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Best p-Simultaneous Approximation in Some Metric Space

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Abstract

Let X be a Banach space, (I, μ) be a finite measure space, and Φ be an increasing subadditive continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. In the present paper, we discuss the best p-simultaneous approximation of $L^{\Phi}(I, G)$ in $L^{\Phi}(I, X)$ where G is a closed subspace of X.

Key Words: Simultaneous, Approximation.

1. Introduction

A function $\Phi : [0, +\infty) \to [0, +\infty)$ is called a modulus function if it satisfies the following conditions:

1. $\Phi(x) = 0$ iff $x = 0$.
2. $\Phi(x + y) \leq \Phi(x) + \Phi(y)$.
3. Φ is a continuous increasing function.

For a modulus function Φ, a finite measure space (I, μ) and a Banach space X,

$$L^{\Phi}(I, X) = \{f : I \to X : \int_I \Phi(||f(t)||)d\mu < +\infty\}.$$

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For $f \in L^{\Phi}(I,X)$, define

$$||f||_{\Phi} = \int_I \Phi(||f(t)||)d\mu.$$

In fact $(L^{\Phi}(I,X), ||\cdot||_{\Phi})$ is a complete metric linear space [4]. Further, it is known that $L^1(I,X) \subseteq L^{\Phi}(I,X)$. For more information about $L^{\Phi}(I,X)$, we refer to [3,5]. For x_1, x_2 in X and $1 < p < +\infty$, we set

$$|(x_1,x_2)|_{\Phi,p} = ((\Phi(||x_1||))^p + (\Phi(||x_2||))^p)^{\frac{1}{p}}.$$

Note that $(X^2, |\cdot|_{\Phi,p})$ is a complete metric space. The diagonal of G^2 is given by $D = \{(g,g) : g \in G\}$. Throughout this paper, X is a Banach space, G is a closed subspace of X and Φ is a modulus function. For f_1 and f_2 in $L^{\phi}(I,X)$, we set

$$|(f_1,f_2)|_{\Phi,p} = [||f_1||_{\Phi}^p + ||f_2||_{\Phi}^p]^{\frac{1}{p}},$$

for all $1 < p < +\infty$. Then $((L^{\Phi}(I,X))^2, |\cdot|_{\Phi,p})$ is a complete metric space. We consider X as a metric space with a metric $d(x,y) = \Phi||x-y||$.

Definition 1.1 *For $x_1, x_2 \in G$, define $dist_{\Phi} : X^2 \to \mathbf{R}$ by*

$$dist_{\Phi}(x_1,x_2,G) := \inf_{z\in G} [(\Phi(||x_1 - z||))^p + (\Phi(||x_2 - z||))^p]^{\frac{1}{p}}.$$

Consequently, for $f_1, f_2 \in L^{\Phi}(I,X)$, we define

$$dist_{\Phi}(f_1,f_2,L^{\Phi}(I,G)) := \inf_{g\in L^{\Phi}(I,G)} [||f_1 - g||_{\Phi}^p + ||f_2 - g||_{\Phi}^p]^{\frac{1}{p}}.$$

Definition 1.2 *We say that $z \in G$ is a best p-simultaneous approximation from G of pair elements $x_1, x_2 \in X$ if*

$$[(\Phi(||x_1 - z||))^p + (\Phi(||x_2 - z||))^p]^{\frac{1}{p}} \le [(\Phi(||x_1 - y||))^p + (\Phi(||x_2 - y||))^p]^{\frac{1}{p}}$$

for every $y \in G$. We say that $g \in L^{\Phi}(I,G)$ is the best p-simultaneous approximation of a pair of elements f_1, f_2 in $L^{\Phi}(I,X)$, if for every $h \in L^{\Phi}(I,G)$, we have

$$||f_1 - g||_{\Phi}^p + ||f_2 - g||_{\Phi}^p \le ||f_1 - h||_{\Phi}^p + ||f_2 - h||_{\Phi}^p.$$

Note that for $g \in G$ is the best p-simultaneous approximation from G of $x_1, x_2 \in X$ iff (g, g) is the best approximation from D of the pair $(x_1, x_2) \in X^2$ where the metric on X^2 is $|\cdot|_{\Phi,p}$. If every pair of elements $x_1, x_2 \in X$ admits a best p-simultaneous approximation from G, then G is said to be p-simultaneous proximinal in X. The problem of best simultaneous approximation has been studied by many authors in [2, 7, 12, 13]. Most of these works have dealt with the characterization of best simultaneous approximation in space of continuous functions with values in a Banach space X. Results of best simultaneous approximation in general Banach space can be found in [1, 8, 10]. Some results were obtained in the spaces of $L^p(I, X)$ have been tackled in [6, 11]. In the present paper, we investigate the best p-simultaneous approximations of $L^\Phi(I, G)$ in $L^\Phi(I, X)$ where G is a closed subspace of X.

2. Main Results

We start with the following technical lemma.

Lemma 2.1 *Suppose* $1 < p < +\infty$. *For* $f_1, f_2 \in L^\Phi(I, X)$ *we have*

1. $\int_I dist_\Phi(f_1(t), f_2(t), G)d\mu \leq 2\, dist_\Phi(f_1, f_2, L^\Phi(I, G))$.
2. $dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I dist_\Phi(f_1(t), f_2(t), G)d\mu$

Proof. For any $g \in L^\Phi(I, G)$ and $t \in I$, we have

$$\begin{aligned}[dist_\Phi(f_1(t), f_2(t), G)]^p &\leq [\Phi||f_1(t) - g(t)||]^p + [\Phi||f_2(t) - g(t)||]^p \\ &\leq [\Phi||f_1(t) - g(t)|| + \Phi||f_2(t) - g(t)||]^p .\end{aligned}$$

Therefore we have,

$$\begin{aligned}\int_I dist_\Phi(f_1(t), f_2(t), G)d\mu &\leq ||f_1 - g||_\Phi + ||f_2 - g||_\Phi \\ &\leq 2\,[||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p]^{\frac{1}{p}} .\end{aligned}$$

After taking the infimum over all g in $L^\Phi(I, G)$, we finish our proof of inequality (1). By the density of simple functions in $L^\Phi(I, X)$, we have for any $\varepsilon > 0$ there are two simple functions f_1^* and f_2^* in $L^\Phi(I, X)$ such that

$$||f_1^* - f_1||_\Phi \leq \frac{\varepsilon}{2^{\frac{1}{p}+1}},$$

and

$$||f_2^* - f_2||_\Phi \le \frac{\varepsilon}{2^{\frac{1}{p}+1}}.$$

We can write $f_i^* = \sum_{k=1}^n \chi_{A_k} x_k^i$, $i = 1, 2$, where $A_k, k = 1, 2, \dots n$ are disjoint measurable subsets of I satisfying $\bigcup_{k=1}^n A_k = I$ and χ_{A_k} is the characteristic function of A_k, and $x_k^i \in X$. We may assume that $\mu(A_k) > 0$, for $k = 1, 2, \dots n$. Since

$$dist_\Phi(x, y, G) = \inf_{z \in G} \left[(\Phi(||x - z||))^p + (\Phi(||y - z||))^p\right]^{\frac{1}{p}},$$

then for any $k > 0$, we can select $y_k \in G$ such that

$$\left[(\Phi(||x_k^1 - y_k||))^p + (\Phi(||x_k^2 - y_k||))^p\right]^{\frac{1}{p}} < dist_\Phi(x_k^1, x_k^2, G) + \frac{\varepsilon}{2n\mu(A_k)}.$$

Let $g = \sum_{k=1}^n \chi_{A_k} y_k$. Clearly $g \in L^\Phi(I, G)$. Now

$$\begin{aligned} dist_\Phi(f_1, f_2, L^\Phi(I, G)) &\le [||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p]^{\frac{1}{p}} \\ &\le [(||f_1 - f_1^*||_\Phi + ||f_1^* - g||_\Phi)^p + (||f_2 - f_2^*||_\Phi + ||f_2^* - g||_\Phi)^p]^{\frac{1}{p}} \\ &\le [||f_1 - f_1^*||_\Phi^p + ||f_2^* - f_2||_\Phi^p]^{\frac{1}{p}} + ||f_1^* - g||_\Phi^p + ||f_2^* - g||_\Phi^p]^{\frac{1}{p}}. \end{aligned}$$

It is easy to show that:

$$\begin{aligned} dist_\Phi(f_1, f_2, L^\Phi(I, G)) &< \frac{\varepsilon}{2} + \left[\left(\int_I \Phi(||f_1^*(t) - g(t)||) d\mu\right)^p + \left(\int_I \Phi(||f_2^*(t) - g(t)||) d\mu\right)^p\right]^{\frac{1}{p}} \\ &= \frac{\varepsilon}{2} + \left[\left(\sum_{k=1}^n \mu(A_k)\Phi(||x_k^1 - y_k||)\right)^p + \left(\sum_{k=1}^n \mu(A_k)\Phi(||x_k^2 - y_k||)\right)^p\right]^{\frac{1}{p}} \\ &\le \frac{\varepsilon}{2} + \sum_{k=1}^n \mu(A_k) \left[(\Phi(||x_k^1 - y_k||))^p + (\Phi(||x_k^2 - y_k||))^p\right]^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2} + \sum_{k=1}^n \mu(A_k) \left[dist_\Phi(x_k^1, x_k^2, G) + \frac{\varepsilon}{2n\mu(A_k)}\right]. \end{aligned}$$

Thus

$$dist_\Phi(f_1, f_2, L^\Phi(I, G)) < \varepsilon + \int_I dist_\Phi(f_1^*(t), f_2^*(t), G) d\mu. \tag{1}$$

Using the subadditivity of Φ, we have

$$dist_\Phi(f_1^*(t), f_2^*(t), G) \leq dist_\Phi(f_1(t), f_2(t), G) + [(\Phi(||f_1^*(t) - f_1(t)||))^p + (\Phi(||f_2^*(t) - f_2(t)||))^p]^{\frac{1}{p}}$$
$$\leq dist_\Phi(f_1(t), f_2(t), G) + \Phi(||f_1^*(t) - f_1(t)||) + \Phi(||f_2^*(t) - f_2(t)||),$$

for all t. Therefore

$$\int_I dist_\Phi(f_1^*(t), f_2^*(t), G)d\mu \leq \int_I dist_\Phi(f_1(t), f_2(t), G)d\mu + ||f_1^* - f_1||_\Phi + ||f_2^* - f_2||_\Phi.$$

Thus

$$dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I dist_\Phi(f_1(t), f_2(t), G)d\mu + \frac{\varepsilon}{2^p},$$

and hence

$$dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I (dist_\Phi(f_1(t), f_2(t), G)d\mu.$$

□

Lemma 2.2 *Suppose that $1 < p < +\infty$. If f_1, f_2 are simple functions in $L^\Phi(I, X)$ and if $\mu(I) = 1$, then*

$$dist_\Phi(f_1, f_2, L^\Phi(I, X)) = \left[\int_I [dist_\Phi(f_1(t), f_2(t), G)]^p d\mu\right]^p.$$

Proof. By Inequality (2) of Lemma 2.1, and the assumption $\mu(I) = 1$, we have

$$dist_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \int_I dist_\Phi(f_1(t), f_2(t), G)d\mu$$
$$\leq \left[\int_I [dist_\Phi(f_1(t), f_2(t), G)]^p d\mu\right]^{\frac{1}{p}}.$$

Given $\varepsilon > 0$. Choose $g_0 \in Ł^\Phi(I, G)$ with the property that

$$||f_1 - g_0||_\Phi^p + ||f_2 - g||_\Phi^p < (dist_\Phi(f_1, f_2, L^\Phi(I, G))^p + \varepsilon. \quad (2)$$

Since f_1, f_2 are simple functions in $L^\Phi(I, X)$, we let

$$f_i(t) = \sum_{k=1}^{n} x_k^i \chi_{A_k}(t), \quad (i = 1, 2)$$

where A_k, $k = 1, 2, \ldots, n$ are disjoint measurable sets with $\mu(A_k) > 0$, and $x_k^i \in X$. For a simple function g in $L^\Phi(I, G)$, we can set $g(t) = \sum_{k=1}^n y_k \chi_{A_k}(t)$. Thus

$$\begin{aligned}
\int_I (dist_\Phi(f_1(t), f_2(t), G))^p d\mu \quad &\leq \quad \int_I (\Phi||f_1(t) - g(t)||)^p d\mu + \int_I (\Phi||f_2(t) - g(t)||)^p d\mu \\
&= \sum_{k=1}^n \int_{A_k} (\Phi||x_k^1 - y_k||)^p d\mu + \sum_{k=1}^n \int_{A_k} (\Phi||x_k^2 - g(t)||)^p d\mu(t) \\
&= \sum_{k=1}^n \mu(A_k)(\Phi||x_k^1 - y_k||)^p + \sum_{k=1}^n \mu(A_k)(\Phi||x_k^2 - g(t)||)^p \\
&\leq \left[\sum_{k=1}^n \mu(A_k)\Phi||x_k^1 - y_k||\right]^p + \left[\sum_{k=1}^n \mu(A_k)\Phi||x_k^2 - g(t)||\right]^p \\
&= \left[\int_I \Phi||f_1(t) - g(t)||d\mu\right]^p + \left[\int_I \Phi||f_2(t) - g(t)||d\mu\right]^p \\
&= ||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p.
\end{aligned}$$

By (2) and the fact that simple functions are dense in $L^\Phi(I, X)$, we have

$$\begin{aligned}
\int_I (dist_\Phi(f_1(t), f_2(t), G))^p d\mu \quad &\leq \quad ||f_1 - g_0||_\Phi^p + ||f_2 - g_0||_\Phi^p \\
&< \quad (dist_\Phi(f_1, f_2, L^\Phi(I, G))^p + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have

$$\left[\int_I (dist_\Phi(f_1(t), f_2(t), G))^p d\mu\right]^{\frac{1}{p}} \leq dist_\Phi(f_1, f_2, L^\Phi(I, G)).$$

□

Theorem 2.1 *Suppose $\mu(I) = 1$. If G is p-simultaneously proximinal in X, then for every pair of simple functions f_1, f_2 in $L^\Phi(I, X)$, there exists $g \in L^\Phi(I, X)$ such that g is the best simultaneous approximation of the pair of elements f_1 and f_2.*

Proof. We can write $f_i = \sum_{k=1}^n \chi_{A_k} x_k^i$, $i = 1, 2$, where A_k, $k = 1, 2, \ldots n$ are disjoint measurable sets such that $\bigcup_{k=1}^n A_k = I$ with $\mu(A_k) > 0$ for $k = 1, 2, \ldots, n$. Pick

$y_k \in G$ such that $y_{_k}$ is the best approximation of the pair of elements $x_k^1, x_k^2 \in X$. Let $g = \sum_{k=1}^{n} \chi_{_{A_k}} y_{_k}$. Then

$$
\begin{aligned}
(||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p)^{\frac{1}{p}} &= \left[\left(\int_I \Phi(||f_1(t) - g(t)||)d\mu\right)^p + \left(\int_I \Phi(||f_2(t) - g(t)||)d\mu\right)^p\right]^{\frac{1}{p}} \\
&= \left[\left(\sum_{k=1}^{n} \mu(A_k)\Phi(||x_k^1 - y_k||)\right)^p + \left(\sum_{k=1}^{n} \mu(A_k)\Phi(||x_k^2 - y_k||)\right)^p\right]^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{n} \mu(A_k) \left[(\Phi(||x_k^1 - y_k||))^p + (\Phi(||x_k^2 - y_k||)^p\right]^{\frac{1}{p}} \\
&= \sum_{k=1}^{n} \mu(A_k) dist_\Phi(x_k^1, x_k^2, G) \\
&= \int_I dist_\Phi(f_1(t), f_2(t), G) d\mu \\
&\leq \left[\int_I (dist_\Phi(f_1(t), f_2(t), G))^p d\mu\right]^{\frac{1}{p}}.
\end{aligned}
$$

Using Lemma 2.2, we have

$$
(||f_1 - g||_\Phi^p + ||f_2 - g||_\Phi^p)^{\frac{1}{p}} = dist_\Phi(f_1, f_2, L^\Phi(I, G)).
$$

□

Theorem 2.2 *Let $g \in L^\Phi(I, G)$ be a best p-simultaneous approximation of a pair $f_1, f_2 \in L^\Phi(I, X)$. Then for every measurable subset A of I and every $h \in L^\Phi(I, G)$, we have*

$$
\int_A \Phi(||f_i(t) - g(t)||)d\mu \quad \leq \quad \int_A \Phi(||f_i(t) - h(t)||)d\mu, \tag{3}
$$

for some $i \in \{1, 2\}$.

Proof. Assume that $\mu(A) > 0$ for some $A \subseteq I$. If there is $h_0 \in L^\Phi(I, G)$ doesn't satisfy (3) for $i = 1, 2$, then we define $g_0 \in L^\Phi(I, G)$ by $g_0(t) = g(t)$ if $t \in I - A$ and $h_0(t)$

if $t \in A$. Thus for $i = 1, 2$ we have

$$\begin{aligned} \int_I \Phi(||f_i(t) - g_0(t)||) d\mu &= \int_A \Phi(||f_i(t) - h_0(t)||) d\mu + \int_{I-A} \Phi(||f_i(t) - g(t)||) d\mu \\ &< \int_I \Phi(||f_i(t) - g(t)||) d\mu. \end{aligned}$$

Hence we have

$$||f_i - g_0||_\Phi^p < ||f_i - g||_\Phi^p, \quad i = 1, 2.$$

This contradicts the fact that g is best p-simultaneous approximation from $L^\Phi(I, G)$ of a pair of the elements f_1, f_2. □

Corollary 2.1 *If g is a best p-simultaneous approximation from $L^\Phi(I, G)$ of a pair of elements $f_1, f_2 \in L^\Phi(I, G)$, then for every measurable subset A of I,*

$$\int_A \Phi(||g(t)||d\mu \leq 2 \max \left\{ \int_A \Phi(||f_1(t)||) d\mu, \int_A \Phi(||f_2(t)||) d\mu \right\}.$$

Proof. It follows from Theorem 2.2 by taking $h = 0$. □

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