Turk J Math
32 (2008) , 83 – 101.

Mixed Type of Integral Equation with Potential Kernel

Mohamed Abdella Ahmed Abdou and Gamal Mohamed Abd Al-Kader

Abstract

This paper presents the solution of an integral equation of a mixed type in three-dimensions in the space $L_2(\Omega) \times C[0,T]$, where $T < \infty$, and Ω is the domain of integration with respect to position. The kernel of position integral term is considered in the potential function form, while the kernel of time is considered as a continuous kernel. A linear system of Fredholm integral equations of the first and second kinds are obtained and solved. Krein's method is used to solve the Fredholm integral equation of the first kind, while the second kind is solved numerically.

Key Words: Fredholm integral equations; Potential kernel; Legendre and Jacobi polynomials; Weber-Sonien integral.

1. Introduction

Many problems of mathematical physics, engineering, and theory of elasticity such as contact problems lead to the Fredholm-Volterra integral equations of the first kind [2, 6, 8]. In recent publication [3], we discussed the existence and uniqueness of the solution of the following mixed integral equation:

$$\int_0^t \int_{-1}^1 F(t,\tau)(\ln|x-y| - d)\phi(y,\tau)dyd\tau + \int_0^t G(t,\tau)\phi(x,\tau)d\tau = f(x,t), \tag{1.1}$$

2000 *AMS Mathematics Subject Classification:* 45B05, 45H05, 45L05, 33C45.

with $(t, \tau \in [0, T], T < \infty$, and d is a constant.

The solution is obtained in the space $L_2[-1, 1] \times C[0, T]$ and depends on the relations between the derivatives of $F(t, \tau)$ and $G(t, \tau)$ with respect to t for $\tau \in [0, T]$.

In the same manner we propose the solution of the following mixed integral equation:

$$\begin{aligned}
&\int_0^t \iint_\Omega F(t,\tau)k(x-\zeta, y-\eta)\psi(\zeta,\eta)d\zeta d\eta d\tau + \int_0^t H(t,\tau)\psi(x,y,\tau)d\tau \\
&\qquad = \Theta[\gamma(t) - \alpha(x,y)] = f(x,y,t); \\
&\qquad \Omega = \{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a, z = 0\}, \\
&\qquad\quad t,\tau \in [0,T], T < \infty, \quad k(x-\zeta, y-\eta) = [(x-\zeta)^2 + (y-\eta)^2]^{-\frac{1}{2}}
\end{aligned} \tag{1.2}$$

under the condition

$$\int_\Omega \psi(x,y,t)dx = P(t) < \infty, \tag{1.3}$$

where Θ and $\gamma(t)$ and their derivatives are continuous functions.

The mixed integral equation (1.2) is examined with respect to position and time for the Fredholm integral equation and the Volterra integral equation terms, respectively. This equation under the condition (1.3) is investigated from the three dimensional semi-symmetric Hertz contact problem in the theory of elasticity of frictionless impression of a rigid surface (G, ν) into an elastic material occupying the domain Ω, where G is the displacement magnitude and ν the Poisson's coefficient. The upper rigid surface (stamp) that has base equation $\alpha(x, y)$ is impressed into the lower elastic surface (plane) by a variable known force $P(t)$. This force exhibits eccentricity of application $e(t)$, $t \in [0, T]$, and in that case $\gamma(t)$ is the rigid displacement. The function $F(t, \tau)$ represents the resistance force of the material in the contact domain Ω through time $t \in [0, T]$; while the function $H(t, \tau)$ is the external force of resistance, which is supplied through the contact domain.

The unknown function $\psi(x, y, t)$ is called the potential function of the integral equation (1.2) and can be obtained in the space $L_2(\Omega) \times C[0, T]$. In order to guarantee the existence of a unique solution of (1.2), under the condition (1.3), we assume the following:

(i) The kernel of position $k(x, y) \in C([\Omega] \times [\Omega])$, $x = \bar{x}(x_1, x_2)$, $\quad y = \bar{y}(y_1, y_2)$ satisfies in the space $L_2(\Omega)$ the following:

$$\left\{ \iint_{\Omega} k^2(x,y)dxdy \right\}^{1/2} = A,$$

where A is a constant.

(ii) The two continuous functions $F(t,\tau)$ and $H(t,\tau)$ belong to $C([0,T]\times[0,T])$ and satisfy $|F(t,\tau)| < B$, $|H(t,\tau)| < D$, where B and D are constants. (iii) The given function $f(x,y,t) \in L_2(\Omega) \times C[0,T]$.

(iv) The unknown function $\psi(x,t)$, $x = \bar{x}(x_1,x_2)$ is assumed to satisfy Hölder condition with respect to time and Lipschitz condition with respect to position.

2. Method of solution

To obtain the solution of the integral equation (1.2) under the condition (1.3), with respect to position, we use polar coordinates:

$$\int_0^t \int_0^a \int_{-\pi}^{\pi} \frac{F(t,\tau)\psi(\rho,\phi,\tau)\rho d\rho d\phi d\tau}{\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\phi)}} + \int_0^t H(t,\tau)\psi(r,\theta,\tau)d\tau = f(r,\theta,t), \qquad (2.1)$$

and

$$\int_0^a \int_{-\pi}^{\pi} \psi(\rho,\phi,t)\rho d\rho d\phi = P(t). \qquad (2.2)$$

To separate the variable, one assumes

$$\psi(r,\theta,t) = \psi_m(r,t)\begin{cases}\cos m\theta \\ \sin m\theta\end{cases}, \qquad f(r,\theta,t) = f_m(r,t)\begin{cases}\cos m\theta \\ \sin m\theta.\end{cases} \qquad (2.3)$$

Using (2.3) in (2.1) and (2.2), we get

$$\int_0^t \int_0^1 F(t,\tau)K_m(r,\rho)\psi_m(\rho,\tau)\rho d\rho d\tau + \int_0^t H(t,\tau)\psi_m(r,\tau)d\tau = f_m(r,t), \quad (a=1) \qquad (2.4)$$

and

$$\int_0^1 \psi_m(\rho,t)\rho d\rho = \begin{cases}\frac{P(t)}{2\pi} & m=0 \\ 0 & m \geq 1,\end{cases} \qquad (2.5)$$

where

$$K_m(r,\rho)=\int_{-\pi}^{\pi}\frac{\cos m\theta d\phi}{\sqrt{r^2+\rho^2-2r\rho\cos\phi}} \tag{2.6}$$

Abdou [2] obtained the kernel of Eq. (2.6) in the form of Weber-Sonien integral form as

$$K_m(r,\rho)=2\pi\int_0^{\infty}J_m(\alpha r)J_m(\alpha\rho)d\alpha, \tag{2.7}$$

where $J_m(x)$ is the Bessel function of order m.

Using (2.7), in (2.4), and assuming $\phi_m(r,t)=\sqrt{r}\psi_m(r,t)$, $g_m(r,t)=\sqrt{r}f_m(r,t)$ in Eqs. (2.4) and (2.5), we have

$$\int_0^t\int_0^1 F(t,\tau)W(r,\rho)\phi_m(\rho,\tau)d\rho d\tau+\int_0^t H(t,\tau)\phi_m(r,\tau)d\tau=g_m(r,t), \tag{2.8}$$

$$\int_0^1\phi_m(\rho,t)\sqrt{\rho}d\rho=\begin{cases}\frac{P(t)}{2\pi} & m=0\\ 0 & m\geq 1,\end{cases} \tag{2.9}$$

where

$$W(r,\rho)=2\pi\sqrt{r\rho}\int_0^{\infty}J_m(\alpha r)J_m(\alpha\rho)d\alpha. \tag{2.10}$$

In Eqs. (2.8), and (2.9), we divide the interval $[0,T]$, $0\leq t\leq T<\infty$ as $0=t_0\leq t_1<\ldots<t_N=T$, where $t=t_l$, $l=1,2,...,N$. In the same way as discussed in [3], we obtain

$$\sum_{j=0}^{l}v_jH_{j,l}\phi_{j,m}(r)+\sum_{j=0}^{l}u_jF_{j,l}\int_0^1 W(r,\rho)\phi_{j,m}d\rho=g_{l,m}(r) \tag{2.11}$$

$$\int_0^1\phi_{l,m}(\rho)\sqrt{\rho}d\rho=\begin{matrix}\frac{P_l}{2\pi} & m=0\\ 0 & m\geq 1,\end{matrix} \tag{2.12}$$

where the following notations are used

$$\begin{aligned}&F(t_l,t_j)=F_{l,j}, \qquad H(t_l,t_j)=G_{l,j},\\ &\phi_m(r,t_j)=\phi_{m,j}(r) \quad \text{and} \quad f_m(r,t_l)=f_{l,m}(r),\ l=0,1,...N;\ 0\leq j\leq l.\end{aligned} \tag{2.13}$$

The characteristic points v_j and u_j [3, 5] depend on the number of derivatives of $H(t,\tau)$ and $F(t,\tau)$, respectively, with respect to $t \in [0,T]$. Thus, we can discuss the following.

(i) When $H(t,\tau)$ has i times derivatives with respect to t, for $i < l$, $l = 1, 2, ...N$, the integral equation (2.11) becomes

$$\sum_{j=0}^{i} v_j H_{j,i}\phi_{j,m}(r) + \sum_{j=0}^{i} u_j F_{j,i} \int_0^1 W(r,\rho)\phi_{j,m} d\rho = g_{i,m}(r), \tag{2.14}$$

$$\sum_{j=0}^{i+1} u_j F_{j,i} \int_0^1 W(r,\rho)\phi_{j,m}(\rho) d\rho = g_{i,m}(r) - \sum_{k=0}^{i} \mu_k \phi_{k,m}(r), \tag{2.15}$$

where $\phi_k(r)$, $0 \le k \le i$ represent the solutions of the linear system of Eqs. (2.14) and μ_k are constants and represent the corresponding coefficients of $\phi_k(r)$. The linear system of Eqs.(2.14) represent Fredholm integral equations of the second kind with a potential kernel given by Eq.(2.10), in the form of a Weber-Sonien integral. On the other hand, the formula (2.15) represents a system of Fredholm integral equations of the first kind with a potential kernel.

(ii) When $H(t,\tau)$ and $F(t,\tau)$ have the same number of derivatives with respect to t, the integral equation (2.4) for all values of $0 \le j \le l$ represents a linear system of Fredholm integral equation of the second kind. Thus, our aim is to solve the linear system of Eqs. (2.14) and (2.15), respectively.

3. Fredholm integral equation of the second kind

To solve the linear system of Eq. (2.11), we use the recurrence relation, and write it in the form

$$\mu_i \phi_{i,m}(r) + \lambda_i \int_0^1 W(r,\rho)\phi_{i,m}(\rho) d\rho = g_{i,m}(r) - \Big[\sum_{k=0}^{i-1} \mu_i \phi_{k,m}(r) + \sum_{k=0}^{i-1} \lambda_i \int_0^1 W(r,\rho)\phi_{k,m}(\rho) d\rho\Big] \tag{3.1}$$

then for $i = 0$,

$$\mu_0 \phi_{0,m}(r) + \lambda_0 \int_0^1 W(r,\rho)\phi_{0,m}(\rho) d\rho = g_{0,m}(r), \tag{3.2}$$

where $\mu_i = v_i H_{i,i}$, $\lambda_i = u_i F_{i,i}$.

Under the condition

$$\int_0^1 \phi_{0,m}(\rho)\sqrt{\rho}d\rho = q, \qquad q = \begin{cases} \frac{P_0}{2\pi} & m = 0 \\ 0 & m \geq 1, \end{cases} \tag{3.3}$$

the solution of (3.2) depends on the kernel (2.10) and the formula of the surface $f_{0,m}(r)$. So, we write $g_{0,m}(r)$ in MacLaurin expansion around $r = 0$:

$$f_{0,m}(r) = \frac{f''_{0,m}(0)}{2!}r^2 + \cdots + \frac{f^{(n)}_{0,m}(0)}{n!}r^n + \cdots \tag{3.4}$$

The formula (3.4) is obtained after considering the initial and the tangent points of the surface that are in contact with the origin O. Also, the formula (3.4) gives the degree of displacement of the surface for any degree. For example, if the displacement is very small such that $A_k = 0$, $A_k = \frac{f^{(k)}_{0,m}(0)}{k!}, \quad k \geq 3$, we obtain the displacement as $f_{0,m}(r) = A_2 r^2$.

In general, we write

$$f_{0,m}(r) = A_{2m}r^{2m}, \quad f_{0,m}(r) = \Theta[\gamma(0) - \alpha_m(r)], \tag{3.5}$$

where m is the harmonic order of the contact problem. Hence, the function $g_{0,m}(r)$ of Eq. (3.2), in view of (3.5), takes the form

$$g_{0,m}(r) = (\Delta_0 - \Theta A_{2m}r^{2m})\sqrt{r}, \quad \Delta_0 = \Theta\gamma(0), \qquad \Theta = G(1-\nu)^{-1}, \tag{3.6}$$

which represents a polynomial of degree $r^{2m+\frac{1}{2}}$. In view of Eq. (3.6) and the boundary condition (3.3) the integral equation (3.2) takes the form

$$\mu_0 Z_{0,m}(r) + \lambda_0 \int_0^1 W(r,\rho)Z_{0,m}(\rho)d\rho = r^{2m+\frac{1}{2}} \tag{3.7}$$

and

$$\Delta_0 \int_0^1 Z_{0,0}(r)\sqrt{r}dr - A_{2m}\int_0^1 Z_{0,m}(r)\sqrt{r}dr = \frac{P_0}{2\pi} \tag{3.8}$$

where

$$\phi_{0,m}(r) = \Delta_0 Z_{0,0}(r) - A_{2m}Z_{0,m}(r), \quad m \geq 1. \tag{3.9}$$

To obtain the value of $\phi_{0,m}(r)$ of Eq. (3.9) or of Eq. (3.2), we must obtain the solution of Eq. (3.7). To do this, we use the formula (7.39-(11)) of [9], and with the aid of [1], we can write the kernel of (2.10) in the form

$$W(r,\rho) = \sqrt{2}\pi(r\rho)^{m+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\Gamma^2(j+m+3/4)P_j^m(r)P_j^m(\rho)}{\Gamma^2(j+1+m)(2j+m+3/4)^{-1}}, \tag{3.10}$$

where $P_j^{(m,-1/2)}(x)$ is the Jacobi polynomial

$$P_j^{(m)}(x) = P_j^{(m,-1/2)}(1-2x^2), \tag{3.11}$$

and [7]

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-y}{2}), \tag{3.12}$$

where $\Gamma(x)$ and ${}_2F_1(\alpha,\beta;\gamma;z)$ are the Gamma and hypergeometric functions, respectively. Hence, the solution of Eq. (3.7) with the kernel (3.10) is equivalent to the solution of the linear system

$$\mu_0 X_i + C_i \sum_{j=0}^{\infty} E_{ij} X_j = f_i, \tag{3.13}$$

where

$$f_i = (2i+m+\frac{3}{4})^{\frac{1}{4}} \int_0^1 f_m(r) r^{m+1} P_i^m(r) dr, \tag{3.14}$$

$$C_i = \frac{\pi\lambda_0 \Gamma^2(i+m+3/4)(2i+m+3/4)^{\frac{1}{4}}}{\Gamma^2(i+m+1)}, \tag{3.15}$$

$$E_{ij} = (2j+m+3/4)(2i+m+3/4) \int_0^1 r^{2m+1} P_i^{(m)}(r) P_j^{(m)}(r) dr. \tag{3.16}$$

The infinite linear system of Eq. (3.13) is solvable under the conditions

$$\sum_{j=0}^{\infty} |C_i E_{ij}| < \mu_0. \tag{3.17}$$

Using the orthogonality of the Jacobi polynomials, the solution of Eq. (3.7) takes the form

$$\mu_0 Z_{0,m}(r) = r^{2m+\frac{1}{2}} - \pi\lambda_0 \sum_{j=0}^{\infty} \frac{\Gamma^2(j+m+3/4) r^m P_j^m(r) X_j}{(j+m+1)(2j+m+3/4)^{-\frac{3}{4}}}. \tag{3.18}$$

Hence, the complete solution of Eq. (3.9) becomes

$$\begin{aligned}\mu_0 \phi_{0,m}(r) =& r^{\frac{1}{2}}(\Theta\gamma(0) - A_{2m} r^{2m}) + \pi\lambda_0 \sum_{j=0}^{\infty} \Big[\frac{\Gamma^2(j+m+3/4) r^m P_j^m(r) X_j^m}{(j+m+1)(2j+m+3/4)^{-\frac{3}{4}}} \\ &- \Theta\gamma(0) \frac{\Gamma^2(j+3/4) P_j^0(r)}{(j+1)(2j+3/4)^{-\frac{3}{4}}} \Big].\end{aligned} \tag{3.19}$$

Hence by mathematical induction, the solution of Eq. (3.1) can be obtained.

4. Fredholm integral equation of the first kind

Many different methods can be used to obtain the solution of Fredholm integral equation. Krein's method is considered one of the best methods in the applied science for solving the singular integral equations, because singularity disappears and the integral equations can be solved directly without singularity [4, 11]. To use Krein's method, firstly we adapt Eq. (2.15) to the form

$$u_l F_{l,l} \int_0^1 W(r,\rho)\phi_{l,m}(\rho) d\rho = g_{l,m}(r) - \sum_{k=0}^{i} \mu_k \phi_{k,m}(r) - \sum_{j=i+1}^{l-1} u_j F_{j,l} \int_0^1 W(r,\rho)\phi_{j,m}(\rho) d\rho. \tag{4.1}$$

The solution of Eq. (4.1) can be obtained using the recurrence relation, where $\phi_{k,m}(r)$, $0 \le k \le i$ can be obtained from Eq.(2.14) with the aid of Eq. (3.19). Hence for $l = i+1$, firstly, we can write Eq. (4.1) as

$$\mu_{i+1} \int_0^1 W(r,\rho)\phi_{i+1,m}(\rho) d\rho = g_{i+1,m}(r) - \sum_{k=0}^{i} \mu_k \phi_{k,m}(r), \qquad (\mu_{i+1} = u_{i+1} F_{i+1,i+1}). \tag{4.2}$$

Secondly, we can write Eq. (4.2) in the form

$$\beta \int_0^1 k(r,\rho)\psi(\rho)\rho d\rho = h(r), \tag{4.3}$$

where

$$\begin{aligned} &\beta = \mu_{i+1}, \qquad \psi(r) = \tfrac{\phi_{i+1,m}(r)}{\sqrt{r}}, \\ &h(r) = \tfrac{1}{\sqrt{r}}[g_{i+1,m}(r) - \textstyle\sum_{k=0}^{i} \mu_k \phi_{k,m}(r)], \end{aligned} \tag{4.4}$$

and

$$k(r,\rho) = 2\pi \int_0^\infty J_m(rt)J_m(\rho t)dt. \tag{4.5}$$

Also, the state condition (2.12) becomes

$$\int_0^1 \psi \rho d\rho = \frac{P}{2\pi}. \tag{4.6}$$

Using Krein's principal method [4, 11], with a kernel in the form of Eq. (4.5) and under the condition (4.6), the general solution of Eq. (4.3) takes the form

$$\beta\phi(r) = \frac{\delta}{\pi^2\sqrt{1-r^2}} - \frac{1}{\pi^2}\int_r^1 \frac{du}{\sqrt{u^2-r^2}}\frac{d^2}{du^2}\int_0^u \frac{th(t)dt}{\sqrt{u^2-t^2}}, \tag{4.7}$$

where

$$\delta = [\frac{d}{du}\int_0^u \frac{th(t)dt}{\sqrt{u^2-t^2}}]_{u=1}. \tag{4.8}$$

The solution of the integral formula (4.7) can be derived in the following theorem.

Theorem When the known function $h(t)$ takes a Legendre polynomial form, the eigenfunctions of Eq. (4.7) have the form

$$\beta\phi(r) = \frac{(-1)^n B_n 2^{2n}(2n+1)(n!)^2}{\pi^2\sqrt{1-r^2}(2n)!} P_{2n}(\sqrt{1-r^2}) \tag{4.9}$$

where $P_{2n}(y)$ is the Legendre polynomial, and

$$B_n = \frac{\sqrt{\pi}\Gamma(n+1)}{2\Gamma(n+3/2)}. \tag{4.10}$$

The proof of this theorem depends on the following two lemmas.

Lemma 1 For all integers $(n > 0)$ the value of the term

$$L_n(r) = \frac{d^2}{dr^2}\int_0^r \frac{tP_{2n}(\sqrt{1-t^2})dt}{\sqrt{r^2-t^2}} \tag{4.11}$$

takes the form

$$L_n(r) = (2n+1)rB_n[3P_{n-1}^{(0,\frac{3}{2})}(2u^2-1) + (2n+3)u^2P_{n-2}^{(1,\frac{5}{2})}(2u^2-1)], \tag{4.12}$$

where B_n is given by (4.10) and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order n (Eq. (3.12)); and $n \geq 0$, where $P_n^{(\alpha,\beta)}(x) = 0$ for $n < 0$.

Proof: To prove lemma 1, we assume the new parameters $\xi = \sqrt{1-r^2}$, $\eta = \sqrt{1-t^2}$, in the integral

$$L_n^{(1)}(r) = \int_0^r \frac{tP_{2n}(\sqrt{1-t^2})dt}{\sqrt{r^2-t^2}}, \tag{4.13}$$

to have

$$L_n^{(1)}(\sqrt{1-\xi^2}) = \int_\xi^1 \frac{\eta P_{2n}(\eta)d\eta}{\sqrt{\eta^2-\xi^2}}, \tag{4.14}$$

which can be adapted to the form

$$L_n^{(1)}(\sqrt{1-\xi^2}) = \int_\xi^1 \frac{P_n^{(0,-\frac{1}{2})}(2\eta^2-1)\eta d\eta}{\sqrt{\eta^2-\xi^2}} \tag{4.15}$$

through using the relation [7]

$$P_{2n}(x) = C_{2n}^{\frac{1}{2}}(x) = P_n^{(0,-\frac{1}{2})}(2x^2-1),$$

where $C_n^\lambda(x)$ is the Gegenbauer polynomial.

Putting in (4.15) $t = 2\xi^2-1$, $v = 2\eta^2-1$, then using the transformation $v = 1-(1-t)\tau$, the formula (4.15) becomes

$$L_n^{(1)}(\sqrt{\frac{1-t}{2}}) = 2^{-\frac{3}{2}}\int_0^1 (1-t)^{\frac{1}{2}}(1-\tau)^{\frac{1}{2}}P_n^{(0,-\frac{1}{2})}(1-(1-t)\tau)d\tau. \tag{4.16}$$

Using the integral relation Eq. (7.39-2), pp.856 of [9])

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}P_n^{(\alpha,\beta)}(1-\gamma t)dt = \frac{\Gamma(n+\alpha+1)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\alpha+1)\Gamma(\lambda+\mu)n!} \times {}_3F_2(-n, n+\alpha+\beta, \lambda; \alpha+1; \lambda+\mu; \gamma/2), \tag{4.17}$$

integral (4.16), becomes

$$L_n^{(1)}(r) = rB_nP_n^{(-1,\frac{1}{2})}(2r^2-1), \tag{4.18}$$

where B_n is given by Eq. (4.10).

In view of the Jacobi differential relation [7],

$$\frac{\partial}{\partial x}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{4.19}$$

Differentiating Eq. (4.19) with respect to r, then using Eq. (4.19), we get

$$\frac{d}{dr}L_n^{(1)}(r) = B_n[P_n^{(-1,\frac{1}{2})}(2r^2-1) + (2n+1)r^2P_{n-1}^{(0,\frac{3}{2})}(2r^2-1)]. \tag{4.20}$$

The value of δ in Eq. (4.8) can be obtained from (4.20) by putting $r = 1$, to have

$$\delta = (2n+1)B_n. \tag{4.21}$$

The required result of Eq.(4.12) is obtained after differentiating Eq. (4.21) with respect to r, and using Eq. (4.20) again. □

Lemma 2 The following relations hold:

$$z\ {}_3F_2(-n+2, n+\frac{5}{2}, 1; 2, \frac{5}{2}; z) = \frac{3}{(2n+3)(n-1)} - \frac{3\sqrt{\pi}(n-2)!}{2(2n+3)\Gamma(n+\frac{1}{2})}P_{n-1}^{(\frac{1}{2},1)}(1-2z); \tag{4.22}$$

$$z\ {}_3F_2(-n+2, n+\frac{5}{2}, 1; 2, \frac{3}{2}; z) = \frac{1}{(2n+3)(n-1)} - \frac{\sqrt{\pi}(n-2)!}{(2n+3)\Gamma(n-\frac{1}{2})}P_{n-1}^{(-\frac{1}{2},2)}(1-2z), \tag{4.23}$$

where ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is the generalized hypergeometric function

Proof: To prove Eq. (4.22) we use Eq. (4.17) to obtain

$$\int_0^1 (1-t)^{\frac{1}{2}} P_{n-2}^{(1,\frac{5}{2})}(1-(1-\xi)t)dt = \frac{2(n-1)}{3}\, {}_3F_2(-n+2, n+\frac{5}{2}, 1; 2; \frac{5}{2}; z). \tag{4.24}$$

Assume

$$f(z) = z\, {}_3F_2(-n+2, n+\frac{5}{2}, 1; 2; \frac{5}{2}; z), \tag{4.25}$$

then differentiate it, to have

$$\frac{df(z)}{dz} = {}_2F_1(-n+2, n+\frac{5}{2}; \frac{5}{2}; z). \tag{4.26}$$

Here, we use the famous relation [7]

$${}_3F_2(-n+2, n+\frac{5}{2}, 1; 2; \frac{5}{2}; z) = \sum_{m=0}^{\infty} \frac{(-n+2)_m (n+\frac{5}{2})_m (1)_m}{m!(2)_m (\frac{5}{2})_m} z^m. \tag{4.27}$$

The formula (4.26) can be adapted with the help of Eq. (3.12) to obtain

$$\frac{df(z)}{dz} = \frac{2\sqrt{\pi}(n-2)!}{4\Gamma(n+\frac{1}{2})} P_{n-2}^{(\frac{3}{2},2)}(1-2z). \tag{4.28}$$

Integrating Eq. (4.28), then using the formula (4.19) with the condition $f(0)=0$, we have

$$f(z) = \frac{3}{(2n+3)(n-1)} - \frac{3\sqrt{\pi}(n-2)!}{2(2n+3)\Gamma(n+\frac{1}{2})} P_{n-1}^{(\frac{1}{2},1)}(1-2z). \tag{4.29}$$

Using the same approach, we can prove

$$\begin{aligned} g(z) =& z\ {}_3F_2(-n+2, n+\frac{5}{2}, 1; 2, \frac{3}{2}; z) \\ =& \frac{1}{(2n+3)(n-1)} - \frac{\sqrt{\pi}(n-2)!}{(2n+3)\Gamma(n-\frac{1}{2})} P_{n-1}^{(-\frac{1}{2},2)}(1-2z) \end{aligned} \tag{4.30}$$

which completes the proof of the lemma. □

Now, the integral term

$$C(r) = \frac{1}{\pi^2} \int_r^1 \frac{du}{\sqrt{u^2-r^2}} \frac{d^2}{du^2} \int_0^u \frac{sh(s)ds}{\sqrt{u^2-s^2}} \tag{4.31}$$

of Eq. (4.7) can be evaluated. To achieve this, we introduce the value of (4.12) into (4.31) to get

$$C_n(t) = \frac{(2n+1)B_n}{\pi^2}\Big[3\int_t^1 \frac{uP_{n-1}^{(0,\frac{3}{2})}(2u^2-1)}{\sqrt{u^2-t^2}}du + (2n+3)\int_t^1 \frac{u^3P_{n-2}^{(1,\frac{5}{2})}(2u^2-1)}{\sqrt{u^2-t^2}}du\Big]. \tag{4.32}$$

Using the substitutions $\xi = 2t^2-1$, $\eta = 2u^2-1$, then taking the parameter $\eta = 1-(1-\xi)\tau$, $0<\tau<1$, the formula (4.32) can be written in the form

$$C_n(t) = \frac{(2n+1)B_n}{2^{\frac{3}{2}}\pi^2}\sqrt{1-\xi}\Big[3G_n(\xi) + \frac{1}{2}(2n+3)(1-\xi)Q_n(\xi) + \frac{1}{2}(2n+3)(1+\xi)H_n(\xi)\Big], \tag{4.33}$$

where

$$\begin{aligned} G_n(\xi) &= \textstyle\int_0^1 (1-\tau)^{-\frac{1}{2}}P_{n-1}^{(0,\frac{3}{2})}(1-(1-\xi)\tau)d\tau \\ Q_n(\xi) &= \textstyle\int_0^1 (1-\tau)^{-\frac{1}{2}}P_{n-2}^{(1,\frac{5}{2})}(1-(1-\xi)\tau)d\tau \\ H_n(\xi) &= \textstyle\int_0^1 (1-\tau)^{-\frac{1}{2}}P_{n-1}^{(1,\frac{5}{2})}(1-(1-\xi)\tau)d\tau, \end{aligned} \tag{4.34}$$

with the aid of the integral formula (4.17) the values of $G_n(\xi)$, $Q_n(\xi)$ and $H_n(\xi)$ can be obtained in the form

$$\begin{aligned} G_n(\xi) &= \tfrac{\sqrt{\pi}(n-1)!}{\Gamma(n+\frac{1}{2})}P_{n-1}^{(\frac{1}{2},1)}(\xi), \\ Q_n(\xi) &= \tfrac{2}{(2n+3)(1-\xi)}\Big[2 - \tfrac{\sqrt{\pi}(n-1)!}{\Gamma(n+\frac{1}{2})}P_{n-1}^{(\frac{1}{2},1)}(\xi)\Big], \\ H_n(\xi) &= \tfrac{4}{(2n+3)(1-\xi)}\Big[1 - \tfrac{\sqrt{\pi}(n-1)!}{\Gamma(n-\frac{1}{2})}P_{n-1}^{(-\frac{1}{2},2)}(\xi)\Big]. \end{aligned} \tag{4.35}$$

Substituting the results of Eq. (4.35) in Eq. (4.32), we get

$$C_n(t) = \frac{(2n+1)\sqrt{1-\xi}B_n}{2^{\frac{3}{2}}\pi^2}\Big\{\frac{2\sqrt{\pi}(n-1)!}{\Gamma(n-\frac{1}{2})}\Big[\frac{1-\xi}{n-\frac{1}{2}}P_{n-1}^{(\frac{1}{2},1)}(\xi) - (1+\xi)P_{n-1}^{(-\frac{1}{2},2)}(\xi)\Big] + 4\Big\}. \tag{4.36}$$

Our aim is to represent the Jacobi polynomial in (4.36) in the form of Legendre polynomial. Firstly, we write

$$E_n^{(1)}(\xi) = \frac{1-\xi}{n-\frac{1}{2}}P_{n-1}^{(\frac{1}{2},1)}(\xi), \tag{4.37}$$

then use the famous relation [7]

$$P_{n-1}^{(\frac{1}{2},1)}(t)=\frac{(-1)^n}{(2n+1)t}\frac{d}{dt}\Big[P_{2n}(\sqrt{1-t^2})\Big], \tag{4.38}$$

to have

$$E_n^{(1)}(\xi)=\frac{4(-1)^n}{(4n^2-1)}\frac{\sqrt{1-t^2}}{t}\Big[(1-y)^2P_{2n}^{'}(y)\Big], \qquad y=\sqrt{1-t^2} \tag{4.39}$$

Using the recurrence relation [7]

$$(1-x^2)P_n^{'}(x)=(n+1)[xP_n(x)-P_{n+1}(x)], \tag{4.40}$$

the final formula of $E_n^{(1)}(\xi)$ becomes

$$E_n^{(1)}(\xi)=\frac{4(-1)^n}{(2n-1)}\frac{y}{1-y^2}[yP_{2n}(y)-P_{2n+1}(y)], \qquad (y=\sqrt{1-t^2}, \quad t=\sqrt{\frac{1+\xi}{2}}). \tag{4.41}$$

Secondly, we write

$$E_n^{(2)}(\xi)=(1+\xi)P_{n-1}^{(-\frac{1}{2},2)}(\xi), \tag{4.42}$$

which can be adapted to the form

$$E_n^{(2)}(\xi)=2t^2P_{n-1}^{(-\frac{1}{2},2)}(2t^2-1). \tag{4.43}$$

Using the famous relation [7]

$$C_{2n}^{\gamma}(x)=(\gamma)_n(\frac{1}{2})_nP_n^{(\gamma-\frac{1}{2},-\frac{1}{2})}(2x^2-1), \tag{4.44}$$

formula (4.43) takes the form

$$E_n^{(2)}(\xi)=\frac{6t^2(-1)^{n-1}}{(2n+1)(2n-1)}C_{n-2}^{\frac{n}{2}}(t). \tag{4.45}$$

Using the relation (see [7])

$$D^mC_n^{\gamma}(x)=2^m(\gamma)_mC_{n-m}^{\gamma+m}(x), \qquad D^m=\frac{d^m}{dx^m}, \quad 0\le m\le n, \tag{4.46}$$

the value of $E_n^{(2)}(\xi)$ becomes

$$E_n^{(2)}(\xi) = \frac{2(-1)^{n-1}(1-y^2)}{(4n^2-1)} \frac{d^2 P_{2n}(y)}{dy^2}. \tag{4.47}$$

To obtain the relation between the second derivatives of Legendre polynomial with zero derivatives, we use the Legendre differential equations [7],

$$(1-y^2)P_{2n}^{''}(y) - 2yP_{2n}^{'}(y) + 2n(2n+1)P_{2n}(y) = 0 \tag{4.48}$$

and

$$(1-y^2)P_{2n}^{'}(y) = (2n+1)[yP_{2n}(y) - P_{2n+1}(y)]. \tag{4.49}$$

Hence Eq. (4.47) becomes

$$E_n^{(2)}(\xi) = \frac{4(-1)^{n-1}}{(2n-1)} \Big[\frac{y^2 P_{2n}(y) - yP_{2n+1}(y)}{(1-y^2)} - 2nP_{2n}(y) \Big]. \tag{4.50}$$

Using the final results of Eqs. (4.40) and (4.50) in (4.36), we obtain

$$C_n(t) = \frac{(2n+1)B_n}{\pi^2\sqrt{1-t^2}} \Big[1 - \frac{(-1)^n\sqrt{\pi}n!}{\Gamma(n+\frac{1}{2})} P_{2n}(\sqrt{1-t^2}) \Big]. \tag{4.51}$$

Introducing the value of δ of Eq. (4.21) and the value of $C_n(t)$ of Eq. (4.51) in (4.7), the formula (4.9) is obtained.

5. Numerical Results

The behavior of the function $\phi_{0m}(r)$ is obtained for the first harmonic $m = 1$ in Figure 1, for the second harmonic $m = 2$ in Figure 2, and for certain values of μ_0, λ, θ, and γ_0. It is seen from these figures that as we move away from origin, oscillations are increasing and there is no damping. In Figure 2, the first zero of the function occurs at larger values of r in comparison with Figure 1.

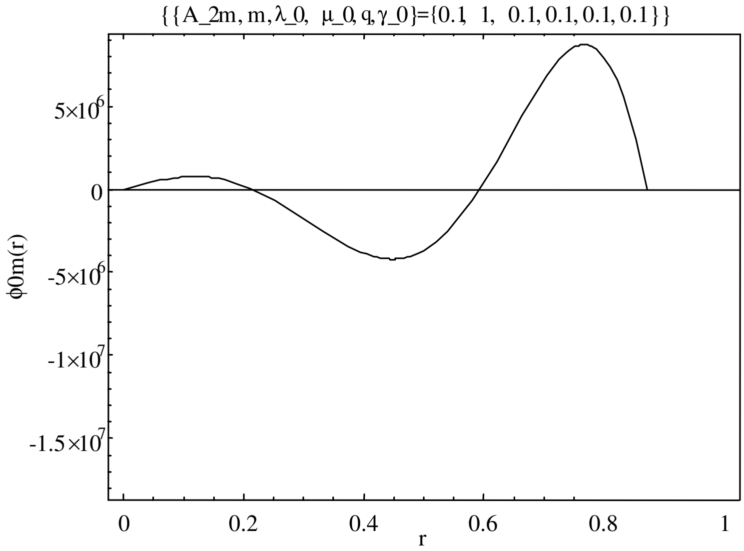

Figure 1. The behavior of the function $\phi_{0m}(r)$ is illustrated for the first harmonic $m = 1$. The parameters are assumed have the values: $\mu_0 = 0.1$, $\lambda = 0.1$, $\theta = 0.1$, and $\gamma_0 = 0.1$.

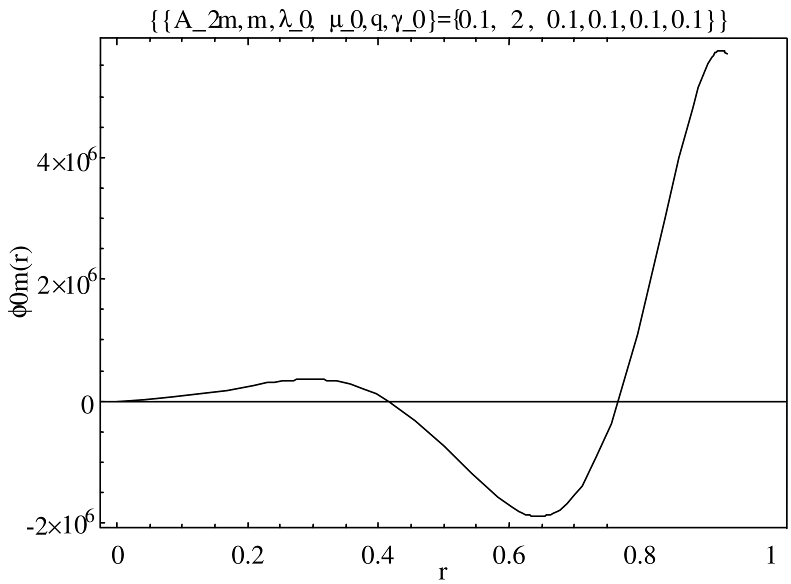

Figure 2. The same of Fig. 1 but $m = 2$.

In Figure 3, the values of the potential function $\psi(x, y, t)$ for the unit values of t are obtained. For $r = 0.1$ the surface has minima and increases as θ increases. The stability of the surface is obtained for higher values of θ.

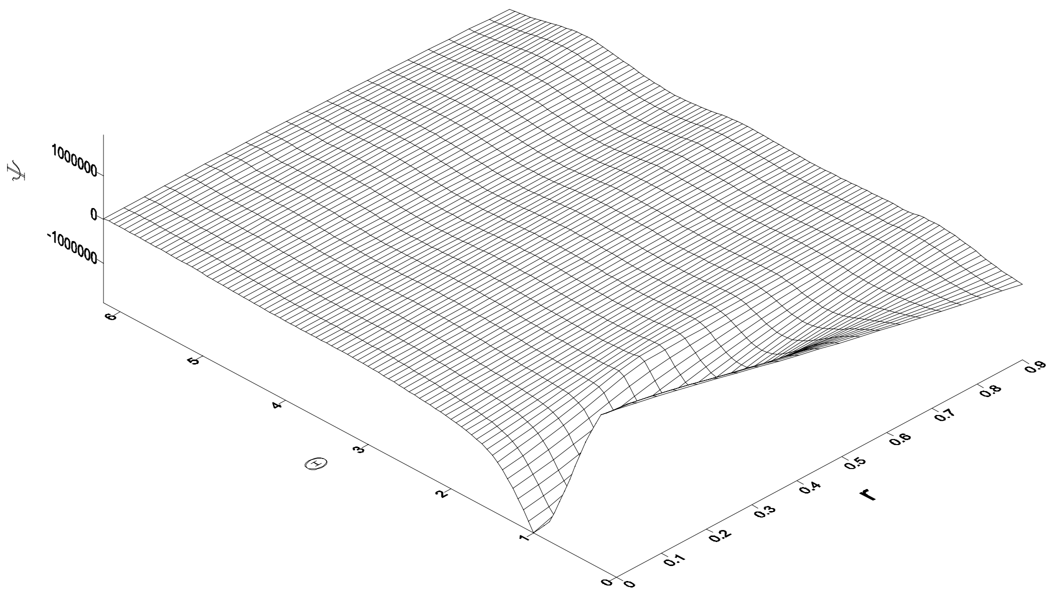

Figure 3. The 3-dimensional plot of potential function $\psi(\theta, r, t)$ for the unit values of t.

6. Conclusions and Remarks

From the above results and discussion the following may be concluded.

(1) In three-dimensional semi-symmetric Hertz contact problem when the impress force is variable with time, $t \in [0, T]$, and when the resistance of the material of contact domain with the external resistance are also variable with time, we get an integral equation of mixed type.

(2) The potential kernel of Fredholm integral term $W(u, v) = 2\pi\sqrt{uv}\int_0^\infty J_m(tu)J_m(tv)dt$, takes the form of a Weber-Sonien integral and its kernel satisfies the form

$$\Big(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\Big)W(u, v) = [h(u) - h(v)]W(u, v), \tag{6.1}$$

where

$$h(x) = (m^2 - \frac{1}{4})x^{-2}.$$

Formula (6.1) represents a nonhomogeneous wave equation.

(3) The elliptic integral form can be obtained as a special case of this work when $m = 0$ in the value of the kernel of Eq. (2.10), as

$$W(x, y) = \frac{1}{\pi(x + y)} E\Big(\frac{2\sqrt{xy}}{(x + y)}\Big). \tag{6.2}$$

Kovalenko [10], developed the Fredholm integral equation of the first kind for the mechanics mixed problem of continuous media and obtained the eigenfunctions of the problem, when the kernel is in the form of elliptic function of the form (6.2).

(4) The integral equation with logarithmic kernel with respect to position is contained as a special case of this work when $m = \pm\frac{1}{2}$ for symmetric and skew symmetric cases, respectively, in Eq. (2.10).

(5) The eigenfunctions for the contact problem of zero harmonic symmetric kernel of the potential function are included when $m = 0$. Also, the eigenfunction for the contact problem of the first and higher order harmonic, $m \geq 1$, is included as a special case.

(6) Krein's method is considered one of the best methods for solving the integral equation of the first kind for contact problems depending on the known function and by avoiding singular point.

Acknowledgements

The authors would like to thank Dr. Hamed El-Mowafy for reading the manuscript.

References

[1] Abdou M. A.: Fredholm integral equation with potential kernel and its structure resolvent, Appl. Math. Comput. **107** 169-180 (2000).

[2] Abdou M. A.: Fredholm-Volterra integral equation of the first kind and contact problem, J. Appl. Math. Comput. **125**, 177-193 (2002).

[3] Abdou M. A. and Abd Al-Kader G. M.: Mixed type of Fredholm-Volterra integral equation, Le Matematiche Vol. **LX**-Fasc.1 41-58 (2005).

[4] Abdou M. A. and Ezzeldin N. Y.: Kreins method with certain singular kernel for solving the integral equation of the first kind, Periodica Math. Hung. **28(2)**, 143-149 (1994).

[5] Atkinson K. E.: *A Survey of Numerical Methods for the Solution of Fredholm Integral Equation of the Second Kind*, SIAM, Philadelphia (1976).

[6] Constanda C.: Integral equation of the first kind in plane elasticity, Quart. Appl. Math. **L111 (4)**,783-793 (1995).

[7] Erdelyi A., Magnus W., Oberhetting F., and Tricomi G.: *Higher Transcendental Functions* , Vol. **2** Mc-Graw Hill, New York (1973).

[8] Frankel J. I.: A Galerkin solution to a regularized Cauchy singular integro-differential equation, Quart. Appl. Math. **53(2)**, 245-258 (1995).

[9] Gradshteyn I. S. and Ryzhik I. M.: *Table of Integrals Series and Products*, fifth edition, Academic Press, New York (1994).

[10] Kovalenko E. V.: Some approximate methods for solving integral equations for mixed problems, J. Appl. Math. Mech. **53(1)**, 85-92 (1989).

[11] Mkhitarian S. M. and Abdou M. A.: On different methods for solving the Fredholm integral equation of the first kind with logarithmic kernel, Dakl. Acad.Nauka, Armenia **90**, 1-10 (1989).

Mohamed Abdella Ahmed ABDOU

Received 15.11.2006

Department of Mathematics, Faculty of Education,
Alexandria University,
Alexandria-EGYPT
Gamal Mohamed ABD AL-KADER
Mathematics Department, Faculty of Science,
Al-Azhar University,
Nasr City 11884, Cairo-EGYPT
e-mail: abdalkaderg@sci-azhar.edu.eg