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On Certain Type of Modular Sequence Spaces

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Abstract

In this paper we consider a particular type of modular sequence spaces defined with the help of a given sequence $\alpha = \{\alpha_n\}$ of strictly positive real numbers α_n's and an Orlicz function M. Indeed, if we define $M_n(x) = M(\alpha_n x)$ and $\tilde{M}_n(x) = M(\frac{x}{\alpha_n})$, $x \in [0, \infty)$, we consider the modular sequence spaces $\ell\{M_n\}$ and $\ell\{\tilde{M}_n\}$, denoted by ℓ_M^α and ℓ_α^M respectively. These are known to be BK-spaces and if M satisfies Δ_2-condition, they are AK-spaces as well. However, if we consider the spaces ℓ_α^M and ℓ_N^α corresponding to two complementary Orlicz functions M and N satisfying Δ_2-condition, they are perfect sequence spaces, each being the Köthe dual of the other. We show that these are subspaces of the normal sequence spaces μ and η which contain α and α^{-1}, respectively. We also consider the interrelationship of ℓ_α^M and ℓ_M^α for different choices of α.

Key Words: Orlicz functions, Modular sequence spaces, Köthe dual.

1. Introduction

An **Orlicz function** is a continuous, convex, non-decreasing function defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Such function M always has the integral representation

$$M(x) = \int_0^x p(t)dt,$$

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where p, known as the **kernel** of M, is right continuous for $t > 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non- decreasing and $p(x) \to \infty$ as $x \to \infty$.

Given an Orlicz function M with kernel p, define $q(s)$ =sup$\{t : p(t) \leq s\}$, $s \geq 0$. Then q possesses the same properties as p and the function N defined as $N(x) = \int_0^x q(t)dt$, is an Orlicz function. The functions M and N are called **mutually complementary Orlicz functions**.

An Orlicz function M is said to satisfy the Δ_2**-condition** for small x or at '0' if for each $k > 1$, there exist $R_k > 0$ and $x_k > 0$ such that

$$M(kx) \leq R_k M(x) \ , \quad \forall \ x \in (0, x_k].$$

Let ω be the family of all real or complex sequences, which is a vector space with the usual pointwise addition and scalar multiplication. We write e^n $(n \geq 1)$ for the n-th unit vector in ω, i.e $e^n = \{\delta_{nj}\}_{j=1}^{\infty}$ where δ_{nj} is the Kronecker delta, and ϕ for the subspace of ω generated by the e^n's , $n \geq 1$, i.e, $\phi = \text{sp}\{e^n \ : \ n \geq 1\}$. A **sequence space** λ is a subspace of ω containing ϕ. It is called **normal or solid** if $y = \{y_i\} \in \lambda$ whenever $|y_i| \leq |x_i|$, $i \geq 1$ for some sequence $x = \{x_i\} \in \lambda$. The α**-dual or cross dual** of λ is the space λ^α or $\lambda^\times$ defined as

$$\lambda^\times \equiv \lambda^\alpha = \{y = \{y_i\} \in \omega \ : \ \sum |x_i y_i| \ \text{ converges for all } \{x_i\} \in \lambda\}.$$

The β-dual of λ is the space λ^β given by

$$\lambda^\beta = \{y = \{y_i\} \in \omega \ : \ \sum x_i y_i \ \text{ converges for all } \{x_i\} \in \lambda\}.$$

Clearly, $\lambda^\times \subset \lambda^\beta$. However, if λ is normal, $\lambda^\times = \lambda^\beta$; cf [3], p.52.

A sequence space λ is said to be **perfect** if $\lambda = \lambda^{\times\times} = (\lambda^\times)^\times$. Every perfect sequence space is normal. A Banach sequence space (λ, S) is called a BK**- space** if the topology S of λ is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps $P_i \ : \ \lambda \to \mathbb{K}, \ \ P_i(x) = x_i$, $i \geq 1$ are continuous, where $\mathbb{K}$ is the scalar field $\mathbb{R}$ (the set of all reals) or $\mathbb{C}$ (the complex plane). For $x = (x_1, ...x_n, ...)$ and $n \in \mathbb{N}$(the set of natural numbers), we write the n^{th} **section** of x as $x^{(n)} = (x_1, ..., x_n, 0, 0, ...)$. If $\{x^{(n)}\}$ tends to x in (λ, S) for each $x \in \lambda$, we say that (λ, S) is an AK**-space**. The norm $\| \cdot \|_\lambda$ generating the topology S of λ is said to be **monotone** if $\|x\|_\lambda \leq \|y\|_\lambda$ for $x = \{x_i\}, y = \{y_i\} \in \lambda$ with $|x_i| \leq |y_i|$, for all $i \geq 1$.

Corresponding to an Orlicz function M, the set

$$\tilde{\ell_M} = \{x \in \omega : \delta(x, M) \equiv \sum_{i \geq 1} M(|x_i|) < \infty\}$$

is known as an **Orlicz sequence class**. If M and N are mutually complementary Orlicz functions, the **Orlicz sequence space** is defined as

$$\ell_M = \{x \in \omega : \sum_{i \geq 1} x_i y_i \text{ converges for all } y \in \tilde{\ell_N}\}.$$

It is a Banach space with respect to the norm $\|\cdot\|_M$ given by

$$\|x\|_M = \sup\{|\sum_{i \geq 1} x_i y_i| : \delta(y, N) \leq 1\}.$$

An equivalent way of defining ℓ_M is

$$\ell_M = \{\ x \in \omega\ :\ \sum_{i \geq 1} M(\frac{|x_i|}{k}) < \infty \quad \text{for some } k > 0\}.$$

In this case, norm $\|\cdot\|_{(M)}$ is defined by

$$\|x\|_{(M)} = \inf\{k > 0 : \sum_{i \geq 1} M(\frac{|x_i|}{k}) \leq 1\}.$$

The norms $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ are equivalent; indeed,

$$\|x\|_{(M)} \leq \|x\|_M \leq 2\|x\|_{(M)} \ \text{ for } \ x \in \ell_M.$$

An important subspace of ℓ_M, which is an AK-apace is the space h_M defined as

$$h_M = \{x \in \ell_M : \sum_{n \geq 1} M(\frac{|x_n|}{k}) < \infty, \quad \text{for each } k > 0\}.$$

The Δ_2-condition of M is equivalent to the equality of the spaces ℓ_M and h_M. Also in this case $(h_M)^\times = \ell_N$; cf [3], p.311.

For a sequence $\{M_n\}$ of Orlicz functions, the modular sequence space $\ell\{M_n\}$ is defined as

$$\ell\{M_n\} = \{\ x \in \omega\ :\ \sum_{n \geq 1} M_n(\frac{|x_n|}{k}) < \infty \quad \text{for some } k > 0\}$$

The space $\ell\{M_n\}$ is a Banach space with respect to the norm $\|\cdot\|_{\{M_n\}}$ defined as

$$\|x\|_{\{M_n\}} = \inf\{k > 0 : \sum_{n\geq 1} M_n(\frac{|x_n|}{k}) \leq 1\}$$

These spaces were introduced by Woo [8] around the year 1973, and generalizes the Orlicz sequence spaces ℓ_M and the modulared sequence spaces considered earlier by Nakano in [7].

An important subspace of $\ell\{M_n\}$, which is an AK-apace, is the space $h\{M_n\}$ defined as

$$h\{M_n\} = \{x \in \ell\{M_n\} : \sum_{n\geq 1} M_n(\frac{|x_n|}{k}) < \infty, \quad \text{for each } k > 0\}.$$

A sequence $\{M_n\}$ of Orlicz functions is said to satisfy uniform Δ_2**-condition** at '0' if there exists $p > 1$ and $n_0 \in \mathbb{N}$ such that for all $x \in (0, 1)$ and $n > n_0$, we have $\frac{xM_n'(x)}{M_n(x)} \leq p$, or equivalently, there exists a constant $K > 1$ and $n_0 \in \mathbb{N}$ such that $\frac{M_n(2x)}{M_n(x)} \leq K$ for all $n > n_0$ and $x \in (0, \frac{1}{2}]$. If the sequence $\{M_n\}$ satisfies uniform Δ_2-condition, then $h\{M_n\} = \ell\{M_n\}$ and vice-versa.

For details of the general theory of sequence spaces and, in particular, of Orlicz and modulared sequence spaces, we refer to [3], [6] and references given therein.

2. The Sequence Spaces ℓ_α^M and ℓ_N^α

Corresponding to two Orlicz functions M and N and a strictly positive sequence $\alpha = \{\alpha_n\}$ of real numbers, let us define

$$\ell_\alpha^M \; = \; \{x \in \omega \; : \; \{\frac{x_n}{\alpha_n}\} \in \ell_M\}$$

and

$$\ell_N^\alpha \; = \; \{x \in \omega \; : \; \{\alpha_n x_n\} \in \ell_N\}.$$

The spaces ℓ_α^M and ℓ_N^α are respectively the modular sequence spaces $\ell\{M_n\}$ and $\ell\{N_n\}$, where $M_n(x) = M(\frac{x}{\alpha_n})$ and $N_n(x) = N(\alpha_n x)$, for $x \in [0, \infty)$ and $n \in \mathbb{N}$. In the sequel, we use the notation $\|\cdot\|_\alpha^M$ for $\|\cdot\|_{\{M_n\}}$ and $\|\cdot\|_N^\alpha$ for $\|\cdot\|_{\{N_n\}}$. Though the modular sequence spaces $\ell\{M_n\}$ and $\ell\{N_n\}$ are known to be Banach spaces; but for the sake of completeness, we give the direct proof of completeness for one of the spaces, namely ℓ_α^M, in this section. Our underlying assumption throughout for the Orlicz functions M and N are $M(1) = 1$ and $N(1) = 1$.

Let us begin with the following theorem.

Theorem 2.1 *The space ℓ_α^M equipped with the norm $\|\cdot\|_\alpha^M$ is a BK-space; and it is AK if M satisfies Δ_2-condition at '0'.*

Proof. For $x \in \ell_\alpha^M$, let us recall

$$\|x\|_\alpha^M = \inf\{\rho > 0 : \sum_{n\geq 1} M(\frac{|x_n|}{\rho\alpha_n}) \leq 1\}.$$

We now prove the completeness of the space $(\ell_\alpha^M, \|\cdot\|_\alpha^M)$. Let us consider a Cauchy sequence $\{x^n\}$ in ℓ_α^M. Then for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x^n - x^m\|_\alpha^M < \varepsilon, \quad \text{for } n, m \geq n_0.$$

Hence there exists $\rho_\varepsilon > 0$ with $\rho_\varepsilon < \varepsilon$ such that

$$\sum_{k\geq 1} M(\frac{|x_k^n - x_k^m|}{\rho_\varepsilon\alpha_k}) \leq 1 \quad \text{for } n, m \geq n_0. \tag{1}$$

As $M(1) = 1$, we get

$$\frac{|x_k^n - x_k^m|}{\rho_\varepsilon\alpha_k} \leq 1 \quad \text{for } n, m \geq n_0,$$

and for each $k \geq 1$. Thus for each given $k \in \mathbb{N}$, the sequence $\{x_k^n\}$ is a Cauchy sequence of scalars and so $x_k^n \to x_k$, as $n \to \infty$. Write $x = \{x_k\}$. Then by continuity of M, we get from (1)

$$\sum_{k\geq 1} M(\frac{|x_k^n - x_k|}{\rho_\varepsilon\alpha_k}) \leq 1 \quad \text{for } n \geq n_0.$$

Hence $x \in \ell_\alpha^M$ and $\|x^n - x\|_\alpha^M < \varepsilon$ for $n \geq n_0$. Consequently, $(\ell_\alpha^M, \|\cdot\|_\alpha^M)$ is a Banach space.

$(\ell_\alpha^M, \|\cdot\|_\alpha^M)$ is a K-space since we have

$$|P_k(x)| = |x_k| \leq \alpha_k\|x\|_\alpha^M$$

for $x \in \ell_\alpha^M$ and for each given k in $\mathbb{N}$. Thus $(\ell_\alpha^M, \|\cdot\|_\alpha^M)$ is a BK-space.

Now if M satisfies Δ_2-condition at '0', then $\ell_M = h_M$ and so for any $\varepsilon > 0$ and $x \in \ell_\alpha^M$,

$$\|x - x^{(n)}\|_\alpha^M = \inf\{\rho > 0 : \sum_{k\geq n} M(\frac{|x_k|}{\rho\alpha_k}) \leq 1\} < \varepsilon$$

for sufficiently large n. Hence the space is AK. □

Note: If M satisfies Δ_2-condition at '0', then $\ell\{M_n\} = h\{M_n\}$, where $M_n(x) = M(\frac{x}{\alpha_n})$, $x \in (0,\infty)$, $n \in \mathbb{N}$. Also, $\ell\{M_n\} = h\{M_n\}$ if and only if $\{M_n\}$ satisfies uniform Δ_2-condition at '0'. Thus if M satisfies Δ_2-condition at '0' then $\{M_n\}$ satisfies uniform Δ_2-condition at '0'. However, it is natural to ask: if the sequence $\{M_n\}$ of Orlicz functions defined with the help of an Orlicz function M and a strictly positive sequence $\{\alpha_n\}$ of scalars, as above, satisfies uniform Δ_2**-condition** at '0', does M satisfy Δ_2-condition at '0'? If it holds, the result proved for modular sequence spaces $\ell\{M_n\}$ with the assumption of $\{M_n\}$ satisfying uniform Δ_2-condition at '0' would be equivalent to proving them with the assumption of M satisfying Δ_2-condition at '0'.

Next, we prove the following proposition.

Proposition 2.2 *If μ is a normal sequence space containing α, then ℓ_α^M is a proper subspace of μ. In addition, if μ is equipped with the monotone norm(quasi-norm) $\|\cdot\|_\mu$, the inclusion map $I : \ell_\alpha^M \to \mu$ is continuous with $\|I\| \leq \|\{\alpha_n\}\|_\mu$.*

Proof. Let $x \in \ell_\alpha^M$. Then $\sum_{n\geq 1} M(\frac{|x_n|}{\rho\alpha_n}) < \infty$ for some $\rho > 0 \Rightarrow \frac{|x_n|}{\rho\alpha_n} \leq K$ for some constant $K > 0$ and each $n \in \mathbb{N}$. Hence $\{x_n\} \in \mu$. As $\{\alpha_n\} \notin \ell_\alpha^M$, it is a proper subspace of μ. Further,

$$\sum_{n\geq 1} M(\frac{|x_n|}{\|x\|_\alpha^M \alpha_n}) \leq 1$$

$$\Rightarrow |x_n| \leq \alpha_n \|x\|_\alpha^M, \text{ for all } n \in \mathbb{N}$$

As $\|\cdot\|_\mu$ is monotone, $\|Ix\|_\mu = \|\{x_n\}\|_\mu \leq \|\{\alpha_n\}\|_\mu \|x\|_\alpha^M$ and so $\|I\| \leq \|\{\alpha_n\}\|_\mu$. □

Remark: Note that one can take μ to be any of the spaces ℓ^∞, δ, ℓ^p, c_0 etc. If we take α in δ, ℓ_α^M are the Banach spaces of the entire sequences, as was revealed in a study carried out earlier in [2] for finding the duals of the spaces $E_M(\gamma)$ which are subspaces of the space of entire functions.

Concerning the spaces ℓ_N^α, we have this next theorem.

Theorem 2.3 *The space $(\ell_N^\alpha, \|\cdot\|_N^\alpha)$ is a BK-space. If N satisfies Δ_2-condition at '0', it is also AK.*

Proof. Omitted, as it is analogous to the proof of Theorem 2.1. □

Proposition 2.4 *If η is a normal sequence space containing $\{\frac{1}{\alpha_n}\} \equiv \alpha^{-1}$, then ℓ_N^α is a proper subspace of η. If the norm (quasi-norm) $\|\cdot\|_\eta$ of η is monotone, then the inclusion map $J : \ell_N^\alpha \to \eta$ is continuous with $\|J\| \le \|\{\alpha^{-1}\}\|_\eta$.*

Proof. In this case, we have

$$|y_n| \le \alpha_n^{-1}\|y\|_N^\alpha$$

for all $n \in \mathbb{N}$ and for $y \in \ell_N^\alpha$. The result now follows as in the case of Proposition 2.2. □

Remark: Note that if $\alpha_n = n,\ n \in \mathbb{N}$, then $\{\alpha_n^{-1}\} \in c_0$; and in this case $\ell_N^\alpha \subset c_0$. On the other hand, if $\{\alpha_n\}$ is such that $\alpha_n^{\frac{1}{n}} \to 0$, then ℓ^∞ is a proper subspace of ℓ_N^α and the inclusion map from ℓ^∞ to ℓ_α^M is continuous. (Indeed, $\{n\} \in \ell_N^\alpha$ and $\{n\} \notin \ell^\infty$; also for $x \in \ell^\infty$, $\frac{|x_n|}{\|x\|_\infty} \le 1$, for all $n \in \mathbb{N}$ where $\|x\|_\infty = \sup_{n\ge1}|x_n|$ and $\alpha_n < \frac{1}{2^n}$ for $n \ge n_0$ $\Rightarrow \sum_{n\ge n_0} N(\frac{|x_n\alpha_n|}{\|x\|_\infty}) \le \sum_{n\ge n_0} N(\frac{1}{2^n}) \le N(1) = 1 \Rightarrow x \in \ell_N^\alpha$ and $\|x\|_N^\alpha \le \|x\|_\infty$.) Thus, we get different sequence spaces for different choices of α. In the next section, we consider this aspect.

3. Interrelationship Between The Spaces ℓ_α^M and ℓ_M^α

In this section we study the interrelationship between spaces ℓ_α^M and ℓ_M^α defined corresponding to the same Orlicz function M; for three different behaviours of the sequence $\alpha = \{\alpha_n\}$. Indeed, we prove this theorem:

Theorem 3.1 *(i) If $\alpha = \{\alpha_n\}$ is such that $a \le \alpha_n \le b$ for all $n \in \mathbb{N}$ for some $a, b > 0$ (i.e both α and α^{-1} are in ℓ^∞), then $\ell_M^\alpha = \ell_\alpha^M = \ell_M$:*

(ii) If $\{\alpha_n\} \in \ell^\infty$ with $c = sup_{n\geq 1}\alpha_n$ and $\{\alpha^{-1}\}$ is unbounded, then ℓ_α^M is properly contained in ℓ_M^α and the inclusion map $J : \ell_\alpha^M \longrightarrow \ell_M^\alpha$ is continuous with $\|J\| \leq c^2$.

(iii) If $\{\alpha_n\}$ is unbounded with $sup_{n\geq 1}\alpha_n^{-1} = d < \infty$ then ℓ_M^α is properly contained in ℓ_α^M and the inclusion map $J_1 : \ell_M^\alpha \longrightarrow \ell_\alpha^M$ is continuous with $\|J_1\| \leq d^2$.

Proof. (i) We first show that $\ell_M = \ell_M^\alpha$. If $x \in \ell_M$, then $\sum_{n\geq 1} M(\frac{|x_n|}{\rho}) < \infty$ for some $\rho > 0$. If $\acute{\rho} = \rho b$, then from the increasing character of M, it follows that $\sum_{n\geq 1} M(\frac{\alpha_n|x_n|}{\acute{\rho}}) \leq \sum_{n\geq 1} M(\frac{|x_n|}{\rho}) < \infty$. Hence $\ell_M \subset \ell_M^\alpha$. Other inclusion, namely $\ell_M^\alpha \subset \ell_M$ follows from the inequality $\sum_{n\geq 1} M(\frac{|x_n|}{\rho/a}) \leq \sum_{n\geq 1} M(\frac{\alpha_n|x_n|}{\rho})$ valid for any $\rho > 0$. Similarly, one can prove $\ell_\alpha^M = \ell_M$. Hence (i) holds.

(ii) For any $\rho > 0$ and $\acute{\rho} = \rho c^2$, we have

$$\sum_{n\geq 1} M(\frac{\alpha_n|x_n|}{\acute{\rho}}) < \sum_{n\geq 1} M(\frac{|x_n|}{\alpha_n\rho})$$

for $x = \{x_n\}$. Hence $\ell_\alpha^M \subset \ell_M^\alpha$.

We now show that the containment $\ell_\alpha^M \subset \ell_M^\alpha$ is proper. From the unboundedness of the sequence $\{\alpha_n^{-1}\}$, choose a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\alpha_{n_k}^{-1} \geq k$. Define $x = \{x_n\}$ as follows:

$$x_n = \begin{cases} 1/k, & n = n_k, \quad k = 1, 2, 3, \ldots \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in \ell_M^\alpha$; but $x \notin \ell_\alpha^M$.

To prove the continuity of inclusion map J, let us first consider the case when $c = 1$. For $x \in \ell_\alpha^M$, write

$$A_\alpha^M(x) = \{\rho > 0 : \sum_{n\geq 1} M(\frac{|x_n|}{\rho\alpha_n}) \leq 1\}$$

and

$$B_M^\alpha(x) = \{\rho > 0 : \sum_{n\geq 1} M(\frac{|x_n|\alpha_n}{\rho}) \leq 1\}.$$

As M is increasing and $c = 1$, we get

$$A_\alpha^M(x) \subseteq B_M^\alpha(x).$$

Hence

$$\|x\|_M^\alpha = \inf B_M^\alpha(x) \leq \inf A_\alpha^M(x) = \|x\|_\alpha^M,$$

i.e, $\|J(x)\|_M^\alpha \leq \|x\|_\alpha^M$. Thus J is continuous with $\|J\| \leq 1 = c^2$.

If $c \neq 1$, define $\beta_n = \frac{\alpha_n}{c}, n \in \mathbb{N}$. Then $\beta_n \leq 1$ and from the above, it follows that

$$\|x\|_M^\beta \leq \|x\|_\beta^M \quad \text{for } x \in \ell_\alpha^M \tag{2}$$

(note that $\ell_\alpha^M = \ell_\beta^M$). Now

$$\|x\|_M^\beta = \frac{1}{c}\|x\|_M^\alpha$$

and

$$\|x\|_\beta^M = c\|x\|_\alpha^M.$$

Hence from (2)

$$\|J(x)\|_M^\alpha = \|x\|_M^\alpha \leq c^2\|x\|_\alpha^M$$

$\Rightarrow$ J is continuous with $\|J\| \leq c^2$. This completes the proof of part (ii). The proof of (iii) is analogous to that of (ii) and so is omitted. □

Note: Observe that for $\alpha = \{\alpha_n\} \in \ell_M$, we have $\ell_\alpha^M \subsetneq \ell_M \subsetneq \ell_M^\alpha$. However, "If the sequence $\{x_n\}$ is such that $\sum M_n(|x_n|) < \infty$ and $\lim_{n\to\infty} x_n \neq 0$, then $\ell\{M_n\}$ contains a subspace isomorphic to ℓ^∞ " (quoted from, and proved in, [8], p.274), we have this proposition.

Proposition 3.2 *If the sequence $\{\alpha_n\}$ is such that $\sum M(\frac{1}{\alpha_n}) < \infty$ (or $\sum M(\alpha_n) < \infty$), then ℓ_α^M (ℓ_M^α) contains a subspace isomorphic to ℓ^∞.*

Proof. Indeed, consider the sequence $\{x_n\}$ with $x_n = 1$, for each $n \in \mathbb{N}$ and use the result stated above. □

4. Perfectness of the Spaces ℓ_α^M and ℓ_N^α

Let M and N be complementary Orlicz functions such that $M(1) = 1$ and $N(1) = 1$. Then we prove the following theorem.

Theorem 4.1 *If M satisfies Δ_2-condition, then $(\ell_\alpha^M)^\times = \ell_N^\alpha$; and if N satisfies Δ_2-condition, then $(\ell_N^\alpha)^\times = \ell_\alpha^M$.*

Proof. Let M satisfy Δ_2-condition. Then for $x \in \ell_N^\alpha$ and $y \in \ell_\alpha^M$, we have

$$\sum_{n\geq 1}|x_n y_n| = \sum_{n\geq 1}|\frac{\alpha_n x_n}{\rho}\frac{\rho y_n}{\alpha_n}| \ \leq \ \sum_{n\geq 1} N(\frac{|\alpha_n x_n|}{\rho}) + \sum_{n\geq 1} M(\frac{|\rho y_n|}{\alpha_n}) \ < \ \infty,$$

where $\rho > 0$ is such that $\sum_{n\geq 1} N(|\frac{\alpha_n x_n|}{\rho}) < \infty$. Thus $x \in (\ell_\alpha^M)^\times$ or $y \in (\ell_N^\alpha)^\times$. Hence $\ell_N^\alpha \subset (\ell_\alpha^M)^\times$ and $\ell_\alpha^M \subset (\ell_N^\alpha)^\times$.

To prove the equality $(\ell_\alpha^M)^\times = \ell_N^\alpha$, let $y \in (\ell_\alpha^M)^\times$. Then

$$\sum_{n\geq 1}|x_n y_n| < \infty \text{ for all } \{x_n\} \text{ with } \{\frac{x_n}{\alpha_n}\} \in \ell_M \tag{3}$$

As M satisfies Δ_2-condition, $\ell_M = h_M$ and so for $\{z_n\} \in h_M$, we get $\sum|\alpha_n z_n y_n| < \infty$ by (3). Hence $\{\alpha_n y_n\} \in (h_M)^\times = \ell_N \Rightarrow y = \{y_n\} \in \ell_N^\alpha$. Thus $(\ell_\alpha^M)^\times = \ell_N^\alpha$ Similarly, one can prove $(\ell_N^\alpha)^\times = \ell_\alpha^M$ if N satisfies Δ_2-condition. □

Finally, we derive the perfectness of the spaces ℓ_α^M and ℓ_N^α in this final corollary:

Corollary 4.2 *If M and N satisfy Δ_2-condition, then the sequence spaces ℓ_α^M and ℓ_N^α are perfect.*

Proof. Immediate from Theorem 4.1. □

Remark: As the dual of a barreled AK-sequence space can be identified with its β-duals (cf [1], p.964 or [3], p.6), the spaces ℓ_α^M and ℓ_N^α are topological duals of each other in the case M and N are complementary Orlicz functions satisfying Δ_2-condition with $M(1) = N(1) = 1$. Besides, if $M_n(x) = M(\frac{x}{\alpha_n})$ and $N_n = N(\alpha_n x)$, $n \in \mathbb{N}$, then one can easily check that for given $n \in \mathbb{N}$, M_n and N_n are mutually complementary Orlicz functions provided M and N are so. Thus the results proved in [8] for modular sequence spaces concerning duality relations shall also be applicable for our spaces ℓ_α^M and ℓ_N^α.

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