hence α is contained in $\mathcal{T}(I)$. In other words, $\mathcal{T}(I)$ is cancellative.

Conversely, let T be an ideal of $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ that is generated by monomials and is cancellative, and define $I \subseteq K[V]$ to be the K-linear span of all locally primitive idempotents of type $\prod_{i=1}^{p}[i]^{n_i}$ with $\prod_{i=1}^{p} \zeta_i^{n_i} \in T$. Then the above argument shows that I is a G-stable ideal of $K[V]$, and certainly $\mathcal{T}(I) \supseteq T$. It remains to show that the latter inclusion is an equality. To this end, let $e = \prod_{k=1}^{n} e(j_k, x_k)$ be a locally primitive idempotent contained in I. Then, by the definition of I, there exists $m \geq n$ such that e is a K-linear sum of idempotents f having types corresponding to elements of T and with the support of each f being a partial product of $W = V_{x_1} \times V_{x_2} \times \cdots \times V_{x_m}$. Suppose some f has its support missing the factor V_{x_a}. Then we can replace f by $f = f{\cdot}1 = \sum_{i=1}^{p} f{\cdot}e(i, x_a)$, and note that the types of the idempotents $f{\cdot}e(i, x_a)$ are each multiples of the type of f and hence correspond to elements of T. Continuing in this manner, we can now assume that e is a K-linear combination of a set $\mathcal{L}$ of primitive idempotents of $K[W]$, with each of the latter having type corresponding to an element of T.

Of course, $L = \sum_{f \in \mathcal{L}} Kf$ is an ideal of $K[W]$, and we know that $e \in L$. In particular, for any integers $j_{n+1}, j_{n+2}, \ldots, j_m$, from 1 to p, we have

$$e' = e{\cdot}e(j_{n+1}, x_{n+1})e(j_{n+2}, x_{n+2}) \cdots e(j_m, x_m) \in L$$

and, since the primitive idempotents of $K[W]$ are linearly independent, it follows that $e' \in \mathcal{L}$. In other words, if γ is the monomial in $\mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ corresponding to the type of e then, by considering the type of e', we see that $\gamma{\cdot}\zeta_{j_{n+1}}\zeta_{j_{n+2}} \cdots \zeta_{j_m} \in T$ for all choices of the subscripts $j_{n+1}, j_{n+2}, \ldots, j_m$. But T is cancellative, so this clearly implies that $\gamma \in T$, and hence $\mathcal{T}(I) = T$, as required. This proves the one-to-one correspondence, and the Noetherian results are now immediate. □

Note that if $I = K[V]$, then $\mathcal{T}(I) = \mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$, so the backmap sends the full polynomial ring to I. Moreover, if $J = (\zeta_1, \zeta_2, \ldots, \zeta_p)$ is the augmentation ideal of the polynomial ring, then it is easy to see that the backmap also sends J to I. Indeed, this is true of any power of J. Thus the above one-to-one correspondence certainly requires that we restrict our attention to cancellative ideals in the polynomial ring. Of course, the cancellative property of $T \triangleleft \mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ can be reformulated by saying that, for all monomials α, the inclusion $\alpha J \subseteq T$ implies that $\alpha \in T$.

Next, we show how to drop the assumption that K contains a primitive pth root of unity. For this, we need the following presumably standard result on algebras and field extensions.

Lemma 2.2 *Let A be an algebra over the field K, let E/K be a finite Galois extension with Galois group $\mathfrak{H}$, and let $B = E \otimes_K A$ be the extended E-algebra. Then extension and intersection yield a one-to-one order-preserving correspondence between the ideals of A and the $\mathfrak{H}$-stable ideals of B.*

Proof. Obviously $\mathfrak{H}$ acts as automorphisms on B and, for each $\sigma \in \mathfrak{H}$, the map $\sigma\colon B \to B$ is a left and right A-module homomorphism. Furthermore, if $\mathrm{tr}\colon E \to K$ denotes the Galois trace determined by $\mathfrak{H}$, then this trace map extends to an A-module homomorphism $\mathrm{tr}\colon B \to A$. In particular, if J is an $\mathfrak{H}$-stable ideal of B, then $\mathrm{tr}(J)$ is an ideal of A contained in J. Hence $\mathrm{tr}(J) \subseteq J \cap A$

Conversely, let I be an ideal of A. Then the extension $I^e = E \otimes_K I = EI$ is an $\mathfrak{H}$-stable ideal of B containing I. Since $\mathrm{tr}\colon E \to K$ is onto, we have $\mathrm{tr}(I^e) = \mathrm{tr}(EI) = \mathrm{tr}(E)I = KI = I$. Thus the extension map is one-to-one with trace as its inverse.

Furthermore, suppose that J is any $\mathfrak{H}$-stable ideal of B. Then $J \supseteq E(J \cap A) = (J \cap A)^e$, so $\mathrm{tr}(J) \supseteq \mathrm{tr}((J \cap A)^e) = J \cap A$, and hence $\mathrm{tr}(J) = J \cap A$.

Finally, let $\{w_1, w_2, \ldots, w_n\}$ be a basis for E over K. Then we know that the discriminant matrix $[\mathrm{tr}(w_i w_j)]$ is nonsingular, and therefore we can solve the system of n linear equations $\sum_{i=1}^n y_i \cdot \mathrm{tr}(w_i w_j) = w_j$, with $j = 1, 2, \ldots, n$, for the n unknowns $y_i \in E$. Since $\{w_1, w_2, \ldots, w_n\}$ is also a basis for B over A, it then follows that $\sum_{i=1}^n y_i \cdot \mathrm{tr}(w_i b) = b$ for all $b \in B$. With this, we see that if J is an $\mathfrak{H}$-stable ideal of B, then $J \subseteq \sum_{i=1}^n y_i \cdot \mathrm{tr}(w_i J) = \sum_{i=1}^n y_i \cdot \mathrm{tr}(J) \subseteq \mathrm{tr}(J)^e$ and hence $J = \mathrm{tr}(J)^e = (J \cap A)^e$, as required. $\square$

Now let K be an arbitrary field of characteristic different from p and let ε be a primitive pth root of 1 in the algebraic closure of K. Since, $\mathrm{char}\, K \neq p$, we know that $K[\varepsilon]/K$ is a finite Galois extension with Galois group, say $\mathfrak{G}$. Then $\mathfrak{G}$ acts faithfully on the cyclic group $\langle \varepsilon \rangle = \{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^p\}$, and hence we can think of $\mathfrak{G}$ as a certain group of permutations on the set of exponents $\{1, 2, \ldots, p\}$. In this way, $\mathfrak{G}$ also acts naturally on the polynomial ring $R = \mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$ by permuting the variables as it permutes the subscripts. With this notation, we have

Lemma 2.3 *Let Ω be an infinite set and let G be a subgroup of $\mathrm{Sym}(\Omega)$ that acts n-transitively for all integers n. Let $V = \mathrm{GF}(p)\Omega$ be the permutation module for G over $\mathrm{GF}(p)$, and let K be a field of characteristic $\neq p$. Now suppose $\mathfrak{G}$ is the Galois group of $K[\varepsilon]/K$, where ε is a primitive pth root of unity, and let $\mathfrak{G}$ act naturally on the polynomial ring $R = \mathbb{Z}[\zeta_1, \zeta_2, \ldots, \zeta_p]$. Then there is a one-to-one order-preserving correspondence between the G-stable ideals of the group algebra $K[V]$ and the $\mathfrak{G}$-stable ideals of the polynomial ring R that are generated by monomials and are cancellative. In particular, the lattice of G-stable ideals of $K[V]$ is Noetherian, and any G-stable ideal is generated by the G-conjugates of finitely many of its elements.*

Proof. Let $A = K[V]$ and let $B = K[\varepsilon] \otimes_K A = K[\varepsilon][V]$. Then, by Lemma 2.2, extension and intersection yield a one-to-one order-preserving correspondence between the ideals of A and the $\mathfrak{G}$-stable ideals of B. Furthermore, it is clear that this correspondence preserves G-stable ideals. Thus, it suffices to understand the ideals in $B = K[\varepsilon][V]$ that are both G-stable and $\mathfrak{G}$-stable.

To this end, we now continue with the proof of Proposition 2.1, using its notation, but replacing K by $E = K[\varepsilon]$. Suppose first that $e(i, x)$ is the primitive idempotent of

$E[V_x]$ corresponding to the linear character $x \mapsto \varepsilon^i$. If $\sigma \in \mathfrak{G}$ satisfies $\sigma(\varepsilon^i) = \varepsilon^j$, then it is clear that $\sigma(e(i,x)) = e(j,x)$. Thus we see that the action of $\mathfrak{G}$ on $E[V]$ corresponds to the natural permutation action of $\mathfrak{G}$ on the idempotent types. Furthermore, since any G-stable ideal I of $E[V]$ is determined by the types of the locally primitive idempotents it contains, we see that I is $\mathfrak{G}$-stable if and only if the collection of these types is stable under the permutation action of $\mathfrak{G}$. By definition of $\mathcal{T}(I)$ and of the action of $\mathfrak{G}$ on R, it therefore follows that I is $\mathfrak{G}$-stable if and only if $\mathcal{T}(I)$ is $\mathfrak{G}$-stable in R. □

Finally, we indicate the modifications needed to handle arbitrary fields F of characteristic $p > 0$. Here, we are concerned with polynomial rings of the form

$$R = \mathbb{Z}[\zeta_{i,d} \mid i \in \{1, 2, \ldots, p\},\, d \in \Delta],$$

for some set Δ, and again with ideals T generated by monomials. We say that such an ideal T is Δ-cancellative if, for any monomial $\alpha \in R$ and any $d \in \Delta$, we have $\alpha \in T$ if and only if $\alpha\zeta_{i,d} \in T$ for all $i = 1, 2, \ldots, p$.

The following is the main result of this section.

Theorem 2.4 *Let Ω be an infinite set and let G be a subgroup of $\mathrm{Sym}(\Omega)$ that acts n-transitively for all integers n. Let F be a field of characteristic $p > 0$ having a basis Δ over $\mathrm{GF}(p)$, let $V = F\Omega$ be the permutation module for G over F, and let K be a field of characteristic $\neq p$. Suppose $\mathfrak{G}$ is the Galois group of $K[\varepsilon]/K$, where ε is a primitive pth root of unity, and let $\mathfrak{G}$ act naturally on the polynomial ring*

$$R = \mathbb{Z}[\zeta_{i,d} \mid i \in \{1, 2, \ldots, p\},\, d \in \Delta]$$

via its permutation action on the first subscript of the variables. Then there is a one-to-one order-preserving correspondence between the G-stable ideals of the group algebra $K[V]$ and the $\mathfrak{G}$-stable ideals of the polynomial ring R that are generated by monomials and are Δ-cancellative. In particular, it follows that the lattice of G-stable ideals of $K[V]$ is Noetherian if and only if $\dim_{\mathrm{GF}(p)} F = |\Delta| < \infty$.

Proof. Let us first assume that K contains a primitive pth root of unity. Since we are only concerned with the additive group structure of V, the multiplicative nature of F is essentially irrelevant. Let Δ denote, as above, a basis for F over $\mathrm{GF}(p)$, and let G act on the set $\Delta \times \Omega$ by acting trivially on the first component. Since $F = \mathrm{GF}(p)\Delta$, it is clear that V is G-isomorphic to the permutation module $\mathrm{GF}(p)(\Delta \times \Omega)$. Hence, without loss of generality, we can assume that $V = \mathrm{GF}(p)(\Delta \times \Omega)$.

In multiplicative notation, V is the weak direct product $V = \prod_{(d,x)\in\Delta\times\Omega} V_{(d,x)}$, where $V_{(d,x)}$ is the cyclic group of order p generated by the element (d,x). Obviously, the nature of K implies that $K[V_{(d,x)}] = \oplus\sum_{i=1}^{p} Ke(i,d,x)$, where $e(i,d,x)$ is the primitive idempotent corresponding to the linear character $(d,x) \mapsto \varepsilon^i$. As in the proof of Proposition 2.1, we let $[i,d]$ denote the type of this idempotent. More generally, if $e = \prod_{k=1}^{n} e(i_k, d_k, x_k)$ is a locally primitive idempotent with the various pairs (d_k, x_k) being distinct elements of $\Delta \times \Omega$, then the type of e is given by the formal product $\prod_{k=1}^{n}[i_k, d_k]$. Since G is n-transitive on Ω for all n, we know that two locally primitive idempotents are G-conjugate if and only if they have the same type. With this, the argument of Proposition 2.1 yields a one-to-one order-preserving correspondence between the G-stable ideals I of $K[V]$ and the ideals $\mathcal{T}(I)$ of R that are generated by monomials and are Δ-cancellative.

Finally, if K is an arbitrary field of characteristic $\neq p$, we extend the field to $K[\varepsilon]$, where ε is a primitive pth root of unity. The argument of Lemma 2.3 then clearly yields the required correspondence. Note that if $|\Delta| = \dim_{\mathrm{GF}(p)} F < \infty$, then R has only finitely many variables and hence is Noetherian. The correspondence then implies that the set of G-stable ideals of $K[V]$ is also Noetherian. On the other hand, if $|\Delta| = \infty$, then R has infinitely many variables fixed by $\mathfrak{G}$, namely all variables of the form $\zeta_{p,d}$ with $d \in \Delta$. In this case, since the ideal $(\zeta_{p,d} \mid d \in \Delta)$ is $\mathfrak{G}$-stable, Δ-cancellative and infinitely generated, the above correspondence now shows that set of G-stable ideals of $K[V]$ is no longer Noetherian, and the result follows. $\square$

Note that if $\mathfrak{Z}$ is the free abelian semigroup on a set of variables, then the ideals of the polynomial ring $\mathbb{Z}[\mathfrak{Z}]$ that are generated by monomials correspond in a one-to-one manner to the semigroup ideals of the semigroup $\mathfrak{Z}$. Thus, the preceding theorem can easily be reformulated, replacing ideals in R with suitably cancellative semigroup ideals in $\mathfrak{Z} = \langle \zeta_{i,d} \mid i \in \{1, 2, \ldots, p\},\, d \in \Delta\rangle$.

3. Linear Groups

We again consider the problem of invariant ideals, but now we assume that G is a locally finite, finitary linear group acting naturally on a vector space V. As we will see, the answer is far less interesting than for permutation groups and modules. Indeed, there is only one nontrivial G-stable ideal of $K[V]$, namely the augmentation ideal. We are concerned below with the finitary versions of the special linear groups, the symplectic groups, and the orthogonal and special unitary groups. These will be defined in a manner that will allow us to quickly and efficiently prove our results.

We start with the finite dimensional situation. These groups are discussed, for example, in [1, Chapter 1], or in [2] or in [4, Chapter 6]. Let F be a locally finite field of characteristic $p > 0$ and let $V \neq 0$ be a finite dimensional F-vector space. Then, as usual, $\mathrm{GL}(V)$ denotes the group of invertible linear transformations on V, and $\mathrm{SL}(V)$ is the subgroup consisting of all such linear transformations of determinant 1. Now let us assume that V is endowed with a nonsingular bilinear form $(\ ,\ )\colon V \times V \to F$ having certain additional properties. An invertible linear transformation g is then said to be an isometry with respect to $(\ ,\ )$ if $(vg, wg) = (v, w)$ for all $v, w \in V$. Certainly, the set of all such g is a subgroup of $\mathrm{GL}(V)$.

We say that the form $(\ ,\ )$ is symplectic if $(v, v) = 0$ for all $v \in V$. In other words, all vectors in V are isotropic, and it follows easily that $(v, w) = -(w, v)$ for all $v, w \in V$. The group of all isometries for such a form is the symplectic group denoted by $\mathrm{Sp}(V)$. It is known that $\det \mathrm{Sp}(V) = 1$ and hence $\mathrm{Sp}(V) \subseteq \mathrm{SL}(V)$. Furthermore, V admits a nonsingular symplectic form if and only if $\dim V$ is even, and then all such forms are equivalent. Hence $\mathrm{Sp}(V)$ exists, and is essentially unique, for all even dimensional vector spaces.

Next, we consider nonsingular symmetric forms defined on V. In general, such forms are not necessarily equivalent, but for locally finite fields there are at most two equivalence classes. The group of isometries of either form is an orthogonal group and, in a somewhat imprecise manner, we will denote either group by $\mathrm{O}(V)$. Furthermore, when $\mathrm{char}\, F = 2$, then for the type of problem we are considering, we can usually assume that the orthogonal groups that are the building blocks for finitary groups are symplectic. Thus, in the case of orthogonal groups, we will restrict our attention to fields F having odd characteristic. It follows that, for the purposes of this paper, $(\ ,\ )\colon V \times V \to F$ is an orthogonal form if it is nonsingular, symmetric, and if V has an orthogonal basis. Indeed, if $\{v_1, v_2, \ldots, v_n\}$ is such a basis, then the nature of the form depends upon whether or not $\prod_i (v_i, v_i)$ is a square in F. Now, it turns out that $\det \mathrm{O}(V) = \pm 1$ and then the commutator subgroup of $\mathrm{O}(V)$, denoted by $\Omega(V)$, is a subgroup of $\mathrm{SL}(V)$. Since F is a locally finite field, the set of norms (v, v), with $v \in V$, is the entire field F, provided $\dim V \geq 2$.

Finally, suppose the locally finite field F admits an automorphism σ of order 2. If V is a finite dimensional F-vector space and if $(\ ,\ )\colon V\times V\to F$ is a nonsingular Hermitian form with respect to σ, then V has an orthonormal basis and hence $(\ ,\ )$ is unique up to equivalence. The set of isometries of V is, of course, the unitary group $\mathrm{U}(V)$, and $\mathrm{SU}(V)=\mathrm{U}(V)\cap\mathrm{SL}(V)$ is the special unitary group. Since F is a locally finite field, the set of norms (v,v), with $v\in V$, is easily seen to be the fixed field F^σ.

The following result is standard. For convenience, we include its simple proof.

Lemma 3.1 *Let F be a locally finite field and let V be a finitely dimensional F-vector space with $\dim V\geq 3$. Assume that V is endowed with a suitable nonsingular bilinear form, and let G be the group of isometries on V.*

i. *V is the linear span of its isotropic vectors.*

ii. *Two nonzero vectors of V are G-conjugate if and only if they have the same norm.*

iii. *If $\dim V\geq 4$, then V contains no proper G-stable subgroup.*

Proof. (i) This is clear if the form is symplectic. Now suppose that $(\ ,\ )\colon V\times V\to F$ is orthogonal, let $\{u,v,w\}$ be an orthogonal set, and choose field elements $a,b\in F$ with $(u,u)+a^2(v,v)+b^2(w,w)=0$. Then $u+av+bw$ and $u-av-bw$ are isotropic and sum to $2u$. The result is now clear since $\operatorname{char}F\neq 2$, since V has an orthogonal basis and since any scalar multiple of an isotropic vector is also isotropic. Similarly, if $(\ ,\ )$ is Hermitian, let $\{u,v\}$ be orthonormal and choose distinct elements $a,b\in F$ having Galois norm -1. Then $au+v$ and $bu+v$ are both isotropic and have difference equal to $(a-b)u$.

(ii) Suppose x and y are nonzero elements of V with the same norm. If $(x,x)\neq 0$, set $X=Fx$, $Y=Fy$, and note that the map $X\to Y$ given by $x\mapsto y$ is an isometry. Furthermore, since F is a finite field, it is easy to see that $(\ ,\ )$ restricted to $X^\perp$ and $Y^\perp$ are equivalent, and hence there is an isometry $X^\perp\to Y^\perp$. Combining these two maps yields an isometry $g\in G$ on $V=X\oplus X^\perp=Y\oplus Y^\perp$ with $xg=y$. On the other hand, if x is isotropic, then by (i) there exists an isotropic vector x' with $(x,x')\neq 0$ and, by scaling x', we can assume that $(x,x')=1$. Similarly, there exists an isotropic vector y' with $(y,y')=1$. If we set $X=Fx\oplus Fx'$ and $Y=Fy\oplus Fy'$, then there is an isometry $X\to Y$ given by $x\mapsto y$ and $x'\mapsto y'$. As above, there is also an isometry from $X^\perp$ to $Y^\perp$, and these two maps yields an isometry $g\in G$ on $V=X\oplus X^\perp=Y\oplus Y^\perp$ with $xg=y$.

(iii) Finally, let W be a nonzero G-stable subgroup of V. If W contains a nonzero isotropic vector then, by (ii), W must contain all such vectors since it is G-stable, and hence $W = V$ since the isotropic vectors span. On the other hand, if $v \in W$ is not isotropic, then $V = Fv \oplus (Fv)^\perp$ and $\dim(Fv)^\perp \geq 3$. Thus $(Fv)^\perp$ contains a nonzero isotropic vector u, and we note that v and $v+u$ have the same norm. Thus, by (ii) again, $v + u = vg \in W$, for some $g \in G$, and hence $u = (v+u) - v \in W$. We conclude that W contains a nonzero isotropic vector, and $W = V$, as required. □

If F is a finite field of characteristic $p > 0$ and if V is a finite dimensional F-vector space, then V is a finite elementary abelian p-group. Thus the additive linear characters on V are additive homomorphisms $\lambda\colon V \to \mathrm{GF}(p)$. Furthermore, if G acts on V, then G acts on $\widehat{V}$, the group of all such characters, by $\lambda^g(v) = \lambda(vg^{-1})$ for all $v \in V$.

Lemma 3.2 *Let F be a finite field of characteristic $p > 0$ and let $\mathrm{tr}\colon F \to \mathrm{GF}(p)$ denote the Galois trace. Suppose V is a nonzero finite dimensional F-vector space endowed with an appropriate nonsingular bilinear form $(\ ,\ )\colon V \times V \to F$ and let G denote the group of isometries on V. Then every linear character $\lambda\colon V \to \mathrm{GF}(p)$ is uniquely of the form $\lambda_w\colon x \mapsto \mathrm{tr}(x,w)$ for some $w \in V$. Furthermore, for all $g \in G$ we have $\lambda_w^g = \lambda_{wg}$.*

Proof. The map $\theta\colon w \mapsto \lambda_w$ is clearly a group homomorphism from V to $\widehat{V}$. Since $|V| = |\widehat{V}| < \infty$, we need only show that θ is one-to-one. But if $\theta(w) = 0$, then $\mathrm{tr}(V,w) = 0$ and hence $w = 0$ since (V,w) is an F-subspace contained in the kernel of the trace map. Finally, if g is an isometry, then $(vg, wg) = (v,w)$ for all $v, w \in V$. Replacing v by vg^{-1} then yields $\lambda_w^g(v) = \mathrm{tr}(vg^{-1}, w) = \mathrm{tr}(v, wg) = \lambda_{wg}(v)$, as required. □

Next, we consider group algebras $K[V]$. The following lemma contains the key idea. The remaining work is then fairly routine.

Lemma 3.3 *Let F be a finite field of characteristic $p > 0$, let V be a finite dimensional F-vector space, and let $(\ ,\ )\colon V \times V \to F$ be a suitable nonsingular bilinear form. Furthermore, let G be the group of isometries on V, let K be a field of characteristic different from p, and let I be a proper G-stable ideal of $K[V]$. If V has a nonsingular subspace U such that $I \cap K[U] \neq 0$ and $\dim U^\perp \geq 2$, then $I = \omega K[V]$ is the augmentation ideal of $K[V]$.*

Proof. Let us first assume that K contains a primitive pth root of unity ε. Of course, when studying the group algebra $K[V]$, we must necessarily think of V as a multiplicative group. Since V is an elementary abelian p-group, the linear characters of V are then homomorphisms $\varphi\colon V \to \langle\varepsilon\rangle$, and these are clearly all of the form $\varphi_\lambda\colon v \to \varepsilon^{\lambda(v)}$ where λ is an additive homomorphism $\lambda\colon V \to \mathrm{GF}(p)$. The corresponding primitive idempotents of $K[V]$ are given by $e_\lambda = |V|^{-1}\sum_{v\in V}\varepsilon^{\lambda(v)}v^{-1}$, and it is easy to see that if g acts on V then $e_\lambda^g = e_{\lambda^g}$.

Now suppose $V = U \oplus U^\perp$ and use f_μ to denote primitive idempotents in $K[U]$ and f'_η to denote primitive idempotents in $K[U^\perp]$. Here, $\mu\colon U \to \mathrm{GF}(p)$ and $\eta\colon U^\perp \to \mathrm{GF}(p)$. Then it is easy to see that $f_\mu f'_\eta = e_{\mu\#\eta}$ where $\mu\#\eta\colon V = U \oplus U^\perp \to \mathrm{GF}(p)$ is the linear functional that extends μ on U and η on $U^\perp$.

By assumption, $I \cap K[U] \neq 0$, and hence $I \cap K[U]$ contains some primitive idempotent f_{μ_0} for some fixed functional $\mu_0\colon U \to \mathrm{GF}(p)$. In particular, since I is an ideal of $K[V]$, I must contains all the idempotents $e_{\mu_0\#\eta} = f_{\mu_0}f'_\eta$ as η runs through all the linear functionals $\eta\colon U^\perp \to \mathrm{GF}(p)$. Now, by the previous lemma, $\mu_0(x) = \mathrm{tr}(x, u_0)$ for some $u_0 \in U$ and $\eta(x) = \mathrm{tr}(x, w)$ for $w \in U^\perp$. Furthermore, since U and $U^\perp$ are orthogonal, it follows easily that $\mu_0\#\eta(x) = \mathrm{tr}(x, u_0 + w)$.

We have therefore shown that I contains all the primitive idempotents e_λ of $K[V]$ where $\lambda = \lambda_v$ and $v = u_0 + w$ for all $w \in U^\perp$. Indeed, since I is G-stable, I contains all the primitive idempotents $e_\lambda^g = e_{\lambda^g}$ and, by Lemma 3.2, $\lambda^g_{u_0+w} = \lambda_{(u_0+w)g}$. Now note that $(u_0 + w, u_0 + w) = (u_0, u_0) + (w, w)$ and, since $\dim U^\perp \geq 2$, (w, w) takes on all allowable values. Thus (u_0+w, u_0+w) also takes on all possible values, and it follows from Lemma 3.1(ii) that I contains all e_{λ_v} with $v \neq 0$. In other words, I contains all primitive idempotents of $K[V]$ except possibly the principal idempotent, and hence $I \supseteq \omega K[V]$ as required.

Finally, let K be an arbitrary field of characteristic different from p and let $\overline{K} = K[\varepsilon]$, where ε is a primitive pth root of unity. If I is a proper G-stable ideal of $K[V]$, then $\bar{I} = \overline{K}\cdot I$ is a proper G-stable ideal of $\overline{K}[V]$ and, by the above, $\bar{I} = \omega\overline{K}[V]$. But $\overline{K}[V]$ is free over $K[V]$, so $I = \bar{I} \cap K[V] = \omega\overline{K}[V] \cap K[V] = \omega K[V]$, and the result follows. $\square$

The simple, locally finite, finitary linear groups are characterized in [3] and described via a pairing of vector spaces. We use a more concrete description to simplify the proof of Theorem 3.4. Let V be an F-vector space and let $\{W_i \mid i \in \mathcal{I}\}$ be an infinite family of finite dimensional subspaces such that $V = \oplus\sum_{i\in\mathcal{I}} W_i$ is the direct sum of the various W_i. We say that $\{W_i \mid i \in \mathcal{I}\}$ is a partition of V and, for convenience, we assume that

$\dim W_i \geq 2$ for each i. For each finite subset $\mathcal{J} \subseteq \mathcal{I}$, we write $W_{\mathcal{J}} = \oplus \sum_{j \in \mathcal{J}} W_j$ and $W'_{\mathcal{J}} = \oplus \sum_{j \notin \mathcal{J}} W_j$, so that $W_{\mathcal{J}}$ is finite dimensional and $V = W_{\mathcal{J}} \oplus W'_{\mathcal{J}}$. A nonsingular linear transformation g on V is said to be finitary if, for some finite $\mathcal{J}$, g stabilizes $W_{\mathcal{J}}$ and acts like the identity on $W'_{\mathcal{J}}$. For fixed $\mathcal{J}$, the set of all such g is clearly isomorphic to $\mathrm{GL}(W_{\mathcal{J}})$, and we let $\mathrm{FGL}(V)$ denote the group of all such finitary transformations. Furthermore, we let $\mathrm{FSL}(V)$ denote the subgroup of all such g that have determinant 1 in their action on $W_{\mathcal{J}}$ for all sufficiently large $\mathcal{J}$.

Now suppose V admits a nonsingular bilinear form that is either symplectic, orthogonal or Hermitian, and assume that it is compatible with the partition. By this we mean that each W_i is a nonsingular subspace and that $(W_i, W_j) = 0$ for all $i \neq j$. We then let $\mathrm{FSp}(V)$, $\mathrm{FO}(V)$ and $\mathrm{FU}(V)$ denote the subgroup of $\mathrm{FGL}(V)$ consisting of all those finitary linear transformations that act as isometries on $W_{\mathcal{J}}$ for all sufficiently large finite $\mathcal{J} \subseteq \mathcal{I}$. In addition, we let $\mathrm{F\Omega}(V)$ and $\mathrm{FSU}(V)$ denote the obvious normal subgroups of $\mathrm{FO}(V)$ and $\mathrm{FU}(V)$, respectively. If F is a locally finite field, then all of the above are infinite locally finite groups with $\mathrm{FSL}(V)$, $\mathrm{FSp}(V)$, $\mathrm{F\Omega}(V)$ and $\mathrm{FSU}(V)$ being infinte simple (see [3]). The main result of this section is

Theorem 3.4 *Let F be a locally finite field of characteristic $p > 0$, let V be a suitably partitioned infinite dimensional F-vector space, and let $G = \mathrm{FSL}(V)$, $\mathrm{FSp}(V)$, $\mathrm{F\Omega}(V)$ or $\mathrm{FSU}(V)$ act on V. If K is a field of characteristic different from p, then $\omega K[V]$ is the unique proper G-stable ideal of $K[V]$.*

Proof. We will consider the special linear group at the end of this argument. For now, let us assume that V is endowed with a suitable bilinear form $(\ ,\ )\colon V \times V \to F$ compatible with the partition $\{W_i \mid i \in \mathcal{I}\}$. We proceed in a series of steps.

First, let F be a finite field and let $G = \mathrm{FSp}(V)$, $\mathrm{FO}(V)$ or $\mathrm{FU}(V)$ according to the nature of the form. In other words, G is the full group of finitary isometries. If I is a nonzero G-stable ideal of $K[V]$, then there exists a finite subset $\mathcal{J}_0 \subseteq \mathcal{I}$ with $I \cap K[W_{\mathcal{J}_0}] \neq 0$. Set $U = W_{\mathcal{J}_0}$ and let $\mathcal{J}$ be any finite subset of $\mathcal{I}$ strictly larger than $\mathcal{J}_0$. Then $I_{\mathcal{J}} = I \cap K[W_{\mathcal{J}}]$ is an ideal of $K[W_{\mathcal{J}}]$ that is stable under the full group of isometries of $W_{\mathcal{J}}$. Furthermore, $I_{\mathcal{J}} \cap K[U] = I \cap K[U] \neq 0$ and the orthogonal complement of U in $W_{\mathcal{J}}$ has dimension at least 2. By the preceding lemma, we conclude that $I \supseteq I_{\mathcal{J}} \supseteq \omega K[W_{\mathcal{J}}]$ and, since $\mathcal{J} > \mathcal{J}_0$ is arbitrary, it follows that $I \supseteq \omega K[V]$.

We continue with F being a finite field, but now we let G_0 be the normal subgroup of G given by $G_0 = \mathrm{FSp}(V)$, $\mathrm{F\Omega}(V)$ or $\mathrm{FSU}(V)$, respectively. Note that, if $\mathcal{J}$ is a finite subset of $\mathcal{I}$, then $|\mathrm{O}(W_{\mathcal{J}}) : \Omega(W_{\mathcal{J}})| \leq 4$ and $|\mathrm{U}(W_{\mathcal{J}}) : \mathrm{SU}(W_{\mathcal{J}})| \leq |F|^{1/2}$. See, for example, [2, Theorems 9.11, 9.12, 11.28 and 11.29]. Since $|F|$ is fixed, it now follows easily that $|G : G_0| < \infty$ with the same bounds as above. Furthermore, in view of Lemma 3.1(iii), G stabilizes no nontrivial finite subgroup of V, and then Lemma 1.1 implies that $S = K[V]$ is G_0-prime. Now let I be a G_0-stable ideal of $K[V]$ and let $\{1 = g_0, g_1, \ldots, g_n\}$ be a transversal for G_0 in G. Then the product $P = I^{g_1} I^{g_2} \cdots I^{g_n}$ is a G-stable ideal of the commutative ring S and each I^{g_i} is G_0-stable since $G_0 \triangleleft G$. In particular, if $P \neq 0$ then, by the above, $I \supseteq P \supseteq \omega K[V]$. On the other hand, if $P = 0$, then the G_0-primeness of S implies that $I^{g_i} = 0$ for some i, and hence $I = 0$. This completes the proof in the case of finite fields.

Next, we let F be an infinite locally finite field. The result in this case actually follows fairly easily from the main theorems of [9] or [6]. However, here we take a simple approach using the fact that F is a union of its finite subfields. To start with, we fix a finite basis $\mathcal{B}_i = \{w_{i1}, w_{i2}, \ldots\}$ for each W_i with $i \in \mathcal{I}$ according to the nature of the form. In the symplectic case, we choose $\mathcal{B}_i$ so that $(w_{ij}, w_{ik}) = 0$ or ± 1, and we let $F_0 = \mathrm{GF}(p)$. In the orthogonal case, we fix a nonsquare $d \in F$ if there is one, let $\mathcal{B}_i$ be orthogonal with norms 1 or d, and we take $F_0 = \mathrm{GF}(p)[d]$. Finally, in the unitary case, we let $\mathcal{B}_i$ be orthonormal and let F_0 be a finite subfield of F not contained in the fixed field F^{σ}. Set $\mathcal{B} = \bigcup_{i \in \mathcal{I}} \mathcal{B}_i$ so that $\mathcal{B}$ is a basis for V. In particular, $W_i = F\mathcal{B}_i$ and $V = F\mathcal{B}$. Now let $G_0 = \mathrm{FSp}(V)$, $\mathrm{F}\Omega(V)$ or $\mathrm{FSU}(V)$, and let I be a nonzero G_0-stable ideal of $K[V]$. Then there exists a finite subfield $\widetilde{F}$ of F with $\widetilde{F} \supseteq F_0$ and $I \cap K[\widetilde{F}\mathcal{B}] \neq 0$. If F' is any finite subfield of F containing $\widetilde{F}$, then $I \cap K[F'\mathcal{B}]$ is a nonzero ideal of $K[F'\mathcal{B}]$ that is stable under the natural subgroup of G_0 corresponding to $V' = F'\mathcal{B}$. By the above, $I \supseteq I \cap K[V'] \supseteq \omega K[V']$ and, since V is the union of all such subspaces V', we conclude that $I \supseteq \omega K[V]$.

Finally, we briefly consider the group $\mathrm{FSL}(V)$ with F an arbitrary locally finite field. In this case, group terms so that each W_i has even dimension, define a nonsingular symplectic form on each such subspace, and extend this to all of V. Then $\mathrm{FSL}(V) \supseteq \mathrm{FSp}(V)$, so any proper $\mathrm{FSL}(V)$-stable ideal is $\mathrm{FSp}(V)$-stable and consequently equal to the augmentation ideal $\omega K[V]$. This completes the proof. $\square$

4. Finitely Equivalent Groups

Theorem 2.4 seems somewhat surprising. The various groups $\mathrm{Sym}(\Omega)$, $\mathrm{FSym}(\Omega)$ and $\mathrm{FAlt}(\Omega)$ have such different structures and yet give rise to the same set of G-stable ideals in $K[F\Omega]$. However, there is a good reason for this to occur. These groups are finitely equivalent, namely they act in the same way on finite subsets of $F\Omega$.

More generally, let V be a group, and let G and H both act on V. We say that G is finitely smaller than H, and write $G \preccurlyeq_V H$, if for all finitely many elements $v_1, v_2, \ldots, v_n \in V$ and $g \in G$, there exists $h \in H$ such that $v_i^g = v_i^h$ for all $i = 1, 2, \ldots, n$. Certainly $G \subseteq H \subseteq \mathrm{Aut}(V)$ implies that $G \preccurlyeq_V H$. Furthermore, for fixed V, it is clear that $\preccurlyeq_V$ is a reflexive and transitive relation. If $G \preccurlyeq_V H$ and $H \preccurlyeq_V G$, then we say that G and H are finitely equivalent and write $G \approx_V H$.

Lemma 4.1 *Let G and H act on V and hence on the group algebra $K[V]$.*

i. *If $G \preccurlyeq_V H$, then any H-stable ideal of $K[V]$ is G-stable.*

ii. *If $G \approx_V H$, then the lattices of G-stable and of H-stable ideals of $K[V]$ are identical.*

Proof. For part (i), let I be an H-stable ideal of $K[V]$ and let $\alpha = \sum_{i=1}^n k_i v_i \in I$. If $g \in G$, then since $G \preccurlyeq_V H$, there exists $h \in H$ with $v_i^g = v_i^h$ for all $i = 1, 2, \ldots, n$. Thus $\alpha^g = \sum_{i=1}^n k_i v_i^g = \sum_{i=1}^n k_i v_i^h = \alpha^h \in I$ and, since this is true for all $\alpha \in I$ and $g \in G$, we conclude that I is G-stable. Part (ii) is now immediate. $\square$

The relevance to Theorem 2.4 is of course given by the following obvious result.

Lemma 4.2 *Let Ω be an infinite set and let G and H be subgroups of* $\mathrm{Sym}(\Omega)$ *that act n-transitively for all integers n. If V is the permutation module $F\Omega$, then $G \approx_V H$.*

There is also an analogous result for finitary linear groups, namely

Lemma 4.3 *Let V be an infinite dimensional F-vector space partitioned by the finite dimensional subspaces $\{W_i \mid i \in \mathcal{I}\}$. The* $\mathrm{GL}(V) \approx_V \mathrm{FSL}(V)$.

Proof. Since $\mathrm{FSL}(V) \subseteq \mathrm{GL}(V)$, we need only show that $\mathrm{GL}(V) \preccurlyeq_V \mathrm{FSL}(V)$. To this end, let $g \in \mathrm{GL}(V)$ and let U be the F-linear span of finitely many elements of V. Then U is a finite dimensional F-subspace of V and $g\colon U \to Ug$ is an F-isomorphism. Now choose a finite subset $\mathcal{J} \subseteq \mathcal{I}$, so that U and Ug are both proper subspaces of

$W = \oplus\sum_{j\in\mathcal{J}} W_j$, and let A and B be subspaces of W that complement U and Ug, respectively. Since $\dim U = \dim Ug$, we have $\dim A = \dim B$, and there is a vector space isomorphism $A \to B$. By combining this map with the map $g\colon U \to Ug$, we obtain a vector space isomorphism $h\colon W = U \oplus A \to Ug \oplus B = W$ that agrees with g on U. Furthermore, since A and B are nonzero, we can easily modify the map $A \to B$ so that $\det h = 1$. If we let h act like the identity on $W' = \oplus\sum_{j\notin\mathcal{J}} W_j$, then $h \in \mathrm{FSL}(V)$ and, since g and h agree on U, we conclude that $\mathrm{GL}(V) \preccurlyeq_V \mathrm{FSL}(V)$. □

Unfortunately, there is no analogous result for the finitary isometry groups. Indeed, if $(\ ,\ )\colon V \times V \to F$ is a suitable bilinear form, then we can certainly choose $g \in \mathrm{GL}(V)$ to send a pair of orthogonal vectors to a pair of non-orthogonal vectors, and then obviously g cannot agree with an isometry on these vectors. We close this paper with an amusing application of the above, namely an alternate proof of the $\mathrm{FSL}(V)$ aspect of Theorem 3.4.

Lemma 4.4 *Let F be a finite field of characteristic $p > 0$, let V be a countably infinite dimensional F-vector space having a partition $\{W_i \mid i \in \mathcal{I}\}$, and let $G = \mathrm{FSL}(V)$. If K is a field of characteristic $\neq p$, then $\omega K[V]$ is the unique proper G-stable ideal of $K[V]$.*

Proof. Since V is countably infinite dimensional, we can identify V with the additive subgroup of $\overline{F}$, the algebraic closure of F. Then $\overline{F}^{\bullet}$, the multiplicative group of $\overline{F}$, acts on $V = \overline{F}^{+}$ by multiplication, so $\overline{F}^{\bullet} \subseteq \mathrm{GL}(V)$. Hence $\overline{F}^{\bullet} \preccurlyeq_V \mathrm{FSL}(V) = G$ by transitivity and the preceding lemma. In particular, it follows from Lemma 4.1(i) that any G-stable ideal of $K[V]$ is $\overline{F}^{\bullet}$-stable. But, we know from [8] that the only proper $\overline{F}^{\bullet}$-stable ideal of $K[V]$ is the augmentation ideal, so the result follows. □

It is, of course, a simple matter to drop the countability assumption in the dimension of the vector space V above. Indeed, all of these problems reduce to the countable case.

References

[1] Carter, R. W.: *Simple Groups of Lie Type*, John Wiley, New York, 1972.

[2] Grove, L.: *Classical Groups and Geometric Algebra*, American Math. Soc., Providence, 2002.

[3] Hall, J. I.: *Periodic simple groups of finitary linear transformations*, Ann. of Math. (2) **163** (2006), 445–498.

[4] Jacobson, N.: *Basic Algebra I.*, W. H. Freeman, New York, 1985.

[5] Passman, D. S.: *The Algebraic Structure of Group Rings*, Krieger, Malabar, 1985.

[6] Passman, D. S.: *Invariant ideals and polynomial forms*, Trans. AMS, **354** (2002), 3379–3408.

[7] Passman, D. S.: *Invariant ideals of abelian group algebras under the action of simple linear groups*, Resenhas IME-USP (Instituto de Matemática e Estatstica da Universidade de São Paulo) **5** (2002), 377–390.

[8] Passman, D. S. and Zalesskiĭ, A. E.: *Invariant ideals of abelian group algebras under the multiplicative action of a field, I*, Proc. AMS **130** (2002), 939–949.

[9] Passman, D. S. and Zalesskiĭ, A. E.: *Invariant ideals of abelian group algebras and representations of groups of Lie type*, Trans. AMS **353** (2001), 2971–2982.

D. S. PASSMAN
Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706, USA
e-mail: passman@math.wisc.edu

Received 04.10.2007