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Some Characterizations of Rectifying Curves in the Euclidean Space $\mathbf{E}^4$

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Abstract

In this paper, we define a rectifying curve in the Euclidean 4-space as a curve whose position vector always lies in orthogonal complement $N^{\perp}$ of its principal normal vector field N. In particular, we study the rectifying curves in $\mathbb{E}^4$ and characterize such curves in terms of their curvature functions.

Key Words: Rectifying curve, Frenet equations, curvature.

1. Introduction

In the Euclidean 3-space, rectifying curves are introduced by B. Y. Chen in [1] as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields T and B of the curve. Accordingly, the position vector with respect to some chosen origin, of a rectifying curve α in $\mathbb{E}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary differentiable functions in arclength parameter $s \in I \subset \mathbb{R}$.

The Euclidean rectifying curves are studied in [1, 2]. In particular, it is shown in [2] that there exist a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential

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geometry in defining the curves of constant precession. The rectifying curves are also studied in [2] as the extremal curves. In the Minkowski 3-space $\mathbb{E}^3_1$, the rectifying curves are investigated in [4].

In this paper, in analogy with the Euclidean 3-dimensional case, we define the rectifying curve in the Euclidean space $\mathbb{E}^4$ as a curve whose position vector always lies in the orthogonal complement $N^\perp$ of its principal normal vector field N. Consequently, $N^\perp$ is given by

$$N^\perp = \{W \in \mathbb{E}^4 \mid < W, N >= 0\},$$

where $< \cdot, \cdot >$ denotes the standard scalar product in $\mathbb{E}^4$. Hence $N^\perp$ is a 3-dimensional subspace of $\mathbb{E}^4$, spanned by the tangent, the first binormal and the second binormal vector fields T, B_1 and B_2 respectively. Therefore, the position vector with respect to some chosen origin, of a rectifying curve α in $\mathbb{E}^4$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B_1(s) + \nu(s)B_2(s), \tag{1}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s. Next, we characterize rectifying curves in terms of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ and give the necessary and the sufficient conditions for arbitrary curve in $\mathbb{E}^4$ to be a rectifying. Moreover, we obtain an explicit equation of a rectifying curve in $\mathbb{E}^4$.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be arbitrary curve in the Euclidean space $\mathbb{E}^4$. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $< \alpha'(s), \alpha'(s) >= 1$, where $< \cdot, \cdot >$ is the standard scalar product of $\mathbb{E}^4$ given by

$$< X, Y >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. In particular, the norm of a vector $X \in \mathbb{E}^4$ is given by $||X|| = \sqrt{< X, X >}$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed curve α, where T, N, B_1 and B_2 denote respectively the tangent, the principal normal, the first binormal and the second binormal vector fields. Then the Frenet formulas are given by (see [3])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2}$$

The functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ are called, respectively, the first, the second and the third curvature of the curve α. If $k_3(s) \neq 0$ for each $s \in I \subset \mathbb{R}$, the curve α lies fully in $\mathbb{E}^4$. Recall that the unit sphere $\mathbb{S}^3(1)$ in $\mathbb{E}^4$, centered at the origin, is the hypersurface defined by

$$\mathbb{S}^3(1) = \{X \in \mathbb{E}^4 \mid < X, X >= 1\}.$$

3. Some Characterizations of Rectifying Curves in $\mathbb{E}^4$

In this section, we firstly characterize the rectifying curves in $\mathbb{E}^4$ in terms of their curvatures. Let $\alpha = \alpha(s)$ be a unit speed rectifying curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. By definition, the position vector of the curve α satisfies the equation (1), for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Differentiating the equation (1) with respect to s and using the Frenet equations (2), we obtain

$$T = \lambda' T + (\lambda k_1 - \mu k_2)N + (\mu' - \nu k_3)B_1 + (\mu k_3 + \nu')B_2.$$

It follows that

$$\begin{aligned} \lambda' &= 1, \\ \lambda k_1 - \mu k_2 &= 0, \\ \mu' - \nu k_3 &= 0, \\ \mu k_3 + \nu' &= 0, \end{aligned} \tag{3}$$

and therefore

$$\begin{aligned} \lambda(s) &= s + c, \\ \mu(s) &= \frac{k_1(s)(s+c)}{k_2(s)}, \\ \nu(s) &= \frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)}, \end{aligned} \tag{4}$$

where $c \in \mathbb{R}$. In this way the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ of the curve α. Moreover, by using the last equation in (3) and relation (4), we easily find that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ satisfy the equation

$$\frac{k_1(s)k_3(s)(s+c)}{k_2(s)} + \Big(\frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)}\Big)' = 0, \quad c \in \mathbb{R}. \quad (5)$$

Conversely, assume that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$, of an arbitrary unit speed curve α in $\mathbb{E}^4$, satisfy the equation (5). Let us consider the vector $X \in \mathbb{E}^4$ given by

$$\begin{aligned} X(s) = \alpha(s) - (s+c)T(s) - \frac{k_1(s)(s+c)}{k_2(s)}B_1(s) \\ -\frac{k_1(s)(k_2(s) - (s+c)k_2'(s)) + k_1'(s)k_2(s)(s+c)}{k_2^2(s)k_3(s)}B_2(s). \end{aligned}$$

By using the relations (2) and (5), we easily find $X'(s) = 0$, which means that X is a constant vector. This implies that α is congruent to a rectifying curve. In this way, the following theorem is proved.

Theorem 3.1 *Let $\alpha(s)$ be unit speed curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then α is congruent to a rectifying curve if and only if*

$$\frac{k_1(s)k_3(s)(s+c)}{k_2(s)} + \Big(\frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)}\Big)' = 0, \quad c \in \mathbb{R}.$$

In particular, assume that all the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ of rectifying curve α in $\mathbb{E}^4$, are constant and different from zero. Then equation (5) easily implies a contradiction. Hence we obtain the following theorem.

Theorem 3.2 *There are no rectifying curves lying fully in $\mathbb{E}^4$, with non-zero constant curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$.*

Moreover, if two of the curvature functions are constant, we may consider the following cases.

Suppose that $k_1(s) = \text{constant} > 0$, $k_2(s) = \text{constant} \neq 0$ and $k_3(s)$ is non-constant function. By using the equation (5), we find differential equation

$$k_3'(s) - k_3^3(s)(s+c) = 0, \quad c \in \mathbb{R}.$$

The solution of the previous differential equation is given by

$$k_3(s) = \frac{1}{\sqrt{|-s^2 - 2cs - 2c_1|}}, \quad c, c_1 \in \mathbb{R}.$$

Similarly, assume that $k_2(s) =$ constant$\neq 0$, $k_3(s) = k_3 =$ constant$\neq 0$ and $k_1(s)$ is non-constant function. Then equation (5) implies differential equation

$$k_3^2 k_1(s)(s+c) + (k_1(s)(s+c))' = 0, \quad c \in \mathbb{R}, \quad k_3 \in \mathbb{R}_0,$$

whose solution has the form

$$k_1(s) = \frac{c_1}{e^{k_3^2 s}(s+c)}, \quad c_1 \in \mathbb{R}^+.$$

Finally, if $k_1(s) =$ constant> 0, $k_3(s) = k_3 =$ constant$\neq 0$ and $k_2(s)$ is non-constant function, by using equation (5) we get differential equation

$$k_3^2 (s+c)/k_2(s) + ((s+c)/k_2(s))' = 0, \quad c \in \mathbb{R}, \quad k_3 \in \mathbb{R}_0.$$

The previous differential equation has the solution

$$k_2(s) = c_1 e^{k_3^2 s}(s+c), \quad c_1 \in \mathbb{R}^+.$$

In this way, we obtain the following theorem.

Theorem 3.3 *Let $\alpha = \alpha(s)$ be unit speed curve in $\mathbb{E}^4$, with curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then α is congruent to a rectifying curve if*

(a) $k_1(s) =$ constant > 0, $k_2(s) =$ constant $\neq 0$ and $k_3(s) = 1/\sqrt{|-s^2 - 2cs - 2c_1|}$, $c, c_1 \in \mathbb{R}$;

(b) $k_2(s) =$ constant $\neq 0$, $k_3(s) = k_3 =$ constant $\neq 0$ and $k_1(s) = c_1/(e^{k_3^2 s}(s+c))$, $c \in \mathbb{R}$, $c_1 \in \mathbb{R}^+$;

(c) $k_1(s) =$ constant > 0, $k_3(s) = k_3 =$ constant $\neq 0$ and $k_2(s) = c_1 e^{k_3^2 s}(s+c)$, $c \in \mathbb{R}$, $c_1 \in \mathbb{R}^+$.

In the next theorem, we give the necessary and the sufficient conditions for the curve α in $\mathbb{E}^4$ to be a rectifying curve.

Theorem 3.4 *Let $\alpha(s)$ be unit speed rectifying curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then the following statements hold:*

(i) The distance function $\rho(s) = \|\alpha(s)\|$ *satisfies* $\rho^2(s) = s^2+c_1s+c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$.

(ii) The tangential component of the position vector of the curve is given by $<\alpha(s), T(s)>= s + c$, $c \in \mathbb{R}$.

(iii) The normal componet $\alpha^N(s)$ *of the position vector of the curve has constant length and the distance function* $\rho(s)$ *is non-constant.*

(iv) The first binormal component and the second binormal component of the position vector of the curve are respectively given by

$$\begin{aligned} &< \alpha(s), B_1(s) >= \frac{k_1(s)(s+c)}{k_2(s)}, \\ &< \alpha(s), B_2(s) >= \frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s)}{k_2^2(s)k_3(s)}, \quad c \in \mathbb{R}. \end{aligned} \tag{6}$$

Conversely, if $\alpha(s)$ *is a unit speed curve in* $\mathbb{E}^4$ *with non-zero curvatures* $k_1(s)$, $k_2(s)$, $k_3(s)$ *and one of the statements (i), (ii), (iii) or (iv) holds, then* α *is a rectifying curve.*

Proof. Let us first suppose that $\alpha(s)$ is a unit speed rectifying curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. The position vector of the curve α satisfies the equation (1), where the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ satisfy relation (3). Multiplying the third equation in (3) with $-\nu'(s)$ and the last equation in (3) with $\mu'(s)$ and adding, we get $k_3(s)(\mu(s)\mu'(s) + \nu(s)\nu'(s)) = 0$. It follows that $\mu(s)\mu'(s) + \nu(s)\nu'(s) = 0$ and consequently

$$\mu^2(s) + \nu^2(s) = a^2, \tag{7}$$

for some constant $a \in \mathbb{R}_0^+$. From relation (1) we have $< \alpha(s), \alpha(s) >= \lambda^2(s) + \mu^2(s) + \nu^2(s)$, which together with (4) and (7) gives $< \alpha(s), \alpha(s) >= (s+c)^2 + a^2$. Therefore, $\rho^2(s) = s^2 + c_1s + c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$, which proves statement (i).

But using the relations (1) and (4) we easily get $< \alpha(s), T(s) >= s + c$, $c \in \mathbb{R}$, so the statement (ii) is proved.

Note that the position vector of an arbitrary curve α in $\mathbb{E}^4$ can be decomposed as $\alpha(s) = m(s)T(s) + \alpha^N(s)$, where $m(s)$ is arbitrary differentiable function and $\alpha^N(s)$ is the normal component of the position vector. If α is a rectifying curve, relation (1) implies $\alpha^N(s) = \mu(s)B_1(s) + \nu(s)B_2(s)$ and therefore $< \alpha^N(s), \alpha^N(s) >= \mu^2(s) + \nu^2(s)$. Moreover, by using (7), we find $||\alpha^N(s)|| = a$, $a \in \mathbb{R}_0^+$. By statement (i), $\rho(s)$ is non-constant function, which proves statement (iii).

Finally, using (1) and (4) we easily obtain (6), which proves statement (iv).

Conversely, assume that statement (i) holds. Then $< \alpha(s), \alpha(s) >= s^2 + c_1 s + c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$. Differentiating the previous equation two times with respect to s and using (2), we obtain $< \alpha(s), N(s) >= 0$, which implies that α is a rectifying curve.

If statement (ii) holds, in a similar way it follows that α is a rectifying curve.

If statement (iii) holds, let us put $\alpha(s) = m(s)T(s) + \alpha^N(s)$, where $m(s)$ is arbitrary differentiable function. Then

$$< \alpha^N(s), \alpha^N(s) >=< \alpha(s), \alpha(s) > -2 < \alpha(s), T(s) > m(s) + m^2(s).$$

Since $< \alpha(s), T(s) >= m(s)$, it follows that

$$< \alpha^N(s), \alpha^N(s) >=< \alpha(s), \alpha(s) > - < \alpha(s), T(s) >^2,$$

where $< \alpha(s), \alpha(s) >= \rho^2(s) \neq$ constant. Differentiating the previous equation with respect to s and using (2), we find

$$k_1(s) < \alpha(s), T(s) >< \alpha(s), N(s) >= 0.$$

It follows that $< \alpha(s), N(s) >= 0$ and hence the curve α is a rectifying.

If statement (iv) holds, by taking the derivative of the equation

$$< \alpha(s), B_1(s) >= \frac{k_1(s)(s + c)}{k_2(s)}$$

with respect to s and using (2), we obtain

$$-k_2(s) < \alpha(s), N(s) > +k_3(s) < \alpha(s), B_2(s) >= \Big(\frac{k_1(s)(s + c)}{k_2(s)}\Big)'.$$

By using (6), the last equation becomes $< \alpha(s), N(s) >= 0$, which means that α is a rectifying curve. This proves the theorem. □

In the next theorem, we find the parametric equation of a rectifying curve.

Theorem 3.5 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve in $\mathbb{E}^4$ given by $\alpha(t) = \rho(t)y(t)$, where $\rho(t)$ is arbitrary positive function and $y(t)$ is a unit speed curve in the unit sphere $\mathbb{S}^3(1)$. Then α is a rectifying curve if and only if*

$$\rho(t) = \frac{a}{\cos(t + t_0)}, \quad a \in \mathbb{R}_0, \quad t_0 \in \mathbb{R}. \tag{8}$$

Proof. Let α be a curve in $\mathbb{E}^4$ given by

$$\alpha(t) = \rho(t)y(t),$$

where $\rho(t)$ is arbitrary positive function and $y(t)$ is a unit speed curve in $\mathbb{S}^3(1)$. By taking the derivative of the previous equation with respect to t, we get

$$\alpha'(t) = \rho'(t)y(t) + \rho(t)y'(t).$$

Hence the unit tangent vector of α is given by

$$T(t) = \frac{\rho'(t)}{v(t)}y(t) + \frac{\rho(t)}{v(t)}y'(t), \tag{9}$$

where $v(t) = ||\alpha'(t)||$ is the speed of α. Differentiating the equation (9) with respect to t, we find

$$T' = \Big(\frac{\rho'}{v}\Big)'y + \Big(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho+\rho'')}{v^3}\Big)y' + \Big(\frac{\rho}{v}\Big)y''. \tag{10}$$

Let Y be the unit vector field in $\mathbb{E}^4$ satisfying the equations $< Y, y >=< Y, y' >=< Y, y \times y' >= 0$. Then $\{y, y', y \times y', Y\}$ is the orthonormal frame of $\mathbb{E}^4$. Therefore, decomposition of y'' with respect to the frame $\{y, y', y \times y', Y\}$ reads

$$y'' =< y'', y > y+ < y'', y' > y'+ < y'', y \times y' > y \times y'+ < y'', Y > Y. \tag{11}$$

Since $< y, y >=< y', y' >= 1$, it follows that $< y'', y >= -1$ and $< y'', y' >= 0$, so the equation (11) becomes

$$y'' = -y+ < y'', y \times y' > y \times y'+ < y'', Y > Y. \tag{12}$$

Substituting (12) into (10) and applying Frenet formulas for arbitrary speed curves in $\mathbb{E}^4$, we find

$$\begin{aligned} \kappa_1 v N = \ & \Big(\Big(\frac{\rho'}{v}\Big)' - \frac{\rho}{v}\Big)y + \Big(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho+\rho'')}{v^3}\Big)y' + \frac{< y'', y \times y' >}{v}\alpha \times y' \\ & +\Big(\frac{\rho}{v}\Big) < y'', Y > Y. \end{aligned} \tag{13}$$

Since $< y, y >= 1$, we have $< y, y' >= 0$ and thus $< \alpha, y' >= 0$. We also have $< \alpha, Y >= 0$. By definition, α is a rectifying curve in $\mathbb{E}^4$ if and only if $< \alpha, N >= 0$.

Therefore, after taking the scalar product of (13) with α, we have $< \alpha, N >= 0$ if and only if

$$\left(\frac{\rho'}{v}\right)' - \frac{\rho}{v} = 0.$$

The previous differential equation is equivalent to the equation

$$\rho\rho'' - 2\rho^{'2} - \rho^2 = 0. \tag{14}$$

whose nontrivial solutions are given by (8). This proves the theorem. □

Example: Let us consider the curve $\alpha(s) = (a/(\sqrt{2}\cos(s+s_0)))(\sin(s), \cos(s), \sin(s), \cos(s))$, $a \in \mathbb{R}_0$, $s_0 \in \mathbb{R}$ in $\mathbb{E}^4$. This curve has a form $\alpha(s) = \rho(s)y(s)$, where $\rho(s) = a/\cos(s+s_0)$ and $y(s) = (1/\sqrt{2})(\sin(s), \cos(s), \sin(s), \cos(s))$. Since $\| y(s) \|= 1$ and $\| y'(s) \|= 1$, $y(s)$ is a unit speed curve in the unit sphere $\mathbb{S}^3(1)$. According to the theorem 3.5, $\alpha(s)$ is a rectifying curve lying fully in $\mathbb{E}^4$.

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