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A Note on a Problem of J. Galambos

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Abstract

For any $x \in (0,1]$, let

$$x = \frac{1}{d_1} + \frac{a_1}{b_1}\frac{1}{d_2} + \cdots + \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}\frac{1}{d_{n+1}} + \cdots$$

be the Oppenheim series expansion of x. In this paper, we investigate the Hausdorff dimension of the set $B_m = \{x : 1 < d_j/h_{j-1}(d_{j-1}) \le m, j \ge 1\}$ which J. Galambos posed as an open question in 1976(see[6]). In [11], it has been considered with the condition $h_j(d) \to \infty$ as $d \to \infty$. In this note, we give a bound estimation of more general case without the former assumption.

Key Words: Oppenheim series expansion; Restricted Oppenheim series expansion; Lüroth series; Hausdorff dimension.

1. Introduction

Let a_n and b_n, $n \ge 1$, be two sequences of positive integer-valued functions defined on $\mathbf{N}$. The algorithm $0 < x \le 1$, $x = x_1$, and, for any $n \ge 1$, with positive integers $d_n(x)$,

$$\frac{1}{d_n(x)} < x_n \le \frac{1}{d_n(x) - 1}, \quad x_n = \frac{1}{d_n(x)} + \frac{a_n(d_n(x))}{b_n(d_n(x))} \cdot x_{n+1} \tag{1}$$

leads to the series expansion

$$x = \frac{1}{d_1(x)} + \sum_{n=1}^{\infty} \frac{a_1(d_1(x)) \cdots a_n(d_n(x))}{b_1(d_1(x)) \cdots b_n(d_n(x))} \frac{1}{d_{n+1}(x)}, \tag{2}$$

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which is called the Oppenheim series expansion of x. Set

$$h_n(j)=\frac{a_n(j)}{b_n(j)}j(j-1),\ j\geq 2. \tag{3}$$

If $h_n(j)$ is integer-valued ($n\geq 1, j\geq 2$), the formula (2) is termed the restricted Oppenheim series expansion of x. Here and in what follows, we always assume h_n is integer-valued, for all $n\geq 1$.

The algorithm (1) implies

$$d_1(x)\geq 2,\ d_{n+1}(x)\geq h_n(d_n(x))+1 \text{ for any } n\geq 1. \tag{4}$$

On the other hand, any $\{d_n,\ n\geq 1\}$ of integer sequence satisfying (4) is an Oppenheim admissible sequence, that is, there exists a unique $x\in(0,1]$ such that $d_n(x)=d_n$ for any $n\geq 1$. The representation (2) under (1) is unique.

The representation (2) under (1) was first studied by A. Oppenheim [8] who established the arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory were laid down by J. Galambos [3], [4], [5], [7], see also the monographs of J. Galambos [6], F. Schweiger [9] and W. Vervaat [10]. The exceptional sets are considered by J. Wu [12]. From [6], Chapter 6, it can be seen that the integer approximation $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_{n-1})$ defined by

$$T_n(x)<\frac{d_n(x)}{h_{n-1}(d_{n-1}(x))}\leq T_n(x)+1, n\geq 1, \tag{5}$$

where $h_0(x)=1$, play an important role in the metric theory of the Oppenheim expansions. $\{T_n(x), n\geq 1\}$ are stochastically independent and they are distributed as the denominators in the Lűroth expansion. J. Galambos (see [6] Page 132), posed the question to calculate the Hausdorff dimension of the set

$$B_m=\{x\in(0,1): 1\leq T_n(x)\leq m, n\geq 1\}$$

and compare this to Lűroth case. In [11], they solved this problem under the assumption $h_j(d)\geq d-1$, for all $j\geq 1$. In this paper, we consider the general case.

We use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_H$ to denote the Hausdorff dimension and 'cl' the closure of a subset of $(0,1]$, respectively. Hausdorff dimension of B_m In this section, we give the main result of this paper.

We start with a result which proved in [11].

Lemma 1.1 *Suppose $h_n(j) \geq j-1$ for all $n \geq 1$ and $j \geq 2$, then for any $m \geq 3$, the set*

$$C_m = \{x \in (0,1]: \ 1 < \frac{d_j(x)}{h_{j-1}(d_{j-1}(x))} \leq m \text{ for any } j \geq 2\}$$

is of Hausdorff dimension 1.

In fact, [11] shows, under the assumption that $h_j(d) \to \infty$ as $d \to \infty$, B_m is of hausdorff dimension 1, as a continuity of [11]. Here, we consider the case that h_j is bounded.

Now we state our main result.

Let

$$B_m = \{x \in (0,1) : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m, n \geq 1\}, \quad \text{for } m \geq 2 \tag{6}$$

then, we have the following theorem.

Theorem 1.2 *Assume that $l \leq h_j \leq L < +\infty$ holds ultimately, then*

$$\inf_{l \leq a \leq L} S(a) \leq \dim_H B_m \leq \sup_{l \leq a \leq L} S(a), \tag{7}$$

where $S(a)$, for any integer $a \geq 1$, is defined as

$$S(a) : \sum_{a < b \leq ma} (\frac{a}{b(b-1)})^{S(a)} = 1. \tag{8}$$

Proof. Assume that $l \leq h_j \leq L$ for all $k \geq t_0$. We will make use of a kind of symbolic space defined as follows: for any $k \geq 1$, let

$$D_k \quad = \quad \{\sigma = (\sigma_1, \cdots, \sigma_k) \in \mathbf{N}^k, \ 1 < \frac{\sigma_j}{h_{j-1}(\sigma_{j-1})} \leq m \text{ for } 1 \leq j \leq k\},$$

and define

$$D^* = \bigcup_{k=0}^{\infty} D_k \ (D_0 := \emptyset \text{ as usual}).$$

For any $k \geq 1$ and $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$, let I_σ and J_σ denote the following closed subintervals of $(0,1]$, respectively,

$$I_\sigma = \text{cl}\{x \in (0,1], d_1(x) = \sigma_1, \cdots, d_k(x) = \sigma_k\}, \quad J_\sigma = \bigcup_{h_k(d_k) < d \leq m h_k(d_k)} I_{\sigma * d}.$$

For each $\sigma \in D_k$, J_σ is called an k-th order interval. It is obvious that

$$B_m = \bigcap_{k=1}^{+\infty} \bigcup_{\sigma \in D_k} J_\sigma. \tag{9}$$

From the proof of Theorem 6.1 in [6], we have, for any $k \geq 1$, for any $\sigma \in D_k$, I_σ is an interval with the endpoints

$$A_\sigma = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{\sigma_k},$$

$$B_\sigma = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{\sigma_k - 1}. \tag{10}$$

As a result

$$|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{\sigma_k(\sigma_k - 1)}, \tag{11}$$

$$\begin{aligned} |J_\sigma| &= \sum_{h_k(\sigma_k) < d \leq m h_k(\sigma_k)} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \frac{1}{d(d-1)} \\ &= (1 - \frac{1}{m}) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{\sigma_k(\sigma_k - 1)}. \end{aligned} \tag{12}$$

As for the upper bound of $\dim B_m$, notice that for any $k \geq 1$, $\bigcup_{\sigma \in D_k} I_\sigma$ is a natural covering system of B_m. Thus, for any $s > \sup_{l \leq a \leq L} S(a)$, by the definition of $S(a)$, we have

$$\begin{aligned} H^s(B_m) &\leq \liminf_{n \to +\infty} \sum_{\sigma \in D_{n+1}} |I_\sigma|^s \\ &= \liminf_{n \to +\infty} \sum_{\sigma \in D_{n+1}} (\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \frac{1}{\sigma_{k+1}(\sigma_{k+1} - 1)})^s \\ &= \liminf_{n \to +\infty} \sum_{\sigma \in D_n} |I_\sigma|^s \cdot \sum_{h_k(\sigma_k) < \sigma_{k+1} \leq m h_k(\sigma_k)} (\frac{h_k(\sigma_k)}{\sigma_{k+1}(\sigma_{k+1} - 1)})^s \\ &\leq \liminf_{n \to +\infty} \sum_{\sigma \in D_n} |I_\sigma|^s \leq \cdots \leq \sum_{\sigma \in D_{t_0}} |I_\sigma|^s < +\infty. \end{aligned} \tag{13}$$

This indicates $\dim_H B_m \le \sup\limits_{l\le a\le L} S(a)$.

Now we investigate the lower bound.

Since $H^s(E) = H^s_\wp(E)$ in R^1, where $H^s_\wp(E)$ denotes in the evaluation of Hausdorff measure of E, any cover of E is restricted to a collection of open intervals. By (9) B_m is a closed set, then by Heine-Borel Theorem, any open covering system U, consisting of an enumerable number of open intervals, can be replaced by a finite number of open intervals; further, these intervals may be closed by the addition of their endpoints, and finally these intervals may be altered to have their endpoints in B_m, without at any stage destroying the property that Uis a covering system of B_m and increasing $\sum_{U_i\in \mathrm{U}} |U_i|^s$.

Let G be an interval in U, of positive length. G is contained in $I_0 = [0,1]$ and is not contained in an I_σ, $\sigma \in D_k$, for k sufficient large. Therefore there exists a largest value of k, say n, for which G belongs to some I_σ, $\sigma \in D_n$. We see then that there exists numbers $n; \sigma_1, \cdots, \sigma_n; k, j$ with $k \ne j$, such that G is contained in $I_{\sigma_1\cdots\sigma_n}$, and

$$GI_{\sigma_1\cdots\sigma_n k} \ne \emptyset,\ GI_{\sigma_1\cdots\sigma_n j} \ne \emptyset.$$

And for the endpoints of G are in B_m, then

$$GJ_{\sigma_1\cdots\sigma_n k} \ne \emptyset,\ GJ_{\sigma_1\cdots\sigma_n j} \ne \emptyset.$$

Therefore $|G|$ is greater than or equal to the gap between $J_{\sigma_1\cdots\sigma_n k}$ and $J_{\sigma_1\cdots\sigma_n j}$, thus, by (10), we know, assume $k < l$,

$$\begin{aligned} |G| &\ge \frac{1}{m}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_n(\sigma_n)}{b_n(\sigma_n)}\frac{1}{k(k-1)} \ge \frac{1}{m^3}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_n(\sigma_n)}{b_n(\sigma_n)}\frac{1}{h_n^2(\sigma_n)} \\ &\ge \frac{1}{m^3 L}|I_{\sigma_1\cdots\sigma_n}|. \end{aligned}$$

Let Ω be the finite set of intervals I_σ corresponding in the above way to intervals G of U. Of course, $\bigcup_{I\in\Omega} I$ is an covering system of B_m.

Let $K_1 = \max\{k : \sigma \in D_k, I_\sigma \in \Omega\}$, $K_2 = \min\{k : \sigma \in D_k, I_\sigma \in \Omega\}$ and define, $W_{K_1} = \{I_\sigma \in \Omega : \sigma \in D_{K_1}\}$.

By the definition of K_1, we know, if $I_{\sigma_1,\cdots,\sigma_{(K_1-1)}*j} \in W_{K_1}$, then, for all $h_{K_1-1}(\sigma_{K_1-1}) < j \le m h_{K_1-1}(\sigma_{K_1-1})$, $I_{\sigma_1,\cdots,\sigma_{K_1-1}*j} \in W_{K_1}$.

For any $s \leq \inf_{l\leq a\leq L} S(a)$,

$$\begin{aligned}&\sum_{h_{K_1-1}(\sigma_{K_1-1})<j\leq mh_{K_1-1}(\sigma_{K_1-1})} |I_{\sigma_1,\cdots,\sigma_{K_1-1}*j}|^s \\ &= |I_{\sigma_1,\cdots,\sigma_{K_1-1}}|^s \sum_{h_{K_1-1}(\sigma_{K_1-1})<j\leq mh_{K_1-1}(\sigma_{K_1-1})} (\frac{h_{K_1-1}(\sigma_{K_1-1})}{j(j-1)})^s \\ &\geq |I_{\sigma_1,\cdots,\sigma_{K_1-1}}|^s.\end{aligned}$$

The argument above shows that we can decrease the basic interval covering to a new one with lower degree (here, the degree of I_σ is defined as the length of σ as a word), and without increasing the sum. As a result, we can replace the covering Ω by a new covering Ω^* in which all basic interval are of the same order satisfying

$$\sum_{I_\sigma\in\Omega} |I_\sigma|^s \geq \sum_{\sigma\in D_{k_2}} |I_\sigma|^s.$$

At same time, since

$$\begin{aligned}&\sum_{\sigma\in D_{k+1}} |I_\sigma|^s = \sum_{\sigma\in D_k} \sum_{h_k(\sigma_k)<j\leq mh_k(\sigma_k)} |I_{\sigma*j}|^s \\ &\geq \sum_{\sigma\in D_k} |I_\sigma|^s \geq \cdots \geq \sum_{\sigma\in D_{t_0}} |I_\sigma|^s.\end{aligned}$$

Thus

$$\sum_{G\in \mathrm{U}} |G|^s \geq \frac{1}{m^3L}\sum_{\sigma\in\Omega} |I_\sigma|^s \geq \frac{1}{m^3L}\sum_{\sigma\in D_{k_2}} |I_\sigma|^s \geq \frac{1}{m^3L}\sum_{\sigma\in D_{t_0}} |I_\sigma|^s. \tag{14}$$

Since s is arbitrary, we have $\dim_H B_m \geq \inf_{l\leq a\leq L} S(a)$. This finishes the proof. □

Applying Theorem 2.2 to the Lüroth series that $h_j = 1$, we have

Corollary 1.3 *For the Lüroth series expansion, we have*

$$\dim B_m = S(1).$$

Lüroth series expansion also stands as a special case to indicate that as a general assertion, this theorem can not be improved.

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