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On Generalized Solution of a Class of Higher Order Operator-Differential Equations

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Abstract

In this paper the sufficient conditions on the existence and uniqueness of a generalized solution on the axis are obtained for higher order operator-differential equations, the main part of which is multi characteristic.

Key Words: Operator-differential equations, Hilbert spaces, existence of generalized solution.

1. Introduction

Let H be a separable Hilbert space, and A be a positive-definite self-adjoint operator in H with domain of definition $D(A)$. Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)\,,\ (\gamma \geq 0)\,,\ (x,y)_\gamma = (A^\gamma x, A^\gamma y)\,,\ x, y \in D(A^\gamma)$.

We denote by $L_2((a,b);H_\gamma)$ $(-\infty \leq a < b \leq +\infty)$ a Hilbert space of vector-functions $f(t)$ determined in (a,b) almost everywhere with values from H measurable, square integrable in the Bochner's sense

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|_\gamma^2 \, dt \right)^{1/2}.$$

Assume

$$L_2((-\infty,+\infty);H) \equiv L_2(R;H).$$

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Further, we define a Hilbert space for natural $m \geq 1$ [1].

$$W_2^m\left((a;b);H\right)=\left\{u\left|u^{(m)}\in L_2\left((a;b);H\right),\ A^m u\in L_2\left((a,b);H_m\right)\right.\right\}$$

with norm

$$\|u\|_{W_2^m((a,b);H)}=\left(\left\|u^{(m)}\right\|^2_{L_2((a,b);H)}+\|A^m u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

Here and in sequel the derivatives are understood in the sense of distributions theory [1]. Here we assume

$$W_2^m\left((-\infty,+\infty);H\right)\equiv W_2^m\left(R;H\right).$$

We denote by $D\left(R;H\right)$a set of infinitely-differentiable functions with values in H.

In the space H we consider the operator − differential equation

$$\begin{gathered}P\left(\frac{d}{dt}\right)u\left(t\right)\equiv\left(-\frac{d^2}{dt^2}+A^2\right)^m u\left(t\right)+\sum_{j=0}^{2m}A_j u^{(2m-j)}\left(t\right)=f\left(t\right),\\ t\in R=(-\infty,+\infty),\end{gathered}\tag{1}$$

where $f\left(t\right)$ and $u\left(t\right)$ are vector-valued functions from H, and coefficients A and A_j $\left(j=\overline{0,2m}\right)$ satisfy the following conditions:

1) A is a positive-definite self-adjoint operator in H ;

2) the operators A_j $\left(j=\overline{0,2m}\right)$ are linear in H.

In this paper we'll give definition of a generalized solution of equation (1) and prove a theorem on the existence and uniqueness of generalized solution (1). Notice that another definition of generalized solution of operator-differential equations and their existence is given in the book [2]. In the paper [3] by S. S. Mirzoyev,and in [4] by M. B. Orazov it was investigated a boundary-value problem when the principal part of equation has the form $(-1)^m\frac{d^{2m}}{dt^{2m}}+A^{2m}$ where A is a self-adjount operator A boundary-value problem when $m=2$ on the semi-axis $R_+=(0,+\infty)$ was studied by the author in [5].

2. Some Auxiliary Facts

First of all we consider some facts that we'll need in future. Denote

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t), \quad u(t)\in D(R;H)$$

and

$$P_1\left(\frac{d}{dt}\right)u(t)=\sum_{j=0}^{2m}A_j u^{(2m-j)}(t), \quad u(t)\in D(R;H).$$

Now let's formulate a lemma that shows the conditions on operator coefficients (1) under which the solution of the equation from the class $W_2^m(R;H)$ has sense.

Lemma 2.1 *Let conditions 1) and 2) be fulfilled, moreover,* $B_j = A_j \times A^{-j}$ $\left(j=\overline{0,m}\right)$ *and* $D_j = A^{-m}A_jA^{m-j}$ $\left(j=\overline{m+1,2m}\right)$ *be bounded in* H*. Then a bilinear functional* $L(u,\psi)\equiv(P_1(d/dt)u,\psi)_{L_2(R;H)}$ *determined for all vector-functions* $u\in D(R;H)$ *and* $\psi\in D(R;H)$ *continues on the space* $W_2^m(R;H)\oplus W_2^m(R;H)$ *that acts in the following way:*

$$L(u,\psi)=(P_1(d/dt)u,\psi)_{L_2(R;H)}=\sum_{j=0}^{2m}\left(A_j u^{(2m-j)},\psi\right)_{L_2(R;H)}$$
$$=\sum_{j=0}^{2m}(-1)^m\left(A_j u^{(m-j)},\psi^m\right)_{L_2(R;H)}+\sum_{j=m+1}^{2m}\left(A_j u^{(2m-j)},\psi\right)_{L_2(R;H)}.$$

Proof. Let $u\in D(R;H)$, $\psi\in D(R;H)$. Then integrating by parts we get

$$L(u,\psi)=(P_1(d/dt)u,\psi)_{L_2(R;H)}=\sum_{j=0}^{2m}\left(A_j u^{(2m-j)},\psi\right)_{L_2(R;H)}$$
$$=\sum_{j=0}^{m}(-1)^m\left(A_j u^{(m-j)},\psi^m\right)_{L_2(R;H)}+\sum_{j=m+1}^{2m}\left(A_j u^{(2m-j)},\psi^{(m)}\right)_{L_2(R;H)}. \tag{2}$$

On the other hand,for $j=\overline{0,m}$ we apply the intermediate derivatives theorem [1] and get

$$\left|\left(A_j u^{(m-j)}, \psi^{(m)}\right)_{L_2(R;H)}\right| = \left|\left(B_j A_j u^{(m-j)}, \psi^{m}\right)_{L_2(R;H)}\right| \leq$$

$$\leq \|B_j\| \cdot \left\|A^j u^{(m-j)}\right\|_{L_2(R;H)} \cdot \left\|\psi^{(m)}\right\|_{L_2(R;H)} \leq C_{m-j} \|D_j\| \cdot \|u\| \cdot \|\psi\|_{W_2^m(R;H)} . \qquad (3)$$

And for $j = \overline{m+1, 2m}$ we again use the theorem on intermediate derivatives [1] and get

$$\left|\left(A_j u^{(2m-j)}, \psi^{m}\right)_{L_2(R;H)}\right| = \left|D_j \left(A^{m-j} u^{(2m-j)}, A^m \psi\right)_{L_2(R;H)}\right| \leq$$

$$\begin{aligned} &\leq \|D_j\| \left\|A^{m-j} u^{(2m-j)}\right\|_{L_2(R;H)} \cdot \|A^m \psi\|_{L_2(R;H)} \leq \\ &\leq C_{2m-j} \|D_j\| \cdot \|u\|_{W_2^m(R;H)} \|\psi\|_{W_2^m(R;H)}. \end{aligned} \qquad (4)$$

Since the set $D(R;H)$ in dense is the space $W_2^m(R;H)$, allowing for inequality (3) and (4) in (2) we get that the inequality

$$\|L(u,\psi)\| \leq const \|u\|_{W_2^m(R;H)} \cdot \|\psi\|_{W_2^m(R;H)}$$

is true for all $u, \varphi \in W_2^m(R;H)$, i.e. $L(u,\psi)$ continues by continuity up to a bilinear functional acting on the spaces $W_2^m(R;H) \oplus W_2^m(R;H)$. We denote this functional by $L(u,\psi)$ as well. The lemma is proved. □

Definition *The vector function $u(t) \in W_2^m(R;H)$ is said to be a generalized solution of (1) if for any vector-function $\psi(t) \in W_2^m(R;H)$ it holds the identity*

$$(u,\psi)_{W_2^m(R;H)} + \sum_{k=1}^{2m-1} C_{2m}^k \left(A^{m-k} u^{(k)}, A^{m-k} \psi^{(k)}\right)_{L_2(R;H)} = (f,\psi)_{L_2(R;H)} , \qquad (5)$$

where $C_{2m}^k = \frac{2m(2m-1)...(2m-k+1)}{k!}$.

To find the solvability conditions of equation (1) we prove the following Lemma by using the method of paper [3].

Lemma 2.2 *For any $u(t) \in W_2^m(R;H)$, hold the following estimates:*

$$\left\|A^{m-j} u^{(j)}\right\|_{L_2(R;H)} \leq d_{m,j}^{m/2} |||u|||_{W_2^m(R;H)} , \quad \left(j = \overline{0,m}\right), \qquad (6)$$

where

$$|||u|||_{W_2^m(R;H)} = \left(\|u\|^2_{W_2^m(R;H_m)} + \sum_{k=1}^{2m-1} c_{2m}^k \left|\left|\left| A^{m-k} u^{(k)} \right|\right|\right|^2_{L_2(R;H)} \right)^{1/2};$$

and the numbers from inequalities (6) are determined as follows

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{\frac{j}{m}} \cdot \left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j = \overline{1, m-1} \\ 1, & j = 0, m \end{cases}$$

Proof. Obviously, the norm $|||u|||_{W_2^m(R;H)}$ is equivalent to the norm $\|u\|_{W_2^m(R;H)}$. Then it follows from the intermediate derivatives theorem that the final numbers

$$b_j = \sup_{0 \neq u \in W_2^m(R;H)} \left\| A^{m-j} u^{(j)} \right\|_{L_2(R;H)} \cdot \|u\|^{-1}_{W_2^m(R;H)}, \quad j = \overline{0, m}.$$

Show that $b_j = d_{m,j}^{m/2}$, $j = \overline{0, m}$. Then $u\,(t) \in D\,(R; H)$.

For all $\beta \in [0, b_j^{-2})$, where

$$b_j = \sup_{\xi \in R} \left| \xi^j \left(\xi^2 + 1\right)^{-m/2} \right| = d_{m,j}^{m/2},$$

we use the Plancherel theorem and get

$$|||u|||_{W_2^m(R;H)} - \beta \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)} = \sum_{k=0}^{2m} C_{2m}^k \left\| A^{m-k} \xi^k \hat{u}\,(\xi) \right\|^2_{L_2(R;H)} -$$

$$-\beta \left\| A^{m-j} \xi^j \hat{u}\,(\xi) \right\|^2_{L_2(R;H)} = \sum_{k=0}^{2m} C_{2m}^k \left(A^{m-k} \xi^k \hat{u}\,(\xi)\,, A^{m-k} \xi^k \hat{u}\,(\xi) \right)_{L_2(R;H)} -$$

$$-\beta \left(A^{m-j} \xi^j \hat{u}\,(\xi)\,, A^{m-j} \xi^j \hat{u}\,(\xi) \right)_{L_2(R;H)} =$$

$$= \int_{-\infty}^{+\infty} \left(\left[(\xi^2 E + A^2)^m - \beta \xi^{2j} A^{(2m-j)} \right] \hat{u}\,(\xi)\,, \hat{u}\,(\xi) \right)_{L_2(R;H)} d\xi, \tag{7}$$

where $\hat{u}(\xi)$ is a Fourier transformation of the vector-function $u(t)$. Since for $\beta \in [0, b_j^{-2})$, it follows from the spectral expansion of the operator A that

$$\left(\left(\left(\xi^2 E + A^2\right)^m - \beta\xi^{2j} A^{2(m-j)}\right) x, x\right) = \int_{-\infty}^{+\infty} \left(\left(\xi^2 + \mu^2\right)^m - \beta\xi^{2j}\mu^{2(m-j)}\right)(dE_\mu x, x) =$$

$$= \int_{\mu_0}^{\infty} \left(1 - \beta\frac{\xi^{2j}\mu^{2(m-j)}}{(\xi^2+\mu^2)^\mu}\right)\left(\xi^2 + \mu^2\right)(dE_\mu x, x) \geq \int_{\mu_0}^{\infty}\left(1 - \beta b_j^2\right)\left(\xi^2+\mu^2\right)(dE_\mu x, x);$$

then equality (6) yields

$$|||u|||^2_{W_2^m(R;H)} \geq \beta \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)}, \tag{8}$$

for all $\beta \in [0, b_j^{-2}$ and $u(t) \in D(R;H)$. Passing to the limit as $\beta \to b_j^{-2}$ we get

$$|||u|||^2_{W_2^m(R;H)} \geq d_{m,j}^{-m/2} \left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)}.$$

Hence it follows

$$\left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)} \leq d_{m,j}^{m/2}\, |||u|||^2_{W_2^m(R;H)}, \quad \left(j = \overline{0,m}\right). \tag{9}$$

Show that inequalities (9) are exact. To this end, for the given $\varepsilon > 0$ we show the existence of the vector-function $u_\varepsilon(t) \in W_2^m(R;H)$, such that

$$E(u_\varepsilon) = |||u|||^2_{W_2^m(R;H)} - \left(d_{m,j}^{-m} + \varepsilon\right)\left\| A^{m-j} u^{(j)} \right\|^2_{L_2(R;H)} < 0. \tag{10}$$

We'll look for $u_\varepsilon(t)$ in the form $u_\varepsilon(t) = g_\varepsilon(t)\varphi_\varepsilon(t)$, where $g_\varepsilon(t)$ is a scalar function from the space $W_2^m(R)$ and $\varphi_\varepsilon \in H_{2m}$, where $\|\varphi_\varepsilon\| = 1$. Using the Plancherel theorem, we write $E(u_\varepsilon)$ in the equivalent form

$$E(u_\varepsilon) = \int_{-\infty}^{+\infty} \left(\left(\left(\xi^2 E + A^2\right)^m - \left(d_{m,j}^{-m} + \varepsilon\right)\xi^{2j} A^{2m-2j}\right)\varphi_\varepsilon, \varphi_\varepsilon\right)\left|\hat{g}_\varepsilon(\xi)\right|^2 d\xi.$$

Note that $\hat{u}(\xi)$ and $\hat{g}_\varepsilon(\xi)$ are the Fourier transformations of the vector-functions $u(t)$ and $g_\varepsilon(t)$, respectively.

Notice that if A has even if one even value μ, then for φ_ε we can choose appropriate eigen-vector $\varphi_\varepsilon = \varphi\left(\|\varphi\| = 1\right)$. Indeed, then at the point $\xi_0 = (j/m)^{1/2} \cdot \mu$

$$\left(\left(\left(\xi_0^2 E + A^2\right)^m - \left(d_{m,j}^{-m} + \varepsilon\right) \xi_0^{2j} A^{2m-2j}\right) \varphi_\varepsilon, \varphi_\varepsilon\right) = \left(\xi_0^2 + \mu^2\right)^m -$$

$$- \left(d_{m,j}^{-m} + \varepsilon\right) \xi_0^{2j} \mu^{2m-2j} = \left(\xi_0^2 + \mu^2\right)^m \left[1 - \left(d_{m,j}^{-m} + \varepsilon\right) \frac{\xi_0^{2j} \mu^{2m-2j}}{\left(\xi^2 + \mu^2\right)^m}\right] < 0. \qquad (11)$$

If the operator A has no eigenvalue, for $\mu \in \sigma(A)$ and for any $\delta > 0$ we can construct a vector φ_δ, $\|\varphi_\delta\| = 1$ such that

$$A^\delta \varphi_\delta = \mu^m \varphi_\delta + 0\left(1, \delta\right), \quad \delta \to 0, \quad m = 1, 2, ...$$

In this case, and for $\xi_0 = (j/m)^{1/2} \mu$ and

$$\left(\left(\left(\xi^2 E + A^2\right)^m - \left(d_{m,j}^{-m} + \varepsilon\right) \xi_0^{2j} A^{2m-2j}\right) \varphi_\delta, \varphi_\delta\right) =$$

$$= \left(\left(\xi_0^2 + \mu^2\right) - \left(d_{m,j}^{-m} + \varepsilon\right) \xi^{2j} \mu^{2m-2j}\right) + 0\left(1, \delta\right).$$

Thus for small $\delta > 0$ it holds inequality (11). Consequently, for any $\varepsilon > 0$ there will be found a vector φ_ε $(\|\varphi_\varepsilon\| = 1)$, for which

$$\left(\left(\left(\xi^2 E + A^2\right)^m - \left(d_{m,j}^{-m} + \varepsilon\right) \xi_0^{2j} A^{2m-2j}\right) \varphi_\varepsilon, \varphi_\varepsilon\right) < 0 \qquad (12)$$

Now for $\xi = \xi_0$ we construct $g_\varepsilon(t)$. Since the left hand side of inequality (12) is a continuous function from the argument ξ, it is true at some vicinity of the point ξ_0. Assume that inequality (12) holds in the vicinity (η_1, η_2). Then we construct $\hat{g}(\xi)$-infinitely-differentiable function of argument ξ with support in (η_1, η_2) and denote it by

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{\eta_1}^{\eta_2} \hat{g}(\xi)\, e^{i\xi t} d\xi.$$

It follows from the Paley-Wienere theorem that $g_\varepsilon(t) \in W_2^m(R)$ and obviously

$$E(u_\varepsilon) = E(g_e, \varphi_\varepsilon) =$$

$$= \int_{\eta_1}^{\eta_2} \left(\left(\left(\xi^2 E + A^2 \right)^m - \left(d_{m,j}^{-m} + \varepsilon \right) \xi^{2j} A^{2m-2j} \right) \varphi_\varepsilon, \varphi_\varepsilon \right) \left| \hat{g}_\varepsilon(\xi) \right|^2 d\xi < 0,$$

i.e. inequalities (9) are exact. The lemma is proved. □

The Main Result

Now let's prove the main theorem.

Theorem 3.1. *Let A be a positive-definite self-adjoint operator in H, the operators $B_j = A_j \cdot A^{-j}$ $\left(j = \overline{0, m}\right)$ and $D_j = A^{-m} A_j A^{m-j}$ $\left(j = \overline{m+1, 2m}\right)$ be bounded in H and it holds the inequality*

$$\gamma = \sum_{j=0}^{m} d_{m,j}^{m/2} \|B_j\| + \sum_{j=m+1}^{2m} d_{m,2m-j}^{m/2} \|D_j\| < 1, \tag{13}$$

where the numbers $d_{m,j}$ are determined from lemma 2.

Then equation (1) has a unique generalized solution and the inequality

$$\|u\|_{W_2^m(R;H)} \le const \|f\|_{L_2(R;H)}$$

holds.

Proof. Show that for $\gamma < 1$ for all vector-functions $\psi \in W_2^m(R;H)$ it holds the inequality

$$(P(d/dt)\psi, \psi)_{L_2(R;H)} \equiv$$

$$\equiv \|\psi\|^2_{W_2^m(R;H)} + \sum_{k=1}^{2m} C_{2m}^k \left\| A^{m-k} \psi^{(k)} \right\|^2_{L_2(R;H)} + L(\psi, \psi) \ge C \|\psi\|^2_{W_2^m(R;H)},$$

where $C > 0$ is a constant number.

Obviously,

$$\left| (P(d/dt)\psi, \psi)_{L_2(R;H)} \right| \ge |||\psi|||^2_{W_2^m(R;H)} - |L(\psi, \psi)|. \tag{14}$$

Since

$$|L(\psi,\psi)| < \sum_{j=0}^{m}\left|\left(A_j\psi^{(m-j)},\psi^{(m)}\right)_{L_2(R;H)}\right| + \sum_{j=m+1}^{2m}\left|\left(A_j\psi^{(2m-j)},\psi\right)_{L_2(R;H)}\right|,$$

then we use lemma 2 and get

$$|L(\psi,\psi)| \le \left(\sum_{j=0}^{m}\|B_j\|\, d_{m,j}^{m/2} + \sum_{j=m+1}^{2m}\|D_j\|\, d_{m,2m-j}^{m/2}\right)|||\psi|||^2_{W_2^m(R;H)} = \\ = \gamma\|\psi\|_{W_2^m(R;H)}. \qquad (15)$$

Allowing for inequality (15) in (14), we get

$$\left|(P(d/dt)\psi,\psi)_{L_2(R;H)}\right| \ge (1-\gamma)\,|||\psi|||^2_{W_2^m(R;H)}. \qquad (16)$$

Further, we consider the problem

$$P_0(d/dt)\,u(t) = f(t),$$

where $f(t)\in L_2(R;H)$. It is easy to see that the vector function

$$u_0(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\xi^2+A^2\right)^m\int_{-\infty}^{+\infty} f(s)\,e^{-2(s-\xi)}ds d\xi \qquad (17)$$

belongs to the space $W_2^m(R;H)$ and satisfies the condition

$$(u_0,\psi) = (f,\psi).$$

Now we'll look for the generalized solution of equation (1) in the form $u = u_0+\xi_0$, where $\xi_0\in W_2^m(R;H)$. Putting this expression into equality (5), we get

$$(P(d/dt)\,u,\psi)_{L_2(R;H)} = -L(u_0,\psi),\ \ \psi\in W_2^m(R;H). \qquad (18)$$

The right hand-side is a continuous functional in $W_2^m(R;H)$, the left hand-side satisfies Lax-Milgram [6] theorem's conditions by inequality (16). Therefore, there exists a unique vector function $u(t) \in W_2^m(R;H)$ satisfying equality (18). On the other hand, for $\psi = u$ it follows from inequality (16) that

$$\left|(P(d/dt)\,u,u)_{L_2(R;H)}\right| = \left|(f,u)_{L_2(R;H)}\right| \geq C\,|||u|||^2_{W_2^m(R;H)} \geq C\,\|u\|^2_{W_2^m(R;H)},$$

then hence it follows

$$\|u\|_{W_2^m(R;H)} \leq const\,\|f\|_{L_2(R;H)}.$$

The theorem is proved. □

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