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Killing and Geodesic Lightlike Hypersurfaces of Indefinite Sasakian Manifolds

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Abstract

In this paper, we study a lightlike hypersurface of indefinite Sasakian manifold, tangent to the structure vector field ξ. Theorems on parallel and Killing distributions are obtained. Necessary and sufficient conditions have been given for lightlike hypersurface to be mixed totally geodesic, D-totally geodesic, $D \perp < \xi >$-totally geodesic and D'-totally geodesic. We prove that, if the screen distribution of lightlike hypersurface M of indefinite Sasakian manifold is totally umbilical, the $D \perp < \xi >$-geodesibility of M is equivalent to the $D \perp < \xi >$-parallelism of the distribution $TM^{\perp}$ of rank 1 (Theorem 4.20). Finally, we give the $D \perp < \xi >$-version (Theorem 4.22) of the Theorem 2.2 ([11], page 88).

Key Words: Lightlike Hypersurfaces; Indefinite Sasakian; Screen Distribution.

1. Introduction

The general theory of degenerate submanifolds of semi-Riemannian (or Riemannian) manifolds is one of the interesting topics of differential geometry. It is well known that semi-Riemannian submanifolds have many similarities with their Riemannian case.

However, lightlike submanifolds [3] are different due to the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. This means that one cannot use, in the usual way, classical submanifold theory to define any induced object on a lightlike submanifold. To deal with this anomaly, lightlike submanifolds were

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introduced and presented in a book by Duggal and Bejancu [11]. They introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle. Several authors have studied a lightlike hypersurface of semi-Riemannian manifold (see [2], [5], [8] and [17], and many more references therein). There are a few papers of general lightlike submanifold of a semi-Riemannian [3], [11], [13]. Concerning the lightlike submanifolds of indefinite Sasakian manifolds, some aspects are studied in [4] and many more references therein. The contact geometry has significant use in differential equations, phase spaces of dynamical systems (see [14] and [16] for examples), and the literature about its lightlike case is very limited. Some specific discussions on this matter can be found in [8], [12], [15] and [19].

Physically, lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons. For instance, the existence of Killing vector fields has often used as the most effective symmetry. In fact, since the Einstein's field equations are a complicated set of nonlinear partial differential equations, many exact solutions have been found by assuming one or more Killing vector fields (see [10] and [1] for more details and many more references therein). In particular, Carter [9] used this information in the study of a null (lightlike) hypersuface which is also a Killing horizon. On the Latter, the relationship between killing and geodesic notions is well specified. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for instance [1], Chapter 8).

All of these motivated us to continue studying the geometry of lightlike hypersurfaces of indefinite Sasakian manifolds, tangent to the structure vector field with specific attention to the Killing and Geodesic lightlike hypersurfaces.

This paper is organized as follows. In section 2, we recall some basic definitions and formulas for indefinite Sasakian manifolds and lightlike hypersurface of semi-Riemannian manifolds.

In section 3, for those lightlike hypersurfaces of indefinite Sasakian manifolds which are tangential to the structure vector field, the decomposition of almost contact metric is given.

In section 4 we study killing, geodesic lightlike hypersurfaces in indefinite Sasakian manifolds and parallel vector field. Some characterization of $D \perp < \xi >$-killing, D-totally geodesic and mixed-totally geodesic lightlike hypersurfaces in indefinite Sasakian manifolds, and D-parallel are given. We obtain a necessary and sufficient condition for integrability of some distributions. We prove that, on the lightlike hypersurface M of Indefinite Sasakian manifold such that the screen distribution is totally umbilical, the $D \perp< \xi >$-geodesibility of M is equivalent to the $D \perp< \xi >$-parallelism of the distribution $TM^{\perp}$ of rank 1. Finally, we give the $D \perp< \xi >$-version of the Theorem 2.2 ([11], page 88) and discuss the effect of the screen distribution on different results found.

2. Preliminaries

2.1. Indefinite Sasakian manifolds

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \overline{\phi} = 0,\ \overline{\phi}\xi = 0 \ \text{and}\ \operatorname{rank} f = 2n. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called a normal contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$

$$\begin{aligned} \overline{g}(\xi, \xi) &= \varepsilon = \pm 1,\ \eta(\overline{X}) = \varepsilon \overline{g}(\xi, \overline{X}),\ \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \varepsilon\, \eta(\overline{X})\, \eta(\overline{Y}) \\ d\eta(\overline{X}, \overline{Y}) &= \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}),\ (\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \varepsilon\, \eta(\overline{Y})\overline{X}, \end{aligned} \tag{2.2}$$

where $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$. In this case, we call $\overline{M}$ an indefinite Sasakian manifold. From the first equation of (2.2), ξ is never a lightlike vector field on $\overline{M}$. Sasakian manifolds with indefinite metrics have been first considered by Takahashi [18]. Their importance for physics have been point out by Duggal [10].

According to the causal character of ξ [10], we have two classes of Sasakian manifolds. Thus in case ξ is spacelike ($\varepsilon = 1$ and the index of $\overline{g}$ is an even number $\nu = 2r$), (respectively, timelike, $\varepsilon = -1$ and the index of $\overline{g}$ is an odd number $\nu = 2r+1$) we say that $\overline{M}$ *is called a space-like almost contact metric manifold* (respectively, *time-like almost contact metric manifold*).

Takahashi [18] shows that it suffices to consider those indefinite almost contact manifolds with spacelike ξ (see [6] for more information). Hence, from now on, we shall restrict ourselves to the case of ξ a spacelike unit vector (that is $\overline{g}(\xi, \xi) = 1$).

$(\overline{\phi}, \xi, \eta, \overline{g})$ on $\overline{M}$ is called an almost contact metric structure if an almost contact structure $(\overline{\phi}, \xi, \eta)$ satisfies the three conditions from above in (2.2). In this case [4]

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X}. \tag{2.3}$$

implies $\overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X})$, ξ is killing vector field, $(\nabla_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})$.
Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(\mathrm{E})$ the smooth sections of the vector bundle E.

2.2. Lightlike hypersurfaces of semi-Riemannian manifolds

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. We say that M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ (see [11]). We consider the vector bundle $TM^{\perp}$ whose fibers are, for any $p \in M$,

$$T_pM^{\perp} \;=\; \left\{Y_p \in T_p\overline{M} : \overline{g}_p(X_p, Y_p) = 0, \forall X_p \in T_pM\right\}. \tag{2.4}$$

Thus, a hypersurface M of $\overline{M}$ is lightlike if and only if $TM^{\perp}$ is a distribution of rank 1 on M. Let $S(TM)$ be the complementary distribution of $TM^{\perp}$ in TM, which is called a screen distribution. From [11], we know that it is non-degenerate. Thus we have direct orthogonal sum decomposition

$$TM = S(TM) \perp TM^{\perp}. \tag{2.5}$$

Since $S(TM)$ is non-degenerate with respect to $\overline{g}$, there exists a complementary orthogonal vector subbundle $S(TM)^{\perp}$ to $S(TM)$ in $T\overline{M}$ over M. Hence, we have the orthogonal decomposition

$$T\overline{M} = S(TM) \perp S(TM)^{\perp}. \tag{2.6}$$

The existence of $S(TM)$ is secured since M is paracompact. Our results will be based on a choice of $S(TM)$. The following normalization result is known.

Theorem 2.1 [11] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$g(N, E) = 1 \;\; \textit{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}).$$

Hence, N is not tangent to M and $\{E, N\}$ is a local field of frames of $S(TM)^{\perp}$. Moreover we have a one-dimensional vector subbundle $N(TM)$ of $T\overline{M}$ over M, which is locally spanned by N. Then we set

$$S(TM)^{\perp} = TM^{\perp} \oplus N(TM), \tag{2.7}$$

where the decomposition is not orthogonal. Thus we have the following decomposition of $T\overline{M}$:

$$T\overline{M} = S(TM) \perp S(TM)^{\perp} = S(TM) \perp (TM^{\perp} \oplus N(TM)) = TM \oplus N(TM). \quad (2.8)$$

Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. If $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$, by using the decomposition (2.5) and (2.8), we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X,\, Y \in \Gamma(TM), \quad (2.9)$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall X,\, Y \in \Gamma(TM),\, V \in \Gamma(N(TM)), \quad (2.10)$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is a symmetric linear connection on M called an induced linear connection, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$. h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

We can also obtain the local version of these formulas for a pair $\{E, N\}$ verifying the properties of the Theorem 2.1. Thus, from the decomposition (2.8), the local Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)\, N, \quad \forall X,\, Y \in \Gamma(TM|_{\mathcal{U}}), \quad (2.11)$$

$$\text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X) N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \quad (2.12)$$

where B, A_N and τ are called the local second fundamental form, the local sharp operator and the transversal 1-form, respectively, for the lightlike immersion of M in $\overline{M}$. It is easy to check that τ is a differential 1-form, A_N is a tensor field of type $(1, 1)$ and ∇ is a torsion-free linear connection on M. It is important to mention that the second fundamental form B is independent of the choice of screen distribution; in fact, from (2.11) and (2.12) we obtain

$$B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E), \quad \forall X,\, Y \in \Gamma(TM|_{\mathcal{U}}).$$

The 1-form τ on $\mathcal{U}$ is defined by

$$\tau(X) = \overline{g}(\nabla^{\perp}_X N, E), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \quad (2.13)$$

Tensors fields B and A_N are not related by g, and therefore, in general A_N is not symmetric with respect to g. The 1-form τ, in general, does not vanish on M as it is in the nondegenerate case. The induced linear connection ∇ is not a metric connection. More precisely, we obtain from (2.11) and the fact that $\overline{\nabla}$ is a metric connection,

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y) \quad (2.14)$$

for any X, $Y \in \Gamma(TM)$, where θ is a differential 1-form locally defined on M by

$$\theta(X) := \overline{g}(X, N), \quad \forall\, X \in \Gamma(TM). \tag{2.15}$$

Denote P as the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition $TM = S(TM) \perp TM^{\perp}$. Taking into account this decomposition, we obtain the following local Gauss and Weingarten formulas with respect to $S(TM)$:

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.16}$$

$$\text{and} \quad \nabla_X E = -A^*_E X - \tau(X)E, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.17}$$

where C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. For more details about these geometric elements, see [11]. C is not symmetric, in general, on $\Gamma(M) \times \Gamma(S(TM))$ and A^*_E is a tensor field of type (1, 1) on M. The following identities are valid:

$$g(A_N X, PY) = C(X, PY), \; g(A_N X, N) = 0, \tag{2.18}$$

$$\text{and} \quad g(A^*_E X, PY) = B(X, PY), \; g(A^*_E X, N) = 0, \tag{2.19}$$

for any X, $Y \in \Gamma(TM)$. From $\overline{g}(\overline{\nabla}_X E, E) = 0$, we get

$$B(X, E) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.20}$$

The induced connection ∇ is torsion-free, but not necessarily metric connection. Also, on the geodesibility of M, we know the following result which does not depend on the screen distribution.

Theorem 2.2 ([11], page 88) *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent:*

(i) *M is totally geodesic.*

(ii) *h (or equivalently B) vanishes identically on M.*

(iii) *A^*_W vanishes identically on M, for any $W \in \Gamma(TM^{\perp})$.*

(iv) *The connection ∇ induced by $\overline{\nabla}$ on M is torsion-free and metric.*

(v) *$TM^{\perp}$ is a parallel distribution with respect to ∇.*

(vi) *$TM^{\perp}$ is a killing distribution on M.*

It turns out that if (M, g) is not totally geodesic, there is no connection that is, at the same time, torsion-free and metric connection. But there is no unicity of such a connection in case there is any.

3. Lightlike hypersurfaces of indefinite Sasakian manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface, *tangent to the structure vector field* ξ *(that is,* $\xi \in TM$*)*. If E is a local section of $TM^{\perp}$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as vector subbundle. We consider local section N of $N(TM)$. Since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}N$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the components of $\overline{\phi}N$ with respect to E vanishes. Thus $\overline{\phi}N \in \Gamma(S(TM))$. From the third equation of (2.2) with $\varepsilon = 1$ we have

$$\overline{g}(\overline{\phi}N, \overline{\phi}E) = 1. \tag{3.21}$$

Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

It is known [8] that if M is tangent to the structure vector field ξ, then ξ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi>, \tag{3.22}$$

where $<\xi> = Span\{\xi\}$.

Proposition 3.1 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, the distribution D_0 is an invariant with respect to $\overline{\phi}$, that is, $\overline{\phi}(D_0) = D_0$.*

Proof. For any $X \in \Gamma(D_0)$ and $Y \in \Gamma(TM)$, we have $\overline{g}(\overline{\phi}X, Y) = -\overline{g}(X, \overline{\phi}Y)$. For $Y = \overline{\phi}E$, we obtain $\overline{g}(\overline{\phi}X, \overline{\phi}E) = \overline{g}(X, E) - \eta(X)\eta(E) = 0$. Thus $\overline{\phi}X \perp \overline{\phi}(TM^{\perp})$. On the other hand we have $\overline{g}(\overline{\phi}X, E) = -\overline{g}(X, \overline{\phi}E) = 0$ for any $E \in \Gamma(TM^{\perp})$. Hence $\overline{\phi}X \perp TM^{\perp}$. Also, we have $\overline{g}(\overline{\phi}X, \xi) = 0$ and $\overline{g}(\overline{\phi}X, \overline{\phi}N) = \overline{g}(X, N) - \eta(X)\eta(N) = 0$ for any $N \in \Gamma(N(TM))$. Thus $\overline{\phi}X \perp \left\{\left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp TM^{\perp} \perp <\xi>\right\}$. Finally we derive $\overline{g}(\overline{\phi}X, N) = -\overline{g}(X, \overline{\phi}N) = 0$, and by summing up these results we deduce

$$\overline{\phi}X \perp \left\{\left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp TM^{\perp} \perp <\xi> \oplus N(TM)\right\},$$

that is $\overline{\phi}(D_0) = D_0$, which proves our assertion. $\square$

Moreover, from (2.6), (2.7), (2.8) and (3.22) we obtain the decomposition

$$TM = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp TM^\perp, \tag{3.23}$$

$$\text{and } T\overline{M} = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp (TM^\perp \oplus N(TM)). \tag{3.24}$$

Example 3.2 Let $\mathbb{R}^7$ be the 7-dimensional real number space. We consider $x = (x_i)_{1\leq i\leq 7}$ as cartesian coordinates on $\mathbb{R}^7$ and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}_{1\leq i\leq 7}$ a tensor field $\overline{\phi}$ of type $(1,1)$ by its matrix:

$$\begin{aligned}
&\overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2},\ \overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_7},\ \overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4},\\
&\overline{\phi}(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_7},\ \overline{\phi}(\frac{\partial}{\partial x_5}) = -\frac{\partial}{\partial x_6},\ \overline{\phi}(\frac{\partial}{\partial x_6}) = \frac{\partial}{\partial x_5},\ \overline{\phi}(\frac{\partial}{\partial x_7}) = 0.
\end{aligned} \tag{3.25}$$

The differential 1-form η is defined by $\eta = dx_7 - x_4dx_1 - x_6dx_3$. The vector field ξ is defined by $\xi = \frac{\partial}{\partial x_7}$. It is easy to check (2.1) and thus $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\mathbb{R}^7$. Finally we define metric $\overline{g}$ by

$$\begin{aligned}
\overline{g} &= (x_4^2-1)dx_1^2 - dx_2^2 + (x_6^2+1)dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2 - x_4dx_1\otimes dx_7\\
&- x_4dx_7\otimes dx_1 + x_4x_6dx_1\otimes dx_3 + x_4x_6dx_3\otimes dx_1 - x_6dx_3dx_7 - x_6dx_7dx_3
\end{aligned} \tag{3.26}$$

with respect to the natural field of frames. It is easy to check that $\overline{g}$ is a semi-Riemannian metric and $(\overline{\phi}, \xi, \eta, \overline{g})$ given by (3.25)–(3.26) is a Sasakian structure on $\mathbb{R}^7$.

We define now a hypersurface M of $(\mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$, with $\xi \in TM$, by $M = \{x \in \mathbb{R}^7\colon x_5 = x_4\}$. Thus the tangent space TM is spanned by $\{U_i\}_{1\leq i\leq 6}$, where $U_1 = \frac{\partial}{\partial x_1}, U_2 = \frac{\partial}{\partial x_2}, U_3 = \frac{\partial}{\partial x_3}, U_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}, U_5 = \frac{\partial}{\partial x_6}, U_6 = \xi$ and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where $E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$. It follows that $TM^\perp \subset TM$. Then M is an 6-dimensional lightlike hypersurface of $\mathbb{R}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}\left(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}\right)$. On the other hand, by using the almost contact structure of $\mathbb{R}^7$ and also by taking into account the decomposition (3.23), the distribution D_0 is spanned by $\left\{F, \overline{\phi}F\right\}$, where $F = U_2$, $\overline{\phi}F = U_1 + x_4\xi$ and the distributions $<\xi>$, $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are spanned by ξ, $\overline{\phi}E = U_3 - U_5 + x_6\xi$, $\overline{\phi}N = \frac{1}{2}(U_3 + U_5 + x_6\xi)$, respectively. Hence M is lightlike hypersurface of $\mathbb{R}^7$.

Let $\overline{M}(\overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite manifold and (M, g) be its lightlike hypersurface with $\xi \in TM$. Now, we consider the distributions on M

$$D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0, \; D' := \overline{\phi}(N(TM)). \tag{3.27}$$

Then D is invariant under $\overline{\phi}$ and

$$TM \quad = \quad D \oplus D' \perp < \xi > . \tag{3.28}$$

Now we consider the local lightlike vector fields $U := -\overline{\phi}\, N, \quad V := -\overline{\phi}\, E$. Then, from (3.28), any $X \in \Gamma(TM)$ is written as

$$X = RX + QX + \eta(X)\xi, \quad QX = u(X)\, U, \tag{3.29}$$

where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(X) := g(X, V)$. Applying $\overline{\phi}$ to (3.29) and (2.1) (note that $\overline{\phi}^2\, N = -N$), we obtain

$$\overline{\phi}\, X = \phi\, X + u(X)\, N, \tag{3.30}$$

where ϕ is a tensor field of type $(1, 1)$ defined on M by $\phi\, X := \overline{\phi}\, RX, \quad X \in \Gamma(TM)$. Again, applying $\overline{\phi}$ to (3.30) and using (2.1), we also have

$$\phi^2\, X = -\, X + \eta(X)\xi + u(X)\, U, \;\; X \in \Gamma(TM). \tag{3.31}$$

Now applying ϕ to the equation (3.31) and since $\phi U = 0$, we obtain $\phi^3 + \phi = 0$, which shows that ϕ is an f-structure [20] of constant rank.

As it was proved in Bejancu-Duggal [1] any nondegenerate real hypersuface of an indefinite almost Hermitian manifold $\overline{M}$ inherits an almost contact metric structure. However, this is not the case for a lightlike hypersurface of the indefinite Sasakian manifold. More precisely, by (2.3) and (3.30) we derive

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.32}$$

for any X, $Y \in \Gamma(TM)$, where v is a 1-form locally defined on M by $v(X) = g(X, U)$, $\forall X \in \Gamma(TM)$.

Lemma 3.3 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then for any X, $Y \in \Gamma(TM)$*

$$\nabla_X \xi \quad = \quad -\phi X \tag{3.33}$$

$$(\nabla_X u)Y \quad = \quad -B(X, \phi\, Y) - u(Y)\tau(X), \tag{3.34}$$

$$(\nabla_X \phi)Y \quad = \quad g(X, Y)\xi - \eta(Y)X - B(X, Y)U + u(Y)A_N X. \tag{3.35}$$

Proof. These expressions are derived by straightforward calculation. □

4. Killing and geodesic lightlike hypersurfaces of indefinite Sasakian manifolds

This section is devoted to some geometric aspects of lightlike hypersurfaces (M, g) of indefinite Sasakian manifolds $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$, with $\xi \in TM$, by using the Lie derivative and the definitions of Killing, totally geodesic and parallel.

Definition 4.1 Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$.

(a) M is D or $D \perp < \xi >$-totally geodesic (respectively, D'-totally geodesic) if its second fundamental form h satisfies $h(X, Y) = 0$ (equivalently $B(X, Y) = 0$), for any X, $Y \in \Gamma(D)$ or $\Gamma(D \perp < \xi >)$ (respectively, X, $Y \in \Gamma(D'))$;

(b) M is mixed totally geodesic if its second fundamental form h satisfies $h(X, Y) = 0$ (equivalently, $B(X, Y) = 0$), for any $X \in \Gamma(D \perp < \xi >)$ and $Y \in \Gamma(D')$.

In connection with the item (b) of this definition, the mixing between $< \xi >$ and the subbundle D' in term of mixed totally geodesic can not be possible because of $B(\xi, U) = -1$. So, only D and D' can be mixed in our case.

Proposition 4.2 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. The Lie derivative with respect to the vector field V is given by, for any X, $Y \in \Gamma(TM)$,*

$$(L_V g)(X, Y) = X.u(Y) + Y.u(X) + u([X, Y]) - 2\overline{g}(h(X, \overline{\phi}Y), E). \tag{4.36}$$

Proof. From a straightforward calculation, we have, for any X, $Y \in \Gamma(TM)$,

$$\begin{aligned} \overline{g}(h(X, \overline{\phi}Y), E) &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, E) = X.g(Y, V) - g(Y, [X, V]) - \overline{g}(Y, \overline{\nabla}_V X) \\ &= X.g(Y, V) - g(Y, [X, V]) - V.g(Y, X) + \overline{g}(\overline{\nabla}_V Y, X) \\ &= X.g(Y, V) - g(Y, [X, V]) - V.g(Y, X) + g([V, Y], X) + \overline{g}(\overline{\nabla}_Y V, X) \\ &= X.g(Y, V) - (L_V g)(X, Y) + Y.g(X, V) - \overline{g}(V, \overline{\nabla}_Y X) \\ &= X.u(Y) - (L_V g)(X, Y) + Y.u(X) + u([X, Y]) - \overline{g}(h(X, \overline{\phi}Y), E). \end{aligned}$$

Thus the proof follows from this equation. □

Definition 4.3 Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$.

(a) A distribution Ξ on M is a Killing distribution if $(L_X g)(Y, Z) = 0$, for any $X \in \Gamma(\Xi)$ and $Y,\, Z \in \Gamma(TM)$.

(b) A distribution Ξ on M is a D or $D \perp\!< \xi >$-Killing distribution (respectively, D'-Killing distribution) if $(L_X g)(Y, Z) = 0$, for any $X \in \Gamma(\Xi)$ and $Y,\, Z \in \Gamma(D)$ or $\Gamma(D \perp\!< \xi >)$ (respectively, $Y,\, Z \in \Gamma(D'))$.

Lemma 4.4 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, for any $X,\, Y \in \Gamma(TM)$*

$$\overline{g}(h(X, \overline{\phi}Y), E) = u(\nabla_X Y). \tag{4.37}$$

Proof. For any $X,\, Y \in \Gamma(TM)$

$$\overline{g}(h(X, \overline{\phi}Y), E) = \overline{g}(\overline{\nabla}_X \overline{\phi}Y, E) = \overline{g}(\overline{\phi}(\overline{\nabla}_X Y), E) = -\overline{g}(\nabla_X Y, \overline{\phi}E) = u(\nabla_X Y), ,$$

which completes the proof. □

Expression (4.37) is equivalent to the expression (3.34) and can be deduced from the definition of $\overline{\phi}$.

Definition 4.5 Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$.

(a) A vector field W is parallel with respect to the connection ∇ if $\nabla_X W = 0$, for any $X \in \Gamma(TM)$.

(b) A vector field W is D or $< \xi >$ or $D \perp\!< \xi >$-parallel (respectively, D'-parallel) with respect to the connection ∇ if $\nabla_X W = 0$, for any $X \in \Gamma(D)$ or $< \xi >$ or $X \in \Gamma(D \perp\!< \xi >)$ (respectively, for any $X \in \Gamma(D'))$.

We note, from this definition, that a parallel vector field with respect to a connection is not necessary $D \perp\!< \xi >$-parallel or D'-parallel vector field. On the other hand, a $D \perp\!< \xi >$-parallel (D and $< \xi >$-parallel) and D'-parallel vector field is a parallel vector field. This because of the following relation, for a linear connection ∇ and for any $X \in \Gamma(TM)$ $(X = RX + u(X)U + \eta(X)\xi)$

$$\begin{aligned} \nabla_{RX+u(X)U+\eta(X)\xi}(\cdot) &= \nabla_{RX+\eta(X)\xi}(\cdot) + u(X)\nabla_U(\cdot) \\ &= \nabla_{RX}(\cdot) + \eta(X)\nabla_\xi(\cdot) + u(X)\nabla_U(\cdot). \end{aligned} \tag{4.38}$$

Lemma 4.6 *The spacelike vector field ξ is D'-parallel with respect to the induced connections ∇.*

Proof. Using (3.33), we have $\nabla_U \xi = \phi U = 0$ which completes the proof. □

Theorem 4.7 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, the distribution M is $D \perp < \xi >$-totally geodesic if and only if the distribution $D \perp < \xi >$ is $D \perp < \xi >$-parallel with respect to the induced connection ∇.*

Proof. Suppose that M is $D \perp < \xi >$-totally geodesic; then, for any X, $Y \in \Gamma(D \perp < \xi >)$, $B(X,Y) = 0$. So, $u(\nabla_X Y) = \overline{g}(h(X, \overline{\phi}Y), E) = B(X, \phi RY) = 0$, since $\phi Y = \overline{\phi} Y = \overline{\phi} RY \in \Gamma(D) \subset \Gamma(D \perp < \xi >)$. Conversely, suppose that $D \perp < \xi >$ is $D \perp < \xi >$-parallel. For X, $Y \in \Gamma(D \perp < \xi >)$, $B(X,Y) = B(X, RY + \eta(Y)\xi) = B(X, RY) + \eta(Y)B(X, \xi) = B(X, RY) - \eta(Y)u(X)$. Since the distribution D is invariant under $\overline{\phi}$, there is $Z \in \Gamma(D)$ such that $RY = \overline{\phi}Z (= \phi Z)$. Thus, $B(X,Y) = B(X, \phi Z) = \overline{g}(h(X, \overline{\phi}Z), E) = u(\nabla_X Z) = 0$, which completes the proof. □

Theorem 4.8 *Let $\overline{M}$ be an indefinite Sasakian manifold and M be a mixed totally geodesic lightlike hypersurface of $\overline{M}$ with $\xi \in TM$. Then, the distributions $D \perp < \xi >$ and D are D'-parallel with respect to the induced connection ∇.*

Proof. Since $D \subset D \perp < \xi >$ is invariant under $\overline{\phi}$, for any $Y \in \Gamma(D \perp < \xi >)$ $(Y = RY + \eta(Y)\xi)$, $u(\nabla_U Y) = \overline{g}(h(U, \overline{\phi}Y), E) = B(U, \phi RY) = 0$. That is, $\nabla_U Y \in \Gamma(D \perp < \xi >)$. Hence $D \perp < \xi >$ is D'-parallel with respect to ∇. On the other hand, $u(\nabla_U RY) = u(\nabla_U Y) = 0$, that is $\nabla_U RY \in \Gamma(D \perp < \xi >)$. So we have $\nabla_U RY = R\nabla_U RY + \eta(\nabla_U RY)\xi$. The component of $\nabla_U RY$ in the direction of ξ is given by $\eta(\nabla_U RY) = g(\nabla_U RY, \xi) = -g(RY, \nabla_U \xi) = -g(RY, \phi U) = 0$, since ∇ is a torsion-free metric connection and $\phi U = 0$. Thus $\nabla_U RY = R\nabla_U RY \in \Gamma(D)$ and the proof is complete. □

Proposition 4.9 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Suppose that the distribution $D \perp < \xi >$ is parallel with respect to the induced connection ∇. If the vector field V is parallel with respect to the connection $\overline{\nabla}$,* *then, for any* $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp <\xi>)$,

$$(L_V g)(X, Y) = 0. \tag{4.39}$$

Proof. From (4.36) we have, for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp <\xi>)$ ($Y = RY + \eta(Y)\xi$), $(L_V g)(X, Y) = Y.u(X) - u([Y, X])$, since $u(Y) = 0$ and $\overline{g}(h(X, \overline{\phi}Y), E) = u(\nabla_X Y) = 0$. If the vector field V is parallel with respect to the connection $\overline{\nabla}$, then, for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp <\xi>)$,

$$\begin{aligned} 0 &= \overline{g}(\overline{\nabla}_Y V, X) = Y.\overline{g}(V, X) - \overline{g}(V, \overline{\nabla}_Y X) \\ &= Y.\overline{g}(V, X) - \overline{g}(V, [Y, X]) - \overline{g}(V, \nabla_X Y) \\ &= Y.\overline{g}(V, X) - \overline{g}(V, [Y, X]) = Y.u(X) - u([Y, X]). \end{aligned}$$

From this expression, we complete the proof. □

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Therefore, we investigate the integrability of the screen distribution.

Proposition 4.10 *Let* $\overline{M}$ *be an indefinite Sasakian manifold and* M *be a* $D \perp <\xi>$*-totally geodesic lightlike hypersurface of* $\overline{M}$ *with* $\xi \in TM$*. Then, for any* $X, Y \in \Gamma(D \perp <\xi>)$

$$(L_V g)(X, Y) = -(L_V g)(Y, X). \tag{4.40}$$

Moreover, the distribution $D \perp <\xi>$ *is integrable if and only if* $\overline{\phi}(TM^{\perp})$ *is a* $D \perp <\xi>$*-Killing distribution on* M*.*

Proof. Since M is a $D \perp <\xi>$-totally geodesic lightlike hypersurface of $\overline{M}$, from Theorem 4.7, the distribution is $D \perp <\xi>$ is $D \perp <\xi>$-parallel. So, the expression (4.36) becomes $(L_V g)(X, Y) = u([X, Y])$, for any $X, Y \in \Gamma(D \perp <\xi>)$ and the equivalence follows from the latter. □

Theorem 4.11 *Let* $\overline{M}$ *be an indefinite Sasakian manifold and* M *be a* $D \perp <\xi>$*-totally geodesic lightlike hypersurface of* $\overline{M}$ *with* $\xi \in TM$*. Then, if the vector field* V *is parallel, then, the distribution* $D \perp <\xi>$ *is integrable and* $\overline{\phi}(TM^{\perp})$ *is a* $D \perp <\xi>$*-Killing distribution on* M*.*

Proof. The proof follows from the equivalences $(L_V g)(X, Y) = \overline{g}(\overline{\nabla}_Y V, X) = g(\nabla_Y V, X) = u([X, Y])$, for any $X, Y \in \Gamma(D \perp < \xi >)$ by using the proof of the Theorem 4.7 and Proposition 4.9. □

Theorem 4.12 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then the distribution $D \perp < \xi >$ is integrable if and only if the tensor field $\overline{\phi}$ of type $(1, 1)$ is symmetric with respect to the local second fundamental form B in the of direction of $D \perp < \xi >$, that is*

$$B(X, \phi Y) = B(\phi X, Y), \tag{4.41}$$

for any X, $Y \in \Gamma(D \perp < \xi >)$.

Proof. Since $\overline{\nabla}$ is the Levi-Civita connection we have , for any X, $Y \in \Gamma(D \perp < \xi >)$,

$$\begin{aligned} \overline{g}([X, Y], \overline{\phi}E) &= \overline{g}(\overline{\nabla}_X Y, \overline{\phi}E) - \overline{g}(\overline{\nabla}_Y X, \overline{\phi}E) \\ &= \overline{g}(\overline{\nabla}_Y \overline{\phi}X, E) - \overline{g}(\overline{\nabla}_X \overline{\phi}Y, E) \\ &= \overline{g}(h(\overline{\phi}X, Y) - h(X, \overline{\phi}Y), E) \\ &= B(X, \phi Y) - B(\phi X, Y). \end{aligned}$$

The Assertion follows from this equation. □

Theorem 4.13 *Let $\overline{M}$ be an indefinite Sasakian manifold and M be a $D \perp < \xi >$-totally geodesic lightlike hypersurface of $\overline{M}$ with $\xi \in TM$. Then $\overline{\phi}(TM^{\perp})$ is a $D \perp < \xi >$-Killing distribution.*

Proof. From expression (4.36) and the proof of Theorem 4.12, we have for any X, $Y \in \Gamma(D \perp < \xi >)$, $(L_V g)(X, Y) - (L_V g)(Y, X) = 2\left(u([X, Y]) - \overline{g}(h(X, \overline{\phi}Y) - h(\overline{\phi}X, Y), E)\right) = 0$. Applying the result of Proposition 4.10, we obtain $(L_V g)(X, Y) = (L_V g)(Y, X) = -(L_V g)(X, Y)$, that is $(L_V g)(X, Y) = 0$. That is, $\overline{\phi}(TM^{\perp})$ is a $D \perp < \xi >$-Killing distribution on M. □

It is well known that the second fundamental form and the shape operators of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.16) and (2.17), in the case of lightlike hypersurfaces, the second fundamental forms on M and their screen distribution $S(TM)$ are related to their respective shape operators A_N and A^*_E. As the shape operator is an information tool in studying the geometry of submanifolds, their studying turns out very important. Next, we study these operators and give their implications in lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$.

Lemma 4.14 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, for any $X \in \Gamma(TM)$*

$$B(X,U) = C(X,V) = u(A_N X), \tag{4.42}$$

$$B(X,V) = u(A^*_E X), \tag{4.43}$$

and $$\nabla^*_X V = \phi(A^*_E X) - C(X,V)E - \tau(X)V. \tag{4.44}$$

Proof. The equality $B(X,U) = C(X,V)$ is trivial and comes from the definition of B and the fact that the $\overline{\nabla}$ is a Levi-Civita connection:

$$\begin{aligned} C(X,V) &= g(A_N X, V) = u(A_N X), \\ \text{and} \quad B(X,V)N &= \overline{\nabla}_X V - \nabla_X V = -\overline{\nabla}_X \overline{\phi} E + \nabla_X \overline{\phi} E \\ &= -(\overline{\nabla}_X \overline{\phi})E - \overline{\phi}(\overline{\nabla}_X E) + \nabla^*_X \overline{\phi} E + C(X, \overline{\phi} E)E \\ &= -\overline{\phi}(\nabla_X E) + \nabla^*_X \overline{\phi} E + C(X, \overline{\phi} E)E \\ &= \overline{\phi}(A^*_E X) + \tau(X)\overline{\phi} E + \nabla^*_X \overline{\phi} E + C(X, \overline{\phi} E)E, \\ \text{that is,} \quad (B(X,V) - u(A^*_E X))\, N &= \phi(A^*_E X) - \tau(X)V + \nabla^*_X \overline{\phi} E + C(X, \overline{\phi} E)E. \end{aligned}$$

The right hand side of this equation belongs to $\Gamma(TM)$, while the left hand side belongs to $\Gamma(N(TM))$. So we have $B(X,V) = u(A^*_E X)$ and $\phi(A^*_E X) = \tau(X)V + \nabla^*_X V + C(X,V)E$, that is, $\nabla^*_X V = \phi(A^*_E X) - C(X,V)E - \tau(X)V$. □

Theorem 4.15 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, M is mixed totally geodesic if and only if, for any $X \in \Gamma(D \perp <\xi>)$,*

$$A_N X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp <\xi>). \tag{4.45}$$

Proof. By definition, M is mixed totally geodesic if and only if, for any $X \in \Gamma(D \perp <\xi>)$, $B(X,U) = 0$. From (4.42) we obtain $u(A_N X) = \overline{g}(A_N X, V) = 0$. i.e. $A_N X \in \Gamma(D \perp <\xi>)$. Since $g(A_N X, N) = 0$, that is, $A_N X$ has no component in $\Gamma(TM^\perp)$, so we have $A_N X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp <\xi>)$. The converse is clear. Thus x the proof. □

Theorem 4.16 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, M is $D \perp < \xi >$-totally geodesic if and only if, for any $X \in \Gamma(D \perp < \xi >)$,*

$$A^*_E X \in \Gamma(\overline{\phi}(TM^\perp)). \tag{4.46}$$

Proof. By the definition, M is $D \perp < \xi >$-totally geodesic if and only if, for any X, $Y \in \Gamma(D \perp < \xi >)$, $B(X,Y) = \overline{g}(h(X,Y),E) = 0$. In particular, for any $X \in \Gamma(D \perp < \xi >)$ and $Y = V$, $B(X,V) = 0$. From (4.43) we obtain $u(A^*_E X) = \overline{g}(A^*_E X, V) = 0$, i.e. $A^*_E X \in \Gamma(D \perp < \xi >)$. Since $g(A^*_E X, N) = 0$, that is, $A^*_E X$ has no component in $\Gamma(TM^\perp)$, so $A^*_E X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp < \xi >)$. If $A^*_E X \in \Gamma(D_0 \perp < \xi >)$ and given that $D_0 \perp < \xi >$ is nondegenerate, then there exists $Z \in \Gamma(D_0 \perp < \xi >)$ such that $\overline{g}(A^*_E X, Z) \neq 0$. From (2.11) and (2.17) we obtain

$$\begin{aligned} g(A^*_E X, Z) &= -\overline{g}(\overline{\nabla}_X E, Z) = -X.\overline{g}(E,Z) + \overline{g}(E, \overline{\nabla}_X Z) \\ &= \overline{g}(E, \nabla_X Z) + B(X,Z)\overline{g}(E,N) = 0. \end{aligned}$$

Thus $A^*_E X \notin \Gamma(D_0 \perp < \xi >)$. Finally $A^*_E X \in \Gamma(\overline{\phi}(TM^\perp))$. Conversely, suppose that, for any $X \in \Gamma(D \perp < \xi >)$, $A^*_E X \in \Gamma(\overline{\phi}(TM^\perp))$. Let $\mathcal{B}_{D\perp<\xi>} = \{E, \overline{\phi}E, \xi, F_i, i = 1, 2, ..., 2n-4\}$ be a local orthonormal field of frames of $D \perp < \xi >$ such that $D_0 = Span\{F_i, i = 1, 2, ..., 2n-4\}$. Now, we want to show that $B(X,.)$ vanishes in each element of $\mathcal{B}_{D\perp<\xi>}$. For any $X \in \Gamma(D \perp < \xi >)$ $(X = RX + \eta(X)\xi)$, $u(A^*_E X) = 0$, i.e. $B(X,V) = 0$. $B(X,\xi) = \overline{g}(\overline{\nabla}_X \xi, E) = -\overline{g}(\overline{\phi}X, E) = -\overline{g}(\overline{\phi}RX, E) = 0$, since D is invariant under $\overline{\phi}$. $B(X, F_i) = \overline{g}(\overline{\nabla}_X F_i, E) = -\overline{g}(F_i, \nabla_X E) = \overline{g}(F_i, A^*_E X) = 0$, since $D_0 \perp \overline{\phi}(TM^\perp)$. Let Y be an element of $\Gamma(D \perp < \xi >)$. Locally, we have $Y = \theta(Y)E + v(Y)V + \eta(Y)\xi + \sum_i \frac{\overline{g}(Y,F_i)}{\overline{g}(F_i,F_i)} F_i \in \Gamma(D \perp < \xi >)$, with $\overline{g}(F_i, F_i) \neq 0$ because of the nondegeneracy of D_0. So $B(X,Y) = \theta(Y)B(X,E) + v(Y)B(X,V) + \eta(Y)B(X,\xi) + \sum_i \frac{\overline{g}(Y,F_i)}{\overline{g}(F_i,F_i)} B(X, F_i) = 0$. Hence M is $D \perp < \xi >$-totally geodesic. □

The expressions of the shape operators A_N and A^*_E can be computed explicitly in the following way. According to decomposition (3.23), we consider a local field of frames on

M, i.e.

$$\{\overline{\phi}E,\, \overline{\phi}N,\, \xi,\, E,\, F_i\}_{1\leq i\leq 2n-4} \tag{4.47}$$

on $\mathcal{U} \subset M$, where $\{F_i\}_{1\leq i\leq 2n-4}$ is an orthonormal field of frames of D_0.

Lemma 4.17 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, for any $X \in \Gamma(TM)$,*

$$A_N X = \sum_{i=1}^{2n-4} \frac{C(X,F_i)}{g(F_i,F_i)} F_i + C(X,\xi)\xi + C(X,U)V + C(X,V)U, \tag{4.48}$$

$$\text{and} \quad A_E^* X = \sum_{i=1}^{2n-4} \frac{B(X,F_i)}{g(F_i,F_i)} F_i + B(X,\xi)\xi + B(X,U)V + B(X,V)U. \tag{4.49}$$

Proof. From the definition of lightlike hypersurface of an indefinite Sasakian manifold through the local field of frames (4.47), we have, for any $X \in \Gamma(TM)$, $A_N X = \sum_{i=1}^{2n-4} \lambda_i\, F_i + \gamma\, \xi + \delta\, E + \alpha\, \overline{\phi}E + \beta\, \overline{\phi}N$. From (2.12) and (2.17) we obtain $\lambda_i\, g(F_i,F_i) = g(A_N X, F_i) = C(X,F_i)$. Since D_0 is nondegenerate distribution on M, $g(F_i,F_i) \neq 0$ and we have $\lambda_i = \frac{C(X,F_i)}{g(F_i,F_i)}$, and $\gamma = g(A_N X,\xi) = \eta(A_N X) = C(X,\xi)$, $\delta = g(A_N X, N) = 0$, $\alpha = -g(A_N X, U) = -C(X,U)$, $\beta = -g(A_N X, V) = -C(X,V)$, which prove (4.48). Similarly we obtain (4.49). □

Theorem 4.18 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, M is $D \perp< \xi >$-totally geodesic if and only if, for any $X \in \Gamma(D \perp< \xi >)$,*

$$A_E^* X = u(A_N X)V. \tag{4.50}$$

Proof. The proof follows from the Theorem 4.16 and the expression (4.49). □

Lemma 4.19 *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, for any $X \in \Gamma(TM)$,*

$$\eta(A_N X) = -v(X), \tag{4.51}$$

$$\text{and} \quad \eta(A_E^* X) = -u(X). \tag{4.52}$$

Proof. With the aid of $\overline{\nabla}_X\xi = -\overline{\phi}X$, we have, for any $X \in \Gamma(TM)$, $\eta(A_N X) = \overline{g}(N, \overline{\nabla}_X\xi) = -\overline{g}(N, \overline{\phi}X)$ and $\eta(A^*_E X) = \overline{g}(E, \overline{\nabla}_X\xi) = -\overline{g}(E, \overline{\phi}X)$. □

It is also well known, in general, that if the lightlike hypersurface (M, g) is totally geodesic, from the Theorem 2.2, the induced connection ∇ on M is torsion-free and g-metric, and, at the same time, the other items of the theorem 2.2 are satisfied too. But, if the lightlike hypersurface M with $\xi \in TM$ is $D \perp <\xi>$-totally geodesic, one of the equivalences in Theorem 2.2 is not satisfied in the direction of the distribution $D \perp <\xi>$ (Theorem 4.18), for instance. We also know that, in general, the induced connection, say ∇, on M is not a Levi-Civita connection and depends on both g and a screen distribution $S(TM)$ of M. This means that only some privileged conditions on the screen distribution of M could allow one to obtain a $D \perp <\xi>$-version of the Theorem 2.2.

Now, we propose a way to heal this missing gap by using the following concept. Say that the screen distribution $S(TM)$ is totally umbilical if on any coordinates neighborhood $\mathcal{U} \subset M$, there exists a smooth function ρ such that

$$C(X, PY) = \rho g(X, PY), \ \ \forall\, X, Y \in \Gamma(TM_{|\mathcal{U}}). \tag{4.53}$$

If we assume that the screen distribution $S(TM)$ of the lightlike hypersurface M with $\xi \in TM$ is totally umbilical, then it follows that C is symmetric on $\Gamma(S(TM)_{|\mathcal{U}})$ and hence according to Theorem 2.3 in [11], the distribution $S(TM)$ is integrable. From the definition of the second fundamental form C and (4.53), we obtain

$$A_N X = \rho PX \ \text{ and } \ C(E, PX) = 0, \forall\, X \in \Gamma(TM_{|\mathcal{U}}). \tag{4.54}$$

Since $\overline{\phi}\xi = 0$, and by using (4.51), we have $\eta(A_N\xi) = \rho\overline{g}(\xi,\xi) = -v(\xi) = 0$, which implies $\rho = 0$, that is the screen distribution $S(TM)$ is totally geodesic. We now have the following theorem.

Theorem 4.20 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$, with $\xi \in TM$, such that $S(TM)$ is totally umbilical. Then M is $D \perp <\xi>$-totally geodesic if and only if the distribution $TM^\perp$ is $D \perp <\xi>$-parallel.*

Proof. Since the screen distribution $S(TM)$ is totally umbilical, $S(TM)$ is totally geodesic, that is, for any $X, Y \in \Gamma(S(TM))$, $C(X, Y) = 0$. In particular, for any $X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp <\xi>)$, $C(X, V) = u(A_N X) = 0$. From the second equation of (4.54), $C(E, V) = 0$, then, for any $X_0 \in \Gamma(D \perp <\xi>)$, $u(A_N X_0) = 0$. From the Theorem 4.18, M is $D \perp <\xi>$-totally geodesic if and only if, for any $X_0 \in \Gamma(D \perp <\xi>_{|\mathcal{U}})$,

$A^*_E X_0 = 0$. To complete the proof of this Theorem, we need the following result. □

Lemma 4.21 *For any* $X_0 \in \Gamma(D \perp < \xi >_{|\mathcal{U}})$, $A^*_E X_0 = 0$ *if and only if* $\nabla_{X_0} Y_0 \in \Gamma(TM^\perp)$, *for any* $Y_0 \in \Gamma(TM^\perp)$.

Proof. Suppose, for any $X_0 \in \Gamma(D \perp < \xi >_{|\mathcal{U}})$, $A^*_E X_0 = 0$. Since the normal bundle $TM^\perp$ is a distribution on M of rank 1 and spanned by E, then, by straightforward calculation, for any $Y_0 = \theta(Y_0)E \in \Gamma(TM^\perp)$, $\nabla_{X_0} Y_0 = (X_0.\theta(Y_0) - \theta(Y_0)\tau(X_0))E \in \Gamma(TM^\perp)$. So, the distribution $TM^\perp$ is $D \perp < \xi >$-parallel. Conversely, suppose the distribution $TM^\perp$ is $D \perp < \xi >$-parallel. Then, for any $X_0 \in \Gamma(D \perp < \xi >)$ and $Y_0 \in \Gamma(TM^\perp_{|\mathcal{U}})$, $\nabla_{X_0} Y_0 \in \Gamma(TM^\perp_{|\mathcal{U}})$. Since $TM^\perp$ is spanned by E, there exist a smooth functions on M $\lambda \neq 0$ such that $\nabla_{X_0} Y_0 = \lambda E$. We have $\lambda = g(\nabla_{X_0} Y_0, N) = \overline{g}(\overline{\nabla}_{X_0} \theta(Y_0)E, N) = X_0.\theta(Y_0) - \theta(Y_0)\tau(X_0)$. On the other hand, $\nabla_{X_0} Y_0 = (X_0.\theta(Y_0) - \theta(Y_0)\tau(X_0))E - A^*_E X_0$. So, we have

$$\nabla_{X_0} Y_0 = (X_0.\theta(Y_0) - \theta(Y_0)\tau(X_0))E - A^*_E X_0 = (X_0.\theta(Y_0) - \theta(Y_0)\tau(X_0))E,$$

that is, $A^*_E X_0 = 0$, for any $X_0 \in \Gamma(D \perp < \xi >)$. This completes the proof. □

We can now state the $D \perp < \xi >$-version of the Theorem 2.2 on the $D \perp < \xi >$-geodesibility of M.

Theorem 4.22 *Let* $(M, g, S(TM))$ *be a lightlike hypersurface of an indefinite Sasakian manifold* $(\overline{M}, \overline{g})$, *with* $\xi \in TM$, *such that* $S(TM)$ *is totally umbilical. Then the following assertions are equivalent:*

(i) M *is* $D \perp < \xi >$*-totally geodesic.*

(ii) h *(or equivalently* B*) vanishes identically on* M *in the direction of the* $D \perp < \xi >$.

(iii) $A^*_W X = 0$, *for any* $W \in \Gamma(TM^\perp)$ *and* $X \in \Gamma(D \perp < \xi >)$.

(iv) *The connection* $\widehat{\nabla} = \nabla|_{D \perp < \xi >}$ *induced by* $\overline{\nabla}$ *on* M *is torsion-tree and metric.*

(v) $TM^\perp$ *is a* $D \perp < \xi >$*-parallel distribution with respect to* ∇.

(vi) $TM^\perp$ *is a* $D \perp < \xi >$*-killing distribution on* M.

As the geometry of a lightlike hypersurface depends on the chosen screen distribution, it is important to investigate the relationship between geometrical objects induced, studied above, by two screen distributions. In this case, it is well known that the local second fundamental form of M on $\mathcal{U}$ is independent of the choice of the screen distribution [11].

Recall the following four local transformation equations (see [11], page 87) of a change in $S(TM)$ to another screen $S(TM)'$:

$$W_i' = \sum_{j=1}^{2n-1} W_i^j (W_j - \epsilon_j c_j E), \tag{4.55}$$

$$N' = N - \frac{1}{2}\left\{\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\right\} E + \sum_{i=1}^{2n-1} c_i W_i, \tag{4.56}$$

$$\tau'(X) = \tau(X) + B(X, N' - N), \tag{4.57}$$

$$\nabla_X' Y = \nabla_X Y + B(X,Y)\left\{\frac{1}{2}\left(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\right) E - \sum_{i=1}^{2n-1} c_i W_i\right\}, \tag{4.58}$$

where $\{W_i\}$ and $\{W_i'\}$ are the local orthonormal basis of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E. c_i and W_i^j are smooth functions, and $\{\epsilon_1, ..., \epsilon_{2n-1}\}$ is the signature of the base $\{W_1, ..., W_{2n-1}\}$. The relationship between the second fundamental forms C and C' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by (using (4.56) and (4.58)) the relation

$$\begin{aligned} C'(X, PY) &= C(X, PY) - \frac{1}{2}\|W\|^2 B(X, Y) + g(\nabla_X PY, W) \\ &= C(X, PY) - \frac{1}{2} g(\nabla_X PY + B(X, Y)W, W) \\ &= C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W), \end{aligned} \tag{4.59}$$

where $W = \sum_{i=1}^{2n-1} c_i W_i$ is the characteristic vector field of the screen change and ω is the dual 1-form of W with respect to the induced metric g of M, that is $\omega(X) = g(X, W), \ \forall X \in \Gamma(TM)$. Therefore Theorem 4.20 and Theorem 4.22 are independent of the screen distribution $S(TM)$ if and only if

$$\omega(\nabla_X PY + B(X, Y)W) = 0, \quad \forall\, X,\ Y \in \Gamma(TM). \tag{4.60}$$

On the other hand, the Theorem 4.18, the expression (4.42) and (4.43) are independent of the screen distribution $S(TM)$.

Before we discuss about the effect of the change of the screen distribution on the Lie derivative (4.36), we need the following Lemma.

Lemma 4.23 *The Lie derivative (4.36) is rewritten as*

$$(L_V g)(X,Y) = (\nabla_X u)Y + (\nabla_Y u)X, \ \ \forall X,\, Y \in \Gamma(TM). \tag{4.61}$$

Proof. By straightforward calculation, we have

$$\begin{aligned}(\nabla_X u)Y + (\nabla_Y u)X &= X.u(Y) + Y.u(X) - u(\nabla_X Y) - u(\nabla_Y X)\\ &= X.u(Y) + Y.u(X) + u([X,Y]) - 2u(\nabla_X Y).\end{aligned}$$

¿From (4.36) and (4.37), we complete the proof. □

First we ask the following question: Is the Lie derivative L_V (4.36) independent of the choice of a screen distribution $S(TM)$? The answer is negative. Indeed, we prove the following with respect to a change in $S(TM)$.

Proposition 4.24 *The Lie derivatives L_V and L'_V of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related through the relation:*

$$(L'_V g)(X,Y) = (L_V g)(X,Y) - B(u(X)Y + u(Y)X, W). \tag{4.62}$$

Proof. Using (3.34), (4.57) and the fact that the local second fundamental form is independent of the choice of a screen distribution, we get

$$\begin{aligned}(L'_V g)(X,Y) &= (\nabla'_X u)Y + (\nabla'_Y u)X\\ &= -B'(X,\phi Y) - B'(Y,\phi X) - u(X)\tau'(Y) - u(Y)\tau'(X)\\ &= -B(X,\phi Y) - B(Y,\phi X) - u(X)\tau(Y) - u(X)B(Y,W)\\ &\quad -u(Y)\tau(X) - u(Y)B(X,W)\\ &= (L_V g)(X,Y) - u(X)B(Y,W) - u(Y)B(X,W)\\ &= (L_V g)(X,Y) - B(u(X)Y + u(Y)X, W),\end{aligned}$$

which is the desired formula. □

¿From this Proposition, we have the following result.

Theorem 4.25 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the Lie derivative L_V is unique, that is, L_V is independent of $S(TM)$, if and only if, the second fundamental form h (or equivalently B) of M vanishes identically on M.*

Proof. The proof follows form (4.58) and Theorem 2.2. □

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