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# Nonexistence of Stable Exponentially Harmonic Maps from or into Compact Convex Hypersurfaces in $\mathbb{R}^{m+1}$

*Jiancheng Liu*

**Abstract**

In this paper, we study the nonexistence problems for stable exponentially harmonic map into or from compact convex hypersurface $M^m \subset \mathbb{R}^{m+1}$, and show that every nonconstant exponentially harmonic map $f$, between $M^m$ and any compact Riemannian manifold, is unstable if (4) holds.

**Key Words:** Exponentially harmonic map, instability, convex hypersurface.

## 1. Introduction and Main Results

Let $M$, $N$ be compact Riemannian manifolds and $f : M \to N$ be a smooth map. Following J. Eells and L. Lemaire [2], $f$ is *an exponentially harmonic map* if it represents a critical point of the exponentially energy integral

$$\boldsymbol{E}(f) = \int_M \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) *1. \tag{1}$$

The Euler-Lagrange equation of the functional $\boldsymbol{E}(f)$ can be written as

$$-\mathrm{d}^*\left(\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\mathrm{d}f\right) = \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\left(\tau(f) + \mathrm{d}f\left(\nabla\left(\frac{|\mathrm{d}f|^2}{2}\right)\right)\right) = 0, \tag{2}$$

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where $\tau(f) = -\mathrm{trace}\nabla \mathrm{d}f = \mathrm{d}^*\mathrm{d}f$ is the tension field along $f$. Hence, if the energy density $|\mathrm{d}f|^2$ is constant, every harmonic map is exponentially harmonic and vice versa.

From the viewpoint of variational calculus, it is natural to study the stability of exponentially harmonic mapping (as well as harmonic or $p$-harmonic mapping), i.e. the exponentially harmonic mapping with non-negative second variation.

For harmonic mapping, Y. L. Xin [6] and P. F. Leung [4] proved the nonexistence of nonconstant stable harmonic mapping between Euclidean sphere $S^m (m > 2)$ and any compact Riemannian manifolds, then H. Takeuchi [5] generalized those results to the case of $p$-harmonic mapping.

Coming into the case of exponentially harmonic mapping, S. E. Koh [3] proved that: *every nonconstant exponentially harmonic map* $f : M \to S^m$*, from compact Riemannian manifold* $M$ *into the standard unit sphere* $S^m$ *in* $\mathbb{R}^{m+1}$*, is unstable if* $|df|^2(x) < m-2$ *for every* $x \in M$*.* Unfortunately, that is uncorrect. For example, let $f : (S^m, 2g_0) \to (S^m, g_0)$ be identity mapping, it is easy to see that $|\mathrm{d}f|^2 = \frac{m}{2} < m - 2$ for $m > 4$. On the other hand, using the same argument as Theorem 7.1 in [1], we know $f$ is a stable exponentially harmonic mapping. That mistake comes from the uncorrected second variation formula. In details, S. E. Koh [3] defined the exponentially energy functional as

$$\int_M \exp(|\mathrm{d}f|^2) *1, \tag{3}$$

using the the corresponding second variation formula

$$\int_M \exp(|\mathrm{d}f|^2)\big\{\langle\widetilde{\nabla}V, \widetilde{\nabla}W\rangle - \sum_i \langle R^{S^m}(V, f_*e_i)f_*e_i, W\rangle + \langle\widetilde{\nabla}V, \mathrm{d}f\rangle\langle\widetilde{\nabla}W, \mathrm{d}f\rangle\big\} *1.$$

In fact, the correct second variation formula corresponding to the exponentially energy functional defined as (3) must be

$$2\int_M \exp(|\mathrm{d}f|^2)\big\{\langle\widetilde{\nabla}V, \widetilde{\nabla}W\rangle - \sum_i \langle R^{S^m}(V, f_*e_i)f_*e_i, W\rangle + 2\langle\widetilde{\nabla}V, \mathrm{d}f\rangle\langle\widetilde{\nabla}W, \mathrm{d}f\rangle\big\} *1.$$

Hence, when we replace the condition $|\mathrm{d}f|^2 < m - 2$ in [3] by $|\mathrm{d}f|^2 < \frac{m}{2} - 1$, the result is also true.

However, in this paper, when the source manifold or the target manifold is the compact convex hypersurface in $\mathbb{R}^{m+1}$, we can prove the following more general results.

**Proposition 1** *Let $M^m \subset \mathbb{R}^{m+1}$ be the compact convex hypersurface, $m > 2$. Its principal curvatures sorted as $0 < \lambda_1 \leq \cdots \leq \lambda_m$, satisfying $\lambda_m < \sum_{j=1}^{m-1} \lambda_j$. If $f : N \to M^m$ is a stable exponentially harmonic map, and*

$$|df|^2 < \frac{1}{\lambda_m^2} \min_{1 \leq i \leq m} \{\lambda_i(\sum_{j=1}^{m} \lambda_j - 2\lambda_i)\}. \tag{4}$$

*then $f$ must be constant.*

**Proposition 2** *With the same assumptions on $M^m$ as in Proposition 1. If $f : M^m \to N$ is a stable exponentially harmonic map satisfying (4). Then $f$ must be constant.*

Combining Proposition 1 and Proposition 2, we have the following main result.

**Theorem 1** *With the same assumptions on $M^m$ as in Proposition 1, then every nonconstant exponentially harmonic map $f$, from $M^m$ into any compact Riemannian manifold $N$, or from any compact Riemannian manifold $N$ into $M^m$, must be unstable if (4) holds.*

In particular, when $M^m = S^m$, the standard unit $m$-sphere, then the condition (4) becomes

$$|\mathrm{d}f|^2 < m - 2. \tag{5}$$

In this case, we have the following

**Corollary 1** *Every nonconstant exponentially harmonic map $f$ between $S^m$ and any compact Riemannian manifold $N$ is unstable if $f$ satisfies (5).*

*Remark 1* The condition $|\mathrm{d}f|^2 < m-2$ is necessary. For example, let $f : (S^m, g_0) \to (S^m, g_0)$ be identity mapping, it is well-known that $f$ is stable exponentially harmonic mapping [2]. However, at that time, $|\mathrm{d}f|^2 = m > m - 2$.

## 2. Proof of the Proposition 1

Throughout this paper, we shall assume that $M^m$ is a compact convex hypersurface in Euclidean space $\mathbb{R}^{m+1}$ and $N$ is compact Riemannian manifold without boundary. $\nabla$, $\overline{\nabla}$, $\nabla^R$ represents the Riemannian connections of $M^m$, $N$ and $\mathbb{R}^{m+1}$, respectively. Choose an orthonormal basis $\{e_i, e_{m+1}\}$, $i = 1, \cdots, m$, in $\mathbb{R}^{m+1}$ such that, restricted to $M^m$, $\{e_i\}$

are tangent to $M^m$ and $e_{m+1}$ is normal to $M^m$. Also, denoted by $h(X,Y) = \nabla^R_X Y - \nabla_X Y$ the second fundamental form of $M^m$ in $\mathbb{R}^{m+1}$ with $h_{ij}$ its components.

In this section, we will prove the Proposition 1, which includes the result in [3] as a special case.

**Proof** *of the Proposition 1* We denote $T^\perp M$ the normal bundle over $M^m$ in $\mathbb{R}^{m+1}$ and $\nabla^\perp$ the normal connection on $T^\perp M$. Then, for $X, Y \in$ X(M) and $\eta \in \Gamma(T^\perp M)$ (the space of sections of the normal bundle $T^\perp M$ of $M$), we have

$$A_\eta X = \nabla^R_X \eta - \nabla^\perp_X \eta,$$

where $A_\eta$ is the Weingarten map corresponding to the normal section $\eta$ which satisfies

$$\langle A_\eta X, Y\rangle = \langle h(X,Y), \eta\rangle.$$

Let $V \in \mathbb{R}^{m+1}$ be a parallel vector field, we decompose $V = V^\top + V^\perp$, where $V^\top$ is the tangential part to $M$ and $V^\perp$ is the normal part to $M$. Then we have a 1-parameter family of mappings $\psi_t : M^m \to M^m$ generated by $V^\top$.

Now, we consider the 1-parametric variation $\psi_t \circ f : N \to M^m$ of the exponentially harmonic map $f : N \to M^m$. Then, for a local orthonormal frame field $\{v_i\}$ on $N$, we have

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \boldsymbol{E}(\psi_t \circ f) = \int_N \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \cdot Q(V,V) \ast 1,$$

where

$$\begin{aligned} Q(V,V) = & \sum_i \langle A_{V^\perp} f_* v_i, f_* v_i\rangle^2 + 2\sum_i \langle A^2_{V^\perp} f_* v_i, f_* v_i\rangle \\ & + \sum_i \langle (\nabla_{V^\perp} A)_{V^\perp} f_* v_i, f_* v_i\rangle - \sum_i \langle A_{h(V^\perp, V^\perp)} f_* v_i, f_* v_i\rangle. \end{aligned}$$

In order to compute the trace of $Q(V,V)$ pointwise, for any $x \in M$, we choose local orthonormal frame field $\{e_1, \cdots, e_{m+1}\}$ such that $\{e_1, \cdots, e_m\}$ tangent to $M$ and $e_{m+1}$

normal to $M$. Then we have

$$\begin{aligned}\operatorname{trace} Q(\cdot,\cdot)(x) &= \sum_i \langle A_{e_{m+1}} f_* v_i, f_* v_i\rangle^2 + \sum_i \langle A^2_{e_{m+1}} f_* v_i, f_* v_i\rangle \\ &\quad - \sum_i \sum_{j=1}^m \langle A_{h(e_j,e_j)} f_* v_i, f_* v_i\rangle \\ &= \sum_i \langle A_{e_{m+1}} f_* v_i, f_* v_i\rangle^2 \\ &\quad + \sum_{j=1}^m \sum_i \langle (2A^2_{e_{m+1}} - \operatorname{trace}(A_{e_{m+1}}) A_{e_{m+1}}) f_* v_i, f_* v_i\rangle.\end{aligned}$$

this completes the proof of Proposition 1. □

## 3. Proof of the Proposition 2

Let $f : M^m \to N$ be an exponentially harmonic map Given two-parameters variations $f_{s,t}$, such that $V = \frac{\partial f_{s,t}}{\partial t}\Big|_{s=t=0}$, $W = \frac{\partial f_{s,t}}{\partial s}\Big|_{s=t=0}$. Then we have the second variation formula

$$\begin{aligned}\frac{\partial^2 \boldsymbol{E}(f_{s,t})}{\partial s \partial t}\Big|_{s=t=0} = \int_M \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \Big\{ &\langle \widetilde{\nabla} V, \widetilde{\nabla} W\rangle - \sum_{i=1}^m \langle R^N(V, f_* e_i) f_* e_i, W\rangle \\ &+ \langle \widetilde{\nabla} V, \mathrm{d}f\rangle \langle \widetilde{\nabla} W, \mathrm{d}f\rangle \Big\} *1,\end{aligned} \tag{6}$$

where, $R^N$ is the curvature tensor of $N$: $R^N(X,Y) = [\overline{\nabla}_X, \overline{\nabla}_Y] - \overline{\nabla}_{[X,Y]}$ for vector fields $X, Y$ on $N$, and $\widetilde{\nabla}$ the induced connection on $f^{-1}TN$ defined by $\widetilde{\nabla}_Z \sigma = \overline{\nabla}_{f_* Z} \sigma$ for tangent vector $Z$ of $M^m$ and the section $\sigma$ of $f^{-1}TN$.

We put

$$I_f(V, W) = \frac{\partial^2 \boldsymbol{E}(f_{s,t})}{\partial s \partial t}\Big|_{s=t=0}. \tag{7}$$

An exponentially harmonic map $f$ is called *stable* if $I_f(V,V) \geq 0$ for any $V \in \Gamma(f^{-1}TN)$.

In order to prove the instability of $f : M^m \to N$, we need to consider some special variational vector fields along $f$. To do this, taking a fixed orthonormal basis $E_A, A =$

$1, \cdots, m+1$ of $\mathbb{R}^{m+1}$, and setting

$$V_A = \sum_{i=1}^{m} v_A^i e_i, \quad v_A^i = \langle E_A, e_i \rangle, \quad v_A^{m+1} = \langle E_A, e_{m+1} \rangle,$$

then $f_* V_A \in \Gamma(f^{-1}TN)$. In the following, we shall use this variational vector field to prove the instability of nonconstant exponentially harmonic map $f$.

Before going to prove the Proposition 2, the following basic facts will be needed:

(1) $\sum\limits_A v_A^i v_A^j = \sum\limits_A \langle E_A, e_i \rangle \langle E_A, e_j \rangle = \delta_{ij}.$ (8)

(2) With respect to the frame field $\{e_i, e_{m+1}\}$ of $\mathbb{R}^{m+1}$, let $\{\omega_i, \omega_{m+1}\}$ be a field of dual frames. Then, we have from the structure equations of $\mathbb{R}^{m+1}$,

$$\mathrm{d}e_i = \sum_{j=1}^{m} (\omega_{ij} e_j + h_{ij} \omega_j e_{m+1}). \tag{9}$$

Furthermore, taking covariant derivative of $v_A^i$ and using (9), we obtain

$$\begin{aligned} \nabla_{e_i} V_A &= \sum (\nabla_{e_i} v_A^j) e_j + v_A^j \nabla_{e_i} e_j \\ &= \sum \mathrm{d}\langle E_A, e_i \rangle e_i = v_A^{m+1} h_{ij} e_j. \end{aligned} \tag{10}$$

(3) $\nabla_{e_i}(\nabla_{e_i} V_A) = -v_A^k h_{ik} h_{ij} e_j + v_A^{m+1} (\nabla_{e_i} h_{ij}) e_j.$ (11)

(4)

$$\begin{aligned} \widetilde{\nabla}_{e_i}(f_*(\nabla_{e_i} V_A)) = &- v_A^k h_{ik} h_{ij} f_* e_j + v_A^{m+1} (\widetilde{\nabla}_{e_i} h_{ij}) f_* e_j \\ &+ v_A^{m+1} h_{ij} \widetilde{\nabla}_{f_* e_i} f_* e_j. \end{aligned} \tag{12}$$

**Proof.** *(of Proposition 2)* Suppose that $f : M^m \to N$ is a nonconstant exponentially harmonic map. Then exponentially harmonicity condition $\mathrm{d}^*\big(\exp\big(\frac{|\mathrm{d}f|^2}{2}\big)\mathrm{d}f\big) = 0$ implies that

$$
\begin{aligned}
&\sum_A \int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \langle \triangle f_* V_A, f_* V_A\rangle *1 \\
=&\sum_A \int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) v_A^i v_A^j \langle \triangle f_* e_i, f_* e_j\rangle *1 \\
=&\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \langle \triangle f_* e_i, f_* e_i\rangle *1 \\
=&\int_{M^m} \left\langle \mathrm{d}^*\mathrm{d}f, \mathrm{d}^*\left(\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\mathrm{d}f\right)\right\rangle *1.
\end{aligned}
\tag{13}
$$

It follows from Weitzenböck formula that

$$
-R^N(f_*V_A, f_*e_i)f_*e_i + f_*\mathrm{Ric}^{M^m}(V_A) = \triangle f_*V_A + \widetilde{\nabla}^2 f_*V_A \tag{14}
$$

with respect to the variational vector fields $f_*V_A$ along $f$. Using (6), (13) and (14), we compute the relation

$$
\begin{aligned}
\sum_A I_f(f_*V_A, f_*V_A) =&\sum_A \int_M \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \big\{ |\widetilde{\nabla} f_*V_A|^2 + \langle \widetilde{\nabla} f_*V_A, \mathrm{d}f\rangle^2 \\
&+ \langle \widetilde{\nabla}^2 f_*V_A, f_*V_A\rangle - \langle f_*\mathrm{Ric}^{M^m}(V_A), f_*V_A\rangle \big\} *1,
\end{aligned}
\tag{15}
$$

where $\widetilde{\nabla}^2 f_*V_A = \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} f_*V_A - \widetilde{\nabla}_{\nabla_{e_i}e_i} f_*V_A$. For any fixed point $P \in M$, choose $\{e_i\}$ such that $\nabla_{e_i}e_i|_P = 0$. Then

$$
\widetilde{\nabla}^2 f_*V_A = \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} f_*V_A - 2\widetilde{\nabla}_{e_i}(f_*(\nabla_{e_i}V_A)) + f_*(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}V_A), \tag{16}
$$

and

$$
\begin{aligned}
&\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) \langle \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} f_*V_A, f_*V_A\rangle *1 \\
=&-\int_{M^m} \left\langle \widetilde{\nabla}_{e_i} f_*V_A, \widetilde{\nabla}_{e_i}\left(\exp\left(\frac{|\mathrm{d}f|^2}{2}\right) f_*V_A\right)\right\rangle *1 \\
=&-\int_{M^m} \left\langle \widetilde{\nabla}_{e_i} f_*V_A, \widetilde{\nabla}_{e_i}\left(\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\right) f_*V_A\right\rangle *1 \\
&-\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right) |\widetilde{\nabla} f_*V_A|^2 *1.
\end{aligned}
\tag{17}
$$

Substituting (16) and (17) into (15), we get therefore

$$
\begin{aligned}
\sum_A I_f(f_*V_A, f_*V_A) = &\sum_A \int_{M^m} \Big\{\exp\Big(\frac{|\mathrm{d}f|^2}{2}\Big)\langle\widetilde{\nabla} f_*V_A, \mathrm{d}f\rangle^2 \\
&- \Big\langle \widetilde{\nabla}_{e_i} f_* V_A, \widetilde{\nabla}_{e_i}\Big(\exp\Big(\frac{|\mathrm{d}f|^2}{2}\Big)\Big) f_*V_A\Big\rangle\Big\} *1 \\
&+ \sum_A \int_{M^m} \exp\Big(\frac{|\mathrm{d}f|^2}{2}\Big)\Big\{\Big\langle -2\widetilde{\nabla}_{e_i}(f_*(\nabla_{e_i}V_A)) \\
&+ f_*(\nabla_{e_i}\nabla_{e_i}V_A) - f_*\mathrm{Ric}^{M^m}(V_A),\ f_*V_A\Big\rangle\Big\} *1 \\
:= &\int_{M^m} \{(\mathrm{I}) + (\mathrm{II})\} *1.
\end{aligned} \tag{18}
$$

In the following, we shall estimate the two parts (I) and (II) on the right hand side of (18), separately. Because trace is independent of the choice of orthonormal basis, we can choose $\{e_i, e_{m+1}\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. It follows from Gauss formula that

$$
\mathrm{Ric}^{M^m} = v_A^i(h_{kk}h_{ij} - h_{il}h_{jl})e_j. \tag{19}
$$

Using (10), (11), (12) and (19), we can easily obtain

$$
\begin{aligned}
\int_{M^m} (\mathrm{II}) *1 = &\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\sum_A \Big\{\big\langle 2v_A^k h_{ik}h_{ij} f_*e_j - v_A^i h_{kk}h_{ij}f_*e_j,\ v_A^l f_*e_l\big\rangle \\
&- \big\langle v_A^{m+1}(\nabla_{e_i}h_{ij})f_*e_j + 2v_A^{m+1}h_{ij}\widetilde{\nabla}_{f_*e_i}f_*e_j,\ v_A^l f_*e_l\big\rangle\Big\} *1 \\
= &\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\langle 2h_{il}h_{ij}f_*e_j - h_{kk}h_{lj}f_*e_j, f_*e_l\rangle *1 \\
= &\int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\left(2\lambda_i - \sum_{k=1}^m \lambda_k\right)\lambda_i|\mathrm{d}f|^2 *1.
\end{aligned} \tag{20}
$$

In order to estimate part (I) in (18), a straightforward computation then shows

$$\begin{aligned}
&\sum_A \left\langle \widetilde{\nabla}_{e_i} f_* V_A, \widetilde{\nabla}_{e_i}\left(\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\right) f_* V_A\right\rangle \\
=&\sum_A \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\widetilde{\nabla}_{e_i}\left(\frac{|\mathrm{d}f|^2}{2}\right)\left\langle v_A^{m+1} h_{ik} f_* e_k + v_A^k \widetilde{\nabla}_{f_* e_i} f_* e_k,\ v_A^j f_* e_j\right\rangle \\
=&\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\left\langle \widetilde{\nabla}_{e_i}\mathrm{d}f,\ \mathrm{d}f\right\rangle^2,
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\sum_A \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\langle \widetilde{\nabla} f_* V_A,\ \mathrm{d}f\rangle^2 \\
=&\sum_A \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\langle v_A^{m+1} h_{ik} f_* e_k + v_A^k \widetilde{\nabla}_{f_* e_i} f_* e_k,\ f_* e_i\rangle^2 \\
=&\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\left\{h_{ik}h_{jl}\langle f_* e_k, f_* e_i\rangle\langle f_* e_l, f_* e_j\rangle + 2\langle \overline{\nabla}_{f_* e_i} f_* e_k, f_* e_i\rangle\langle \overline{\nabla}_{f_* e_j} f_* e_k, f_* e_j\rangle\right\} \\
=&\exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\left\{\lambda_i\lambda_j\langle f_* e_i, f_* e_i\rangle\langle f_* e_j, f_* e_j\rangle + \langle \widetilde{\nabla}_{e_i}\mathrm{d}f,\ \mathrm{d}f\rangle^2\right\}.
\end{aligned} \tag{22}$$

Then, it follows from (21) and (22) that

$$\begin{aligned}
\int_{M^m} (\mathrm{I}) *1 &= \int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\lambda_i\lambda_j\langle f_* e_i, f_* e_i\rangle\langle f_* e_j, f_* e_j\rangle *1 \\
&\leq \int_{M^m} \exp\left(\frac{|\mathrm{d}f|^2}{2}\right)\lambda_m^2|\mathrm{d}f|^4 *1.
\end{aligned} \tag{23}$$

Finally, substituting (20), (23) into (18), we get

$$\sum_A I(f_* V_A, f_* V_A) \leq \int_{M^m} \exp\left(\tfrac{|\mathrm{d}f|^2}{2}\right)|\mathrm{d}f|^2\left\{\lambda_m^2|\mathrm{d}f|^2 + \left(2\lambda_i - \sum_{j=1}^m \lambda_j\right)\lambda_i\right\} *1, \tag{24}$$

which implies that $\sum_A I_f(f_* V_A, f_* V_A) < 0$ if $f$ is nonconstant and satisfying (4). Thus, there exists at least one $V_0 \in \{V_1, \cdots, V_{m+1}\}$ such that

$$I_f(f_* V_0, f_* V_0) < 0.$$

That is, a nonconstant exponentially harmonic map $f$ is unstable if (4) holds. This completes the proof of Proposition 2. □

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Jiancheng LIU
Department of Mathematics,
Northwest Normal University,
Lanzhou 730070, CHINA
e-mail: liujc@nwnu.edu.cn

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