Turk J Math
32 (2008) , 197 – 211.

Posner's Second Theorem and an Annihilator Condition with Generalized Derivations

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Abstract

Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, $\delta \neq 0$ a non-zero generalized derivation of R, $f(x_1,..,x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $a \in R$ such that $a[\delta(f(r_1,..,r_n)), f(r_1,..,r_n)] = 0$, for any $r_1,..,r_n \in R$. Then one of the following holds:

1. $a = 0$;
2. there exists $\lambda \in C$ such that $\delta(x) = \lambda x$, for all $x \in R$;
3. there exist $q \in U$ and $\lambda \in C$ such that $\delta(x) = (q+\lambda)x + xq$, for all $x \in R$, and $f(x_1,..,x_n)^2$ is central valued on R.

Key Words: Prime rings, derivations, left Utumi quotient rings, two-sided Martindale quotient ring, differential identities.

1. Introduction

The well known theorem of Posner in [18] asserts that if R is a prime ring and d a non-zero derivation of R such that $[d(x), x] \in Z(R)$, the center of R, for all $x \in R$, then R must be commutative. Starting from this result, several authors studied the relationship between the structure of a prime ring R and the behaviour of an additive mapping f which satisfies the Engel-type condition $[f(x), x]_k = 0$ which, for $k > 1$, is defined by $[f(x), x]_k = [[f(x), x]_{k-1}, x]$. In [10] Lanski shows that if d is a derivation of R such that $[d(x), x]_k = 0$, for all x in a Lie ideal L of R, then either L is central

2000 *AMS Mathematics Subject Classification:* 16N60, 16W25.

in R or $char(R) = 2$ and R satisfies the standard identity $s_4(x_1, .., x_4)$ of degree 4. On the other hand, in a prime ring R of characteristic different from 2, any non-central Lie ideal contains the set $\{[x_1, x_2] : x_1, x_2 \in I\}$ of all evaluations of the polynomial $[x_1, x_2]$ in a two-sided ideal I of R. For this reason, many researchers in this area analysed in detail the case when the Lie ideal is replaced by the set of all evaluations of a polynomial $f(x_1, .., x_n)$ and $[d(f(x_1, .., x_n)), f(x_1, .., x_n)]_k$ is a differential identity for some ideal of R. In particular we refer the reader to the results obtained by P. H. Lee and T. K. Lee in [12] and [13]. In case $f(x_1, .., x_n)$ is a multilinear polynomial, they prove that it must be central-valued in R unless $char(R) = 2$ and R satisfies $s_4(x_1, .., x_4)$. In a recent paper, we consider another related generalization; more precisely in [4] we describe what happens if the derivation d is replaced by an additive mapping δ defined as follows: for all $x, y \in R, \delta(xy) = \delta(x)y + xg(y)$, for some derivation g of R. Such a mapping δ is called a generalized derivation. Obviously any derivation is a generalized derivation. We like to remark that one of the leading roles in the development of the theory of generalized derivations is played by the maps defined as $\delta(x) = bx + xc$ for $b, c \in R$; in this case δ is called inner generalized derivation.

In the light of these definitions, in [4] we prove that if I is a right ideal of R, U the two-sided Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R and $[\delta(f(r_1, .., r_n)), f(r_1, .., r_n)] = 0$, for any $r_1, .., r_n \in I$, then either $\delta(x) = ax$, with $(a-\gamma)I = 0$ and suitable $a \in U$, $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds: (i) $f(x_1, .., x_n)$ is central valued in $eRCe$; (ii) $\delta(x) = cx + xb$, where $(c+b+\alpha)e = 0$, for $b, c \in U$, $\alpha \in C$, and $f(x_1, .., x_n)^2$ is central valued in $eRCe$; (iii) $char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.

In particular the following result holds:

Theorem A *Let R be a prime ring with extended centroid C, U its two-sided Utumi quotient ring, $\delta \neq 0$ a non-zero generalized derivation of R, $f(x_1, .., x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, such that $[\delta(f(r_1, .., r_n)), f(r_1, .., r_n)] = 0$, for all $r_1, .., r_n \in R$. Then one of the following holds:*

1. *$f(x_1, .., x_n)$ is central valued in R;*
2. *there exists $\lambda \in C$ such that $\delta(x) = \lambda x$, for all $x \in R$;*
3. *there exist $q \in U$ and $\lambda \in C$ such that $\delta(x) = (q + \lambda)x + xq$, for all $x \in R$, and $f(x_1, .., x_n)^2$ is central valued on R;*
4. *$char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for R.*

Here we will continue the study of the set

$$S = \{[\delta(f(x_1,..,x_n)), f(x_1,..,x_n)], x_1,..,x_n \in R\}$$

for a generalized derivation δ of R. An approach that can be used in studying S is to examine its size and a reasonable criteria for studying the size of S is to examine its left annihilator $Ann_R(S) = \{x \in R, \quad xs = 0, \forall s \in S\}$: if S is large we would expect that $Ann_R(S) = 0$. In fact we will prove this theorem:

Theorem 1 *Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, $\delta \neq 0$ a non-zero generalized derivation of R, $f(x_1,..,x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $a \in R$ such that $a[\delta(f(r_1,..,r_n)), f(r_1,..,r_n)] = 0$, for all $r_1,..,r_n \in R$. Then one of the following holds:*

1. *$a = 0$;*
2. *there exists $\lambda \in C$ such that $\delta(x) = \lambda x$, for all $x \in R$;*
3. *there exist $q \in U$ and $\lambda \in C$ such that $\delta(x) = (q + \lambda)x + xq$, for all $x \in R$, and $f(x_1,..,x_n)^2$ is central valued on R.*

Of course we do not consider the case when R is a domain; in fact, in this case, either $Ann_R(S) = 0$ or $[\delta(f(r_1,..,r_n)), f(r_1,..,r_n)] = 0$, for all $r_1,..,r_n \in R$. In this condition we conclude by Theorem A.

We also would like to remark that in case δ is an usual derivation of R, the conclusion of Theorem 1 follows directly from the following result we proved in [6]:

Theorem B *Let R be a prime ring of characteristic different from 2 with extended centroid C, U its two-sided Utumi quotient ring, $d \neq 0$ a non-zero derivation of R, $f(x_1,..,x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $a \in R$ such that $a[d(f(r_1,..,r_n)), f(r_1,..,r_n)] = 0$, for any $r_1,..,r_n \in R$. Then $a = 0$.*

In [14], T. K. Lee extended the definition of a generalized derivation to the Utumi quotient ring U of R as follows: by a generalized derivation we mean an additive mapping $\delta : I \to U$ such that $\delta(xy) = \delta(x)y + xd(y)$, for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U.

In all that follows let U be the two-sided Utumi quotient ring of R and $C = Z(U)$

the center of U, $T = U *_C C\{X\}$ the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, .., x_n, \ldots$. We refer the reader to [1] for the definitions and the related properties of these objects.
Of course U is a prime ring and, by replacing R by U, we may assume, without loss of generality, $R = U$, $C = Z(R)$ and R is a C-algebra centrally closed. Moreover we will use the following notation:

$$f(x_1, .., x_n) = x_1 x_2 .. x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$. We also assume $char(R) \neq 2$ and $f(x_1, ..., x_n)$ non-central valued.

2. The case of inner generalized derivations

In this section we consider the case when δ is an inner generalized derivation induced by the elements $b, c \in R$, that is $\delta(x) = bx + xc$, for all $x \in R$. In this sense, our aim will be to prove the following:

Proposition 1 *Let R be a prime ring of characteristic different from 2, $f(x_1, .., x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $a, b, c \in R$.*
If $a[bf(r_1, .., r_n) + f(r_1, .., r_n)c, f(r_1, .., r_n)] = 0$, for all $r_1, .., r_n \in R$, then one of the following holds:

1. *$a = 0$;*
2. *$b, c \in C$;*
3. *there exists $\lambda \in C$ such that $c - b = \lambda$ and $f(x_1, .., x_n)^2$ is central valued on R.*

2.1. The matrix case

By first we will study the case when $R = M_m(F)$ is the algebra of $m \times m$ matrices over a field F of characteristic different from 2. Notice that the set $f(R) = \{f(r_1, .., r_n) : r_1, .., r_n \in R\}$ is invariant under the action of all inner automorphisms of R. Hence if denote we $r = (r_1, .., r_n) \in R \times R \times R \times ... \times R = R^n$, then for any inner automorphism φ of $M_m(F)$, we have that $\underline{r} = (\varphi(r_1), .., \varphi(r_n)) \in R^n$ and $\varphi(f(r)) = f(\underline{r}) \in f(R)$. Let us denote as usual by e_{ij} the matrix unit with 1 in (i, j)-entry and zero elsewhere.

Since $f(x_1,..,x_n)$ is not central then, by [16], there exist $u_1,..,u_n \in M_t(F)$ and $\alpha \in F-\{0\}$, such that $f(u_1,..,u_n)=\alpha e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(v_1,..,v_n) : v_1,..,v_n \in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i \neq j$ there exist $r_1,..,r_n \in M_t(F)$ such that $f(r_1,..,r_n)=\alpha e_{ij}$.

We start by studying the case of 2×2 matrices:

Lemma 1 *Let $R=M_2(F)$ be the algebra of 2×2 matrices over the field F of characteristic different from 2. If there exist $a,b,c \in R$ such that $a[bu+uc,u]=0$ for any $u \in f(R)$ then either $a=0$ or $c-b$ is a diagonal matrix.*

Proof. Suppose $a \neq 0$. Let $u=\alpha e_{ij} \in f(R)$, for any $i \neq j$ and $\alpha \neq 0$:

$$0=a[b\alpha e_{ij}+\alpha e_{ij}c, \alpha e_{ij}] = -\alpha^2 a e_{ij}(c-b)e_{ij}.$$

In other words, either the i-th column of the matrix a is zero or, for all j different from i, the (j,i)-entry γ_{ji} of $(c-b)$ is zero. Suppose that $(c-b)$ is not a diagonal matrix, say $\gamma_{12} \neq 0$. In this case, as we said above, the 2-nd column of a is zero. Of course we may assume $\gamma_{21}=0$, otherwise the first column of a is zero too, and we are done. In other words we are in the situation:

$$c-b=\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \quad \gamma_{12}\neq 0, \qquad a=\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}.$$

Let φ be an element of $Aut(M_2(F))$, defined as $\varphi(x)=(1-e_{12})x(1+e_{12})$. Now choose $v=(1-e_{12})u(1+e_{12})=\begin{bmatrix} -\alpha & -\alpha \\ \alpha & \alpha \end{bmatrix} \in f(R)$. Hence

$$0=a[bv+vc,v]=-\alpha^2\begin{bmatrix} a_{11}(\gamma_{11}-\gamma_{12}-\gamma_{22}) & a_{11}(\gamma_{11}-\gamma_{12}-\gamma_{22}) \\ a_{21}(\gamma_{11}-\gamma_{12}-\gamma_{22}) & a_{21}(\gamma_{11}-\gamma_{12}-\gamma_{22}) \end{bmatrix},$$

which implies

$$\gamma_{11}-\gamma_{12}-\gamma_{22}=0 \qquad (1).$$

Let now χ be another element of $Aut(M_2(F))$, defined as $\chi(x)=(1+e_{12})x(1-e_{12})$ and choose $w=(1+e_{12})u(1-e_{12})=\begin{bmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{bmatrix} \in f(R)$. Hence we have

$$0=a[bw+wc,w]=\alpha^2\begin{bmatrix} a_{11}(\gamma_{11}+\gamma_{12}-\gamma_{22}) & a_{11}(-\gamma_{11}-\gamma_{12}+\gamma_{22}) \\ a_{21}(\gamma_{11}+\gamma_{12}-\gamma_{22}) & a_{21}(-\gamma_{11}-\gamma_{12}+\gamma_{22}) \end{bmatrix},$$

that is

$$\gamma_{11} + \gamma_{12} - \gamma_{22} = 0 \qquad (2).$$

By equations (1) and (2) and since $char(R) \neq 2$, it follows $\gamma_{12} = 0$, a contradiction. Therefore, if $a \neq 0$, $(c - b)$ is a diagonal matrix. □

Lemma 2 *Let $R = M_t(F)$ be the algebra of $t \times t$ matrices over the field F of characteristic different from 2 and $t \geq 3$. If there exist $a, b, c \in R$ such that $a[bu + uc, u] = 0$ for any $u \in f(R)$ then either $a = 0$ or $c - b$ is a diagonal matrix.*

Proof. As above, for any $i \neq j$ there exist $r_1, .., r_n \in M_t(F)$ such that $f(r_1, .., r_n) = \alpha e_{ij}$, with $\alpha \neq 0$. Hence, for all $i \neq j$,

$$0 = a[bf(r_1, .., r_n) + f(r_1, .., r_n)c, f(r_1, .., r_n)] = 0 = -\alpha^2 a e_{ij}(c - b)e_{ij}.$$

In other words, since $char(R) \neq 2$ and $\alpha \neq 0$, either the i-th column of the matrix a is zero or, for all j different from i, the (j, i)-entry q_{ji} of $(c - b)$ is zero.

Suppose by contradiction that $(c-b)$ is not a diagonal matrix, then there exists some non-zero entry q_{ji} of $(c - b)$, for $i \neq j$. As we said above, the i-th column of a is zero. Let $m \neq i, j$ and $\varphi_{mi}(x) = (1 + e_{mi})x(1 - e_{mi})$. Consider the following valutations of $f(x_1, .., x_n)$:

$$f(\underline{r}) = \gamma e_{ij}, \quad f(\underline{s}) = \varphi_{mi}(f(\underline{r})) = \gamma e_{ij} + \gamma e_{mj}, \quad \gamma \neq 0.$$

Since $f(\underline{s})^2 = 0$ we have $0 = a[bf(\underline{s}) + f(\underline{s})c, f(\underline{s})] = \gamma^2 a(e_{ij} + e_{mj})(c - b)(e_{ij} + e_{mj})$. Moreover, since the i-th column of a is zero, we obtain $\gamma^2 a(q_{ji} + q_{jm})e_{mj} = 0$. Notice that if $q_{ji} + q_{jm} = 0$, then $q_{jm} = -q_{ji} \neq 0$, so the m-th column of a is zero. On the other hand, if $q_{ji} + q_{jm} \neq 0$, it follows again that the m-th column of a is zero. Hence we can say that a has at most one non-zero column, the j-th one.

Let now ψ any F-automorphism of $M_t(F)$, then

$$0 = \psi(a)[\psi(b)\psi(f(r_1, .., r_n)) + \psi(f(r_1, .., r_n))\psi(c), \psi(f(r_1, .., r_n))] =$$

$$\psi(a)[\psi(b)f(s_1, .., s_n) + f(s_1, .., s_n)\psi(c), f(s_1, .., s_n)]$$

for all $s_1, .., s_n \in M_t(F)$. Therefore, as above, we can conclude that, if the (j, i)-entry of $\psi(c - b)$ is non-zero, for some $j \neq i$, then $\psi(a)$ has at most one non-zero column, the j-th one.

Let now $\psi(x) = (1 + e_{jm})x(1 - e_{jm}))$, with $m \neq j, i$. Hence $\psi(c - b) = (c - b) + e_{jm}(c - b) - (c - b)e_{jm} - e_{jm}(c - b)e_{jm}$ and so its (j, i)-entry is $q_{ji} + q_{mi}$.

If $q_{ji} + q_{mi} = 0$ then $q_{ji} = -q_{mi} \neq 0$, that is the (m, i)-entry of $(c - b)$ is non-zero. In this case a has at most one non-zero column, the m-th one; but $m \neq j$ and so any column of a is zero.

If $q_{ji} + q_{mi} \neq 0$ then the (j, i)-entry of $\psi(c - b)$ is non-zero, hence $\psi(a)$ has at most one non-zero column, the j-th one.

Since $\psi(a) = (\sum_h a_{hj}e_{hj} + a_{mj}e_{jj}) - (\sum_h a_{hj}e_{hm} + a_{mj}e_{jm})$, then, for any $h \neq j$ must be $a_{hj} = 0$ and also $a_{jj} + a_{mj} = 0$. But in this situation we get $a = 0$. Therefore, if $a \neq 0$, then $q_{ji} = 0$, for all $j \neq i$. □

Lemma 3 *Let $R = M_t(F)$ be the algebra of $t \times t$ matrices over the field F of characteristic different from 2 and $t \geq 2$. If there exist $a, b, c \in R$ such that $a[bu + uc, u] = 0$ for any $u \in f(R)$ then one of the following holds:*

1. $a = 0$;
2. $b, c \in Z(R)$;
3. $c - b \in Z(R)$ *and* $u^2 \in Z(R)$, *for all* $u \in f(R)$.

Proof. Suppose $a \neq 0$. By previous two lemmas, $c - b$ is a diagonal matrix, say $c - b = \sum q_{kk}e_{kk}$. Moreover if φ is an automorphism of $M_t(F)$, the same conclusion holds for $\varphi(c - b)$, since as above

$$0 = \varphi(a)[\varphi(b)\varphi(f(r_1, .., r_n)) + \varphi(f(r_1, .., r_n))\varphi(c)] =$$

$$\varphi(a)[\varphi(b)f(s_1, .., s_n) + f(s_1, .., s_n)\varphi(c), (f(s_1, .., s_n))]$$

and $\varphi(a) \neq 0$. Therefore, for any $i \neq j$, $\varphi(c - b) = (1 + e_{ij})(c - b)(1 - e_{ij})$ must be a diagonal matrix. Thus $(q_{jj} - q_{ii})e_{ij} = 0$, that is $q_{jj} = q_{ii}$ and $c - b$ is a central element, that is there exists an element $\lambda \in Z(R)$, such that $b = c + \lambda$. Thus we have that, for all $r_1, .., r_n \in R$,

$$0 = a[cf(r_1, .., r_n) + f(r_1, .., r_n)c, f(r_1, .., r_n)] = a[c, f(x_1, .., x_n)^2].$$

Let A be the additive subgroup generated by the polynomial $f(x_1, .., x_n)^2$. By [3], since $char(R) \neq 2$, either $f(x_1, .., x_n)^2$ is central valued on R, or there exists a non-central Lie ideal L of R such that $L \subseteq A$. In this last case we get $a[c,u]=0$ for all $u \in L$. Since R is simple, L is not central and $char(R) \neq 2$, it is well known that $0 \neq [R,R] \subseteq L$ (see pp 4-5 in [8], Lemma 2 and Proposition 1 in [7], Theorem 4 in [11]). It follows that $a[c,[x_1,x_2]]=0$ for all $x_1,x_2 \in R$. As a consequence of Theorem 1 in [5], we get $c \in Z(R)$. Therefore either $f(x_1,..,x_n)^2$ is central on R, or $b,c \in Z(R)$. □

2.2. The proof of Proposition 1

In order to continue our investigation, we need to fix some well known facts.

Remark 1 Recall that if B is a basis of U over C, then any element of $T = U *_C C\{x_1,..,x_n\}$ can be written in the form $g = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials, that is $m_i = q_0 y_1 \cdots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1,..,x_n\}$. In [2] it is showed that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, if $a_1,a_2 \in U$ are linearly independent over C and $a_1 g_1(x_1,..,x_n) + a_2 g_2(x_1,..,x_n) = 0 \in T$, for some $g_1,g_2 \in T$, then both $g_1(x_1,..,x_n)$ and $g_2(x_1,..,x_n)$ are the zero element of T

Lemma 4 *If R does not satisfy any non-trivial generalized polynomial identity, then either $a=0$ or b and c are elements of C.*

Proof. Suppose that either $b \notin C$ or $c \notin C$, if not we are done. Since R does not satisfy any non-trivial generalized polynomial identity, we have that

$$a[bf(x_1,..,x_n)+f(x_1,..,x_n)c, f(x_1,..,x_n)]$$

is the zero element in the free product $T = U *_C C\{x_1,..,x_n\}$, that is

$$a\left(bf(x_1,..,x_n)^2 - f(x_1,..,x_n)^2 c + f(x_1,..,x_n)(c-b)f(x_1,..,x_n)\right) = 0 \in T.$$

Suppose a and ab are linearly independent over C. We have

$$abf(x_1,..,x_n)^2 - a(f(x_1,..,x_n)^2 c + f(x_1,..,x_n)(c-b)f(x_1,..,x_n)) = 0 \in T.$$

By Remark 1, $abf(x_1,..,x_n)^2 = 0 \in T$. Since R does not satisfy any non-trivial generalized polynomial identity, this forces $ab=0$, which is a contradiction.

Thus we assume a and ab are linearly C-dependent, that is $ab = \beta a$, with $\beta \in C$. In this case we have:

- if $\beta \neq 0$:

$$\beta a f(x_1,..,x_n)^2 - a f(x_1,..,x_n)^2 c + a f(x_1,..,x_n)(c-b) f(x_1,..,x_n) = 0 \in T;$$

- if $\beta = 0$:

$$-a f(x_1,..,x_n)^2 c + a f(x_1,..,x_n)(c-b) f(x_1,..,x_n) = 0 \in T.$$

In any case, if a and $(c-b)$ are linearly independent over C, by Remark 1 we obtain that in particular $af(x_1,..,x_n)(c-b)f(x_1,..,x_n)$ is the zero element in T. Since $(c-b) \neq 0$, we get the required conclusion $a = 0$.
Finally, let a and $(c-b)$ be linearly dependent over C. If $(c-b) = 0$, it follows that

$$\beta a f(x_1,..,x_n)^2 - a f(x_1,..,x_n)^2 c = 0 \in T,$$

in particular $af(x_1,..,x_n)^2 c$ is the zero element in T. If $a \neq 0$, then $c = 0$, that is also $b = 0$, a contradiction.
In case $(c-b) \neq 0$ then, for some $0 \neq \gamma \in C$, we have

$$\beta\gamma(c-b) f(x_1,..,x_n)^2 - \gamma(c-b)(f(x_1,..,x_n)^2 c$$
$$-\gamma(c-b) f(x_1,..,x_n)(c-b) f(x_1,..,x_n) = 0 \in T$$

in particular $\gamma(c-b)(f(x_1,..,x_n)^2 c = 0 \in T$, which implies $c = 0$. By our assumption it follows that $b \notin C$, which implies the contradiction that

$$-\beta\gamma b f(x_1,..,x_n)^2 - \gamma b f(x_1,..,x_n) b f(x_1,..,x_n)$$

is a non-trivial generalized polynomial identity for R. □

In order to prove the next Lemma, here we premit the following:

Fact 1 *Let R be a dense ring of linear transformations over an infinite dimensional right vector space V over a division ring D. Then for any linearly D-independent subset $\{v_1,\dots,v_k\}$ of V, and for any subset $\{w_1,\dots,w_k\}$ of V, there exist $r_1,\dots r_n \in R$ such that $f(r_1,\dots,r_n)v_i = w_i$, for all $i = 1,\dots,k$.*

Proof. Let $U_1 = \{v_1,\dots,v_k\} \subseteq V$ be a subset of linearly D-independent vectors. Since $dim_D V = \infty$, there exists a set

$$U_2 = \{v_{ij}; \quad i = 1,\dots,k; \quad j = 1,\dots,n-1\}$$

of linearly D-independent vectors such that $U_1 \cup U_2$ is also a linearly D-independent subset of V. In other words, if denote $v_i = v_{in}$, for all $i = 1, \dots, k$, we have that

$$\{v_{ij}; \quad i = 1, \dots, k; \quad j = 1, \dots, n\}$$

is a subset of independent vectors in V. By the density of R, there exist $r_1, \dots, r_n \in R$ such that

$$r_1 v_{i1} = w_i, \quad r_i v_{ji} = v_{j,i-1}, \quad i = 2, \dots, n; \quad j = 1, \dots k$$

$$r_i v_{jl} = 0, \quad \text{for} \quad l \neq i.$$

Easy calculations show that $f(r_1, \dots, r_n) v_i = w_i$, for all $i = 1, \dots, k$. □

Lemma 5 *Let R be a dense ring of linear transformations over an infinite dimensional right vector space V over a division ring D. Then either $a = 0$ or b and c are central elements.*

Proof. Suppose that a is non-zero.

Our first aim is to show that for any $v \in V$, if $av \neq 0$ then v, cv are linearly D-dependent.

By contradiction let v, cv be D-independent. There exist $w, w_1, .., w_{n-1}, v_1, .., v_{n-1} \in V$ such that $v, cv = u, w, w_1 .., w_{n-1}, v_1, .., v_{n-1}$ are linearly independent. By Fact 1, there exist $r_1, .., r_n \in R$ such that

$$f(r_1, .., r_n) v = 0, \quad f(r_1, .., r_n) u = w, \quad f(r_1, .., r_n) w = v.$$

Hence, if av is non-zero, then we get the contradiction

$$0 = a[bf(r_1, .., r_n) + f(r_1, .., r_n)c, f(r_1, .., r_n)]v =$$

$$-af(r_1, .., r_n)^2 u = -av \neq 0.$$

Now suppose $av = 0$.

Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$. Hence $a(w - v) = aw \neq 0$. By the previous argument we have that w, cw are linearly D-dependent and $(w - v)$, $c(w - v)$ is as well.

Thus there exist $\alpha, \beta \in D$ such that $cw = w\alpha$ and $c(w - v) = (w - v)\beta$. Moreover v, w are linearly independent.

Suppose first that there exist $\lambda, \mu \in D$ such that $b(w - v) = w\lambda + v\mu$. Since $dim_D V = \infty$, there exist $w_3, .., w_{n-1} \in V$ such that $v, w, w_3, .., w_{n-1}$ are linearly independent and by Fact 1 there exist $s_1, .., s_n \in R$ such that

$$f(s_1, .., s_n) v = 0, \quad f(s_1, .., s_n) w = w - v, \quad f(s_1, .., s_n)^2 w = w - v$$

and

$$0 = a(bf(s_1,..,s_n)^2 - f(s_1,..,s_n)^2c + f(s_1,..,s_n)(c-b)f(s_1,..,s_n))w =$$

$$a(w\lambda + v\mu + (w-v)(\beta-\alpha) - (w-v)\lambda) = aw(\beta-\alpha).$$

Suppose now that $\{b(w-v), w, v\}$ are linerly D-independent. Hence there exist $w_3, ..., w_{n-1}, u_1, ..., u_{n-1} \in V$ such that $b(w-v), v, w, w_3, .., w_{n-1}, u_1, .., u_{n-1}$ are linearly independent and $t_1, .., t_n \in R$ such that

$$f(t_1,..,t_n)v = 0, \quad f(t_1,..,t_n)w = w-v, \quad f(t_1,..,t_n)^2w = w-v$$

and in this case we also have

$$f(t_1,..,t_n)b(w-v) = b(w-v)$$

which implies that

$$0 = a(bf(t_1,..,t_n)^2 - f(t_1,..,t_n)^2c + f(t_1,..,t_n)(c-b)f(t_1,..,t_n))w =$$

$$a(b(w-v) + (w-v)(\beta-\alpha) - b(w-v)) = aw(\beta-\alpha).$$

In any case, we have that $aw(\beta-\alpha) = 0$. Because $aw \neq 0$ then $\alpha = \beta$ and $cv = v\alpha$. This means that for any choice of $v \in V$, v, cv are linearly D-dependent. Standard arguments prove that there exists $\beta \in D$ such that $cv = v\beta$, for all $v \in V$ and also, by using this fact, that $c \in Z(R)$. Therefore for all $r_1, .., r_n \in R$

$$abf(r_1,..,r_n)^2 - af(r_1,..,r_n)bf(r_1,..,r_n) = 0.$$

Our final aim is to prove that $b \in Z(R)$. To do this we repeat the same above argument: we want to show that for any $v \in V$ then v, bv are linearly D-dependent.

Suppose by contradiction that v, bv are D-independent. There exist $w, w_1, .., w_{n-1}, v_1, .., v_{n-1} \in V$ such that $v, bv = u, w, w_1.., w_{n-1}, v_1, .., v_{n-1}$ are linearly independent. By Fact 1, there exist $r_1, .., r_n \in R$ such that

$$f(r_1,..,r_n)v = 0, \quad f(r_1,..,r_n)u = v, \quad f(r_1,..,r_n)w = v.$$

Hence, for $av \neq 0$, we get the contradiction

$$0 = a[bf(r_1,..,r_n), f(r_1,..,r_n)]w =$$

$$abf(r_1,..,r_n)^2w - af(r_1,..,r_n)bf(r_1,..,r_n)w = -av \neq 0.$$

Let now $av = 0$. As above there exists $w \in V$ such that $aw \neq 0$. Hence $a(w - v) = aw \neq 0$. Again it follows that w, bw are linearly D-dependent and is also the case for $(w - v)$, $b(w - v)$.

Thus there exist $\alpha', \beta' \in D$ such that $bw = w\alpha'$ and $b(w-v) = (w-v)\beta'$. Moreover v, w are linearly independent. Moreover there exist $w_3, .., w_{n-1} \in V$ such that $v, w, w_3, .., w_{n-1}$ are linearly independent and again by Fact 1 there exist $s_1, .., s_n \in R$ such that

$$f(s_1, .., s_n)v = 0, \quad f(s_1, .., s_n)w = w - v, \quad f(s_1, .., s_n)^2 w = w - v$$

and

$$0 = a(bf(s_1, .., s_n)^2 - f(s_1, .., s_n)bf(s_1, .., s_n))w = a(w\lambda + v\mu + (w - v)(\beta - \alpha).$$

Since $aw \neq 0$, one has $\alpha = \beta$ and $qv = v\alpha$.

Therefore in any case v, bv are linearly D-dependent, and $b \in Z(R)$. □

2.3. Proof of Proposition 1

As we said above, we assume $C = Z(R)$ and R is a C-algebra centrally closed, that is $R = RC$. If R does not satisfy any non-trivial generalized polynomial identity then, by lemma 4, we are done. Thus we may suppose that R satisfies a non-trivial generalized polynomial identity. By Martindale's theorem in [17], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. If $dim_D V = \infty$, then, by lemma 5, we get the required conclusion.

Therefore we consider the case $dim_D(V) = k$, with k finite positive integer. Of course $k \geq 2$, because R is not a domain. In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity. By lemma 2 in [10] (see also theorem 2.3.29 in [19]), $R \subseteq M_t(F)$, for a suitable field F and $t \geq 2$, moreover $M_t(F)$ satisfies the same generalized identity of R, hence $a[bf(r_1, .., r_n) + f(r_1, .., r_n)c, f(r_1, .., r_n)] = 0$, for all $r_1, .., r_n \in M_t(F)$ and moreover $f(x_1, .., x_n)$ is a non-central polynomial for $M_t(F)$. In this case we are done by lemma 3. □

3. The proof of the Theorem

In [14] Lee proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtained the following result: (Theorem 3 in

[14]) Every generalized derivation δ on a dense right ideal of R can be uniquely extended to U and assumes the form $\delta(x) = bx + d(x)$, for some $b \in U$ and a derivation d on U.

In this section we denote by $f^d(x_1, .., x_n)$ the polynomial obtained from $f(x_1, .., x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write $d(f(r_1, .., r_n)) = f^d(r_1, .., r_n) + \sum_i f(r_1, .., d(r_i), .., r_n)$, for all $r_1, r_2, .., r_n$ in R.

In light of this, we finally prove the main result:

Theorem 1 *Let R be a prime ring of characteristic different from 2, C its extended centroid, U its two-sided Utumi quotient ring, $\delta \neq 0$ a non-zero generalized derivation of R, $f(x_1, .., x_n)$ a non-central multilinear polynomial over C in n non-commuting variables, $a \in R$ such that*

$$a[\delta(f(r_1, .., r_n)), f(r_1, .., r_n)] = 0$$

for any $r_1, .., r_n \in R$. Then one of the following holds:

1. *$a = 0$;*
2. *there exists $\lambda \in C$ such that $\delta(x) = \lambda x$, for all $x \in R$;*
3. *there exist $q \in U$ and $\lambda \in C$ such that $\delta(x) = (q + \lambda)x + xq$, for all $x \in R$, and $f(x_1, .., x_n)^2$ is central valued on R.*

Proof. Suppose by contradiction that $a \neq 0$. Since R satisfies the generalized differential identity

$$a[\delta(f(x_1, .., x_n)), f(x_1, .., x_n)],$$

the above cited Lee's result says that R satisfies

$$a[bf(x_1, .., x_n) + d(f(x_1, .., x_n)), f(x_1, .., x_n)] \tag{3}$$

If d is an inner derivation induced by an element $c \in U$, then R satisfies the generalized polynomial identity

$$a\left([bf(x_1, .., x_n) + cf(x_1, .., x_n) - f(x_1, ..x_n)c, f(x_1, .., x_n)]\right)$$

which is

$$a\left([(b + c)f(x_1, .., x_n) - f(x_1, ..x_n)c, f(x_1, .., x_n)]\right).$$

In this case we are done by proposition 1.
Hence let d be an outer derivation of R. In this case R satisfies the differential identity

$$a[\delta(f(x_1, .., x_n)), f(x_1, .., x_n)] =$$

$$a\left([bf(x_1,..,x_n)+f^d(x_1,..,x_n)+\sum_i f(x_1,..,d(x_i),..,x_n),f(x_1,..,x_n)]\right).$$

By Kharchenko's theorem (see [9] and [15]), R satisfies the generalized polynomial identity

$$a[bf(x_1,..,x_n)+f^d(x_1,..,x_n)+\sum_i f(x_1,..,y_i,..,x_n),f(x_1,..,x_n)],$$

and in particular R satisfies the blended component

$$a[bf(x_1,..,x_n),f(x_1,..,x_n)].$$

Applying Proposition 1, since $a \neq 0$ and $f(x_1,..,x_n)$ is not central, one has that $b \in C$. Therefore, by (3), R satisfies the differential identity

$$a[d(f(x_1,..,x_n)),f(x_1,..,x_n)].$$

In this situation, by Theorem B, we conclude that $d = 0$ and $\delta(x) = bx$, with $b \in C$, for all $x \in R$. □

Acknowledgement

The author wishes to thank the referee; by following her/his valuable suggestions, he has been able to shorten the proof of Lemma 5.

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Received 28.02.2007