A Condition for Warped Product Semi-Invariant Submanifolds to be Riemannian Product Semi-Invariant Submanifolds in Locally Riemannian Product Manifolds

Mehmet Atçeken

Abstract

In this article, we give a necessary and sufficient condition for warped product semi-invariant submanifolds to be Riemannian product semi-invariant submanifolds in a locally Riemannian product manifold whose factor manifolds are real space form.

Key Words: Real space form, Riemannian product and Warped product.

1. Introduction

It is well-known that the notion of warped products plays some important role in differential geometry as well as physics. The geometry of warped product was introduced by B.Y. Chen and it has been studied in the different manifold types by many geometers[see references].

Recently, B.Y. Chen have introduced the notion of CR-warped product in Kaehlerian manifolds and showed that there exist no warped product CR-submanifolds in the form $M = M_\perp \times_f M_T$ in Kaehlerian manifolds. Therefore, he considered warped product CR-submanifolds in the form $M = M_T \times_f M_\perp$, which is called CR-warped product, where

M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of Kaehlerian manifold $\bar{M}$[5].

We note that CR-warped products in Kaehlerian manifold correspond to semi-invariant warped products in Riemannian product manifolds. Recently, we showed that there exist no warped product semi-invariant submanifolds in the form $M = M_T \times_f M_\perp$ in contrast to Kaehlerian manifolds [1, 5]. So, in the remainder of this paper we consider warped product semi-invariant submanifolds in the form $M = M_\perp \times_f M_T$, where $M_\perp$ is an anti-invariant submanifold and M_T is an invariant submanifold of Riemannian product manifold M and it is called warped product semi-invariant submanifold in the rest of this paper.

2. Preliminaries

Let $\bar{M}$ be a Riemannian manifold and M be an isometrically immersed submanifold in $\bar{M}$. Then the formulas of Gauss and Weingarten for submanifold M in $\bar{M}$ are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V, \tag{2}$$

for any vector fields X, Y tangent to M and V normal to M, where $\bar{\nabla}$, ∇ denote the Levi-Civita connections on $\bar{M}$, M, respectively, $\nabla^\perp$ is the normal connection in $TM^\perp$, A_V is the shape operator of M with respect to V and h is also the second fundamental form of M. Moreover, the second fundamental form h and shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), V), \tag{3}$$

where g denotes the Riemannian metric on M as well as $\bar{M}$.

Moreover, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned}
\bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\bar{\nabla}_X h)(Y, Z) \\
&\quad - (\bar{\nabla}_Y h)(X, Z)
\end{aligned} \tag{4}$$

and

$$\{\bar{R}(X,Y)Z\}^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{5}$$

for any X, Y, Z and W tangent to M, where $\bar{R}$ and R denote the Riemannian curvature tensors of $\bar{M}$ and M, respectively, and $\{\bar{R}(X,Y)Z\}^{\perp}$ is the normal component of $\bar{R}(X,Y)Z$.

The covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{6}$$

Now, let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and $f > 0$ be a smooth function on M_1. We consider the product manifold $M_1 \times M_2$ with its projections $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\eta : M_1 \times M_2 \longrightarrow M_2$. The warped product $M = M_1 \times_f M_2$ is a manifold $M_1 \times M_2$ equipped with the Riemannian metric tensor such that

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (f o \pi)^2 g_2(\eta_* X, \eta_* Y) \tag{7}$$

for any $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ means the set of all differentiable vector fields on M and $*$ the symbol stands for differential. Thus we have $g = g_1 + f^2 g_2$.

The function f is called warping function of the warped product manifold $M = M_1 \times_f M_2$. If warping function is constant, then warped product is called Riemannian product. Furthermore, if we denote the Levi-Civita connection on M by ∇, then we have the following proposition for the warped product manifold.

Proposition 2.1 *Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in \Gamma(TM_1)$ and $V, W \in \Gamma(TM_2)$, then we have*

(1) $\nabla_X Y \in \Gamma(TM_1)$ *is the lift of $\nabla_X Y$ on M_1; and*

(2) $\nabla_X V = \nabla_V X = X(\ln f)V$;

(3) $\nabla_V W = \nabla_V^{M_2} W - \frac{g(W,V)}{f} grad f$

where, ∇^{M_2} is the Levi-Civita connection on M_2[7].

Let $\bar{M}$ be a m-dimensional Riemannian manifold with Riemannian metric g and $\{e_1, e_2, ..., e_m\}$ be an orthonormal frame on $\bar{M}$. For a smooth function ϕ on $\bar{M}$, the Gradient gradϕ, the Hessian H^{ϕ} and the Laplacian $\triangle \phi$ of ϕ are defined, respectively, by

$$g(\mathrm{grad}\phi, X) = X\phi \tag{8}$$

$$H^\phi(X,Y) = XY\phi - (\nabla_X Y)\phi = g(\nabla_X \mathrm{grad}\phi, Y) \tag{9}$$

and

$$\triangle\phi = \sum_{k=1}^{m}\{(\nabla_{e_k}e_k)\phi - e_k e_k\phi\} \tag{10}$$

for any vector fields X, Y tangent to $\bar{M}$. From (9) and (10), it is easily seen that the Laplacian is negative of the Hessian as a specific case.

Let $\bar{M}$ be a compact orientable Riemannian manifold without boundary. Then we have

$$\int_{\bar{M}} \triangle\phi dV = 0, \tag{11}$$

where dV is the volume element of $\bar{M}$[10].

A real space form is a connected Riemannian manifold of constant sectional curvature c, denoted by $\bar{M}(c)$. Then the Riemannian curvature tensor of $\bar{M}(c)$ is given by

$$R(X,Y)Z = c\{g(X,Z)Y - g(Y,Z)X\} \tag{12}$$

for any vector fields X, Y, Z tangent to $\bar{M}$.

Let $\bar{M}$ be an m-dimensional manifold with a tensor F of type (1,1) such that $F^2 = I$, $F \neq \mp I$ then $\bar{M}$ is said to be an almost product manifold with almost product structure F. If an almost product manifold M has a Riemannian metric g such that $g(FX,Y) = g(X,FY)$, for any $X, Y \in \Gamma(T\bar{M})$, then $\bar{M}$ is called an almost Riemannian product manifold. We denote the Levi-Civita connection on $\bar{M}$ by ∇ with respect to g. If $(\bar{\nabla}_X F)Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$, then $\bar{M}$ is called a locally Riemannian product manifold [10].

Let $\bar{M}$ be a Riemannian manifold with almost Riemannian product structure F and M be an isometrically immersed submanifold in $\bar{M}$. For each $x \in M$, we denote by D_x the maximal invariant subspace of the tangent space $T_x M$ of M. If the dimension of D_x is the same for all x in M, then D_x gives an invariant distribution on M.

Definition 2.1 *M is called a semi-invariant submanifold of a locally Riemannian product manifold $\bar{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that*

1) TM has the orthogonal direct sum $TM = D_1 \oplus D_2$;

2) The distribution D_1 is invariant, i.e., $F(D_1) = D_1$;

3) The distribution D_2 is anti-invariant, i.e., $F(D_2) \subset TM^\perp$.

Furthermore, let $\bar{M} = \bar{M}_1 \times \bar{M}_2$ be a locally Riemannian product manifold. If the factor manifolds $\bar{M}_2$ and $\bar{M}_2$ have constant sectional curvatures c_1 and c_2, denoted $\bar{M}_1(c_1)$ and $\bar{M}_2(c_2)$, respectively, then the Riemannian curvature tensor of locally Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$ is given by

$$
\begin{aligned}
R(X,Y)Z &= \frac{1}{4}(c_1 + c_2)\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX - g(FX,Z)FY\} \\
&+ \frac{1}{4}(c_1 - c_2)\{g(Y,Z)FX - g(X,Z)FY + g(FY,Z)X - g(FX,Z)Y\} \quad (13)
\end{aligned}
$$

for any vector fields X, Y, Z tangent to $\bar{M}$[10].

Let M be a semi-invariant submanifold of a locally Riemannian product manifold $\bar{M}$. From the definition we have

$$TM = D \oplus D^\perp. \tag{14}$$

We denote the orthogonal complementary subbundle of $F(D^\perp)$ in $TM^\perp$ by ν, then we have direct sum

$$TM^\perp = F(D^\perp) \oplus \nu. \tag{15}$$

It is easily to see that ν is an invariant subbundle of $\Gamma(T\bar{M})$ with respect to F. Moreover, for any vector field X tangent to M we put

$$FX = TX + \omega X, \tag{16}$$

where TX and ωX are the tangent part and normal part of FX, respectively. Also, for any vector field V normal to M we put

$$FV = BV + CV, \tag{17}$$

where BV and CV are the tangent part and normal part of FV, respectively. According to the decomposition (15), we consider the projections P_1 and P_2 of $TM^\perp$ on $F(D^\perp)$ and ν, respectively, then we can write the second fundamental form h of M as

$$h(X,Y) = h_1(X,Y) + h_2(X,Y) \tag{18}$$

for any $X, Y \in \Gamma(TM)$, where we have put

$$h_1(X, Y) = P_1 h(X, Y) \ \ and \ \ h_2(X, Y) = P_2 h(X, Y).$$

3.   Warped Product Submanifolds In Locally Riemannian Product Manifolds

Next we will give the following lemma for later use.

Lemma 3.1 *Let $M = M_\perp \times_f M_T$ be a warped product submanifold of a locally Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. Then we have*

$$\frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 \ = \ \|h(X, Y)\|^2 - H^{\ln f}(Y, Y)\|X\|^2 - \|X\|^2(Y \ln(f))^2$$
$$- \ g(h(X, X), h(Y, Y)) \qquad (19)$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

Proof.   From (5), Proposition 2.1(3) and Levi-Civita connection $\overline{\nabla}$ connection, we have

$$
\begin{aligned}
g(\bar{R}(X, Y)FX, FY) \ &= \ g((\bar{\nabla}_X h)(Y, FX) - (\bar{\nabla}_Y h)(X, FX), FY) \\
&= \ g(\nabla_X^\perp h(Y, FX) - h(\nabla_X Y, FX) - h(\nabla_X FX, Y), FY) \\
&- \ g(\nabla_Y^\perp h(X, FX) - h(\nabla_Y X, FX) - h(\nabla_Y FX, X), FY) \\
&= \ g(\bar{\nabla}_X h(Y, FX), FY) - g(h(\nabla_X Y, FX), FY) \\
&- \ g(h(\nabla_X FX, Y), FY) - g(\bar{\nabla}_Y h(X, FX), FY) \\
&+ \ g(h(\nabla_Y X, FX), FY) + g(h(\nabla_Y FX, X), FY) \\
&= \ Xg(h(Y, FX), FY) - g(h(Y, FX), \bar{\nabla}_X FY) \\
&- \ g(h(\nabla_X FX, Y), FY) - Yg(h(X, FX), FY) \\
&+ \ g(h(X, FX), \bar{\nabla}_Y FY) + g(h(\nabla_Y FX, X), FY) \qquad (20)
\end{aligned}
$$

for any $X \in \Gamma(TM_T)$ and $Y \in \Gamma(TM_\perp)$.

On the other hand, taking account of $\bar{M}$ being a locally Riemannian product manifold and $M = M_\perp \times_f M_T$ is a warped product, then we have

$$
\begin{aligned}
g(h(Y, FX), FY) \ &= \ g(\bar{\nabla}_Y FX, FY) = g(\bar{\nabla}_Y X, Y) = g(\nabla_Y X, Y) \\
&= \ Y \ln f g(X, Y) = 0 \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
g(h(X, FX), FY) &= g(\bar\nabla_X FX, FY) = g(\nabla_X X, Y) = -g(\nabla_X Y, X) \\
&= -Y \ln f g(X, X).
\end{aligned} \tag{22}
$$

Moreover, considering the ambient space is locally Riemannian product, with Proposition 2.1(3), and using $M_\perp$, is totally geodesic in M we get

$$
\begin{aligned}
g(h(X, FX), F\bar\nabla_Y Y) &= g(h(X, FX), Fh(Y, Y) + F\nabla_Y Y) \\
&= g(Fh(X, FX), h(Y, Y)) + g(Fh(X, FX), \nabla_Y Y) \\
&= g(F\bar\nabla_X FX - F\nabla_X FX, h(Y, Y)) \\
&+ g(F\bar\nabla_X FX - F\nabla_X FX, \nabla_Y Y) \\
&= g(\bar\nabla_X X, h(Y, Y)) + g(X, FX)g(\frac{F\operatorname{grad}f}{f}, h(Y, Y)) \\
&+ g(\bar\nabla_X X, \nabla_Y Y) - g(F\nabla_X FX, \nabla_Y Y) \\
&= g(h(Y, Y), h(X, X)) + \frac{1}{f}g(X, FX)g(h(Y, Y), F\operatorname{grad}f) \\
&+ g(\nabla_X X, \nabla_Y Y) + \frac{1}{f}g(X, FX)g(\operatorname{grad}f, F\nabla_Y Y) \\
&= g(h(X, X), h(Y, Y)) + \frac{1}{f}g(X, FX)g(h(Y, Y), F\operatorname{grad}f) \\
&- \frac{1}{f}g(X, X)g(\nabla_Y Y, \operatorname{grad}f).
\end{aligned} \tag{23}
$$

Furthermore, using (1), (2), taking account of A and h being symmetric, we infer

$$
\begin{aligned}
g(h(Y, Y), \operatorname{grad}f) &= g(\bar\nabla_Y Y, F\operatorname{grad}f) = g(\bar\nabla_Y FY, \operatorname{grad}f) = -g(A_{FY} Y, \operatorname{grad}f) \\
&= Y(Y, F\operatorname{grad}f) - g(Y, \bar\nabla_Y F\operatorname{grad}f) = -g(\bar\nabla_Y \operatorname{grad}f, FY) \\
&= -g(h(Y, \operatorname{grad}f), FY) = -g(\bar\nabla_{\operatorname{grad}f} FY, Y) \\
&= g(A_{FY}\operatorname{grad}f, Y),
\end{aligned} \tag{24}
$$

which implies that

$$
g(h(Y, Y), F\operatorname{grad}f) = 0. \tag{25}
$$

Thus equation (23) becomes

$$
\begin{aligned}
g(h(X, FX), F\bar{\nabla}_Y Y) &= g(h(X, X), h(Y, Y)) - g(X, X)g(\nabla_Y Y, \frac{\mathrm{grad}f}{f}) \\
&= g(h(X, X), h(Y, Y)) - g(X, X)(\nabla_Y Y) \ln f. \quad (26)
\end{aligned}
$$

In the (20), our assertion becomes

$$
\begin{aligned}
g(h(Y, FX), \bar{\nabla}_X FY) &= g(h(Y, FX), Fh(X, Y) + F\nabla_X Y) \\
&= g(h(Y, FX), Fh(X, Y)) + g(h(Y, FX), Y \ln f(FX)) \\
&= g(Fh(Y, FX), h(X, Y)) \\
&= g(F\bar{\nabla}_Y FX - F\nabla_Y FX, h(X, Y)) \\
&= g(\bar{\nabla}_Y X, h(X, Y)) - Y \ln f g(X, h(X, Y)) \\
&= \|h(X, Y)\|^2. \quad (27)
\end{aligned}
$$

Also, considering (21) and Proposition 2.1(3), we arrive at

$$
\begin{aligned}
g(h(\nabla_X FX, Y), FY) &= g(h(\nabla_X^{M_T} FX - \frac{\mathrm{grad}f}{f}g(X, FX), Y), FY) \\
&= g(h(\nabla_X^{M_T} FX, Y), FY) \\
&\quad - g(X, FX)\frac{1}{f}g(h(\mathrm{grad}f, Y), FY) \\
&= -g(X, FX)\frac{1}{f}g(h(\mathrm{grad}f, Y), FY), \quad (28)
\end{aligned}
$$

and here, taking account of (25) we get

$$
\begin{aligned}
g(h(\mathrm{grad}f, Y), FY) &= g(\bar{\nabla}_Y \mathrm{grad}f, FY) = -g(\bar{\nabla}_Y FY, \mathrm{grad}f) \\
&= -g(\bar{\nabla}_Y Y, F\mathrm{grad}f) = -g(h(Y, Y), F\mathrm{grad}f) = 0. \quad (29)
\end{aligned}
$$

356

Substituting these equations into (20), consider (22) again and using (8) and (9), we find

$$
\begin{aligned}
g(\bar{R}(X,Y)FX, FY) &= -\|h(X,Y)\|^2 + Y((Y \ln f)g(X,X)) \\
&+ g(h(X,X), h(Y,Y)) - g(X,X)(\nabla_Y Y) \ln f \\
&+ Y \ln f g(h(X,FX), FY) \\
&= -\|h(X,Y)\|^2 + Y(Y \ln f g(X,X)) \\
&+ g(h(X,X), h(Y,Y)) - g(X,X)(\nabla_Y Y) \ln f \\
&- (Y \ln f)^2 g(X,X) \\
&= -\|h(X,Y)\|^2 + Y(Y \ln f)g(X,X) + Y(g(X,X))Y \ln f \\
&+ g(h(X,X), h(Y,Y)) - (\nabla_Y Y) \ln f g(X,X) \\
&- (Y \ln f)^2 g(X,X) \\
&= -\|h(X,Y)\|^2 + Y(Y \ln f)g(X,X) + 2Y \ln f g(\nabla_Y X, X) \\
&+ g(h(X,Y), h(X,Y)) - (\nabla_Y Y) \ln f g(X,X) \\
&- (Y \ln f)^2 g(X,X) \\
&= -\|h(X,Y)\|^2 + Y(Y \ln f)g(X,X) + (Y \ln f)^2 g(X,X) \\
&+ g(h(X,X), h(Y,Y)) - (\nabla_Y Y) \ln f g(X,X) \\
&= -\|h(X,Y)\|^2 + H^{\ln f}(Y,Y)g(X,X) + (Y \ln f)^2 g(X,X) \\
&+ g(h(X,X), h(Y,Y)). \tag{30}
\end{aligned}
$$

Taking account (13) and (30), we conclude that

$$
\begin{aligned}
&- \frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 - \frac{1}{4}(c_1 - c_2)\|Y\|^2 g(FX, X) = -\|h(X,Y)\|^2 \\
&+ H^{\ln f}(Y,Y)g(X,X) + (Y \ln f)^2 g(X,X) + g(h(X,X), h(Y,Y)). \tag{31}
\end{aligned}
$$

Here $g(FX, X) = 0$ can be choosen. Because, in the locally Riemannian product

357

manifolds, choosing the vector fields $\pi_* X$ and $\eta_* X$ which have the same length, then from (7), X and FX be orthogonal vector which gives our assertion and (31) becomes

$$
\begin{aligned}
\|h(X,Y)\|^2 &= \frac{1}{4}(c_1 + c_2)\|X\|^2\|Y\|^2 + H^{\ln f}(Y,Y)g(X,X) \\
&+ (Y\ln f)^2 g(X,X) + g(h(X,X), h(Y,Y)).
\end{aligned}
\tag{32}
$$

This completes proof of the lemma. $\qquad\qquad\square$

Theorem 3.1 *Let* $M = M_\perp \times_f M_T$ *be a compact orientiable warped product semi-invariant submanifold in a locally Riemannian product manifold*
$\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. *Then* M *is a Riemannian product submanifold if and only if*

$$
\sum_{j=1}^{q}\sum_{i=1}^{p}\sum_{k=1}^{s}(h_{ij}^k)^2 = \sum_{j=1}^{q}\sum_{i=1}^{p}\sum_{k=1}^{s}h_{ii}^k h_{jj}^k + \frac{pq}{4}(c_1 + c_2),
\tag{33}
$$

where $h_{ij}^k = g(h(e^j, e_i), N_k)$ *and* $\{e_1, e_2, ..., e_p\}$, $\{e^1, e^2, ..., e^q\}$ *and* $\{N_1, N_2, ..., N_s\}$ *are orthonormal bases for* $\Gamma(TM_\perp)$, $\Gamma(TM_T)$ *and* $\Gamma(\nu)$, *respectively.*

Proof. From (10) and (9), respectively, we have

$$
\triangle \ln f = -\sum_{i=1}^{p} g(\nabla_{e_i}\operatorname{grad}\ln f, e_i) - \sum_{j=1}^{q} g(\nabla_{e^j}\operatorname{grad}\ln f, e^j).
\tag{34}
$$

Making use of ∇ being Levi-Civita connection and (9) we have

$$
\begin{aligned}
H^{\ln f}(e_i, e_i) &= e_i(e_i \ln f) - (\nabla_{e_i} e_i)\ln f \\
&= e_i g(e_i, \operatorname{grad}\ln f) - g(\nabla_{e_i} e_i, \operatorname{grad}\ln f) \\
&= g(\nabla_{e_i} e_i, \operatorname{grad}\ln f) + g(\nabla_{e_i}\operatorname{grad}\ln f, e_i) - g(\nabla_{e_i} e_i, \operatorname{grad}\ln f) \\
&= g(\nabla_{e_i}\operatorname{grad}\ln f, e_i).
\end{aligned}
\tag{35}
$$

Moreover, taking account of $\operatorname{grad} f \in \Gamma(TM_\perp)$, Proposition 2.1(3) and consider (34), (35) we arrive at

358

$$\triangle \ln f = -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{q} g(\nabla_{e^j} \operatorname{grad} \ln f, e^j)$$

$$= -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{q} \{e^j g(\operatorname{grad} \ln f, e^j) - g(\operatorname{grad} \ln f, \nabla_{e_j} e^j)\}$$

$$= -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{q} \{e^j \frac{g(\operatorname{grad} f, e^j)}{f} - \frac{g(\operatorname{grad} f, \nabla_{e_j} e^j)}{f}\}$$

$$= -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{q} \{0 - \frac{g(\operatorname{grad} f, -g(e^j, e^j)\frac{1}{f}\operatorname{grad} f)}{f}\}$$

$$= -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \frac{1}{f^2} \sum_{j=1}^{q} g(e^j, e^j) g(\operatorname{grad} f, \operatorname{grad} f)$$

$$= -\sum_{i=1}^{p} H^{\ln f}(e_i, e_i) - \frac{q}{f^2} \|\operatorname{grad} f\|^2. \tag{36}$$

Now, let $\{Fe_1, Fe_2, ..., Fe_p, N_1, N_2, ..., N_s\}$ be an orthonormal basis for $TM^{\perp} = F(TM_{\perp}) \oplus \nu$ such that $\{Fe_1, Fe_2, ..., Fe_p\}$ and $\{N_1, N_2, ..., N_s\}$ are orthonormal bases for $F(TM_{\perp})$ and ν, respectively. By direct calculations, we have

$$h(X, X) = \sum_{i=1}^{p} g(h(X, X), Fe_i)Fe_i + \sum_{j=1}^{s} g(h(X, X), N_j)N_j$$

$$= -\sum_{i=1}^{p} g(X, FX)(e_i \ln f)Fe_i + \sum_{j=1}^{s} g(h(X, X), N_j)N_j \tag{37}$$

$$h(Y, Y) = \sum_{k=1}^{p} g(h(Y, Y), Fe_k)Fe_k + \sum_{l=1}^{s} g(h(Y, Y), N_l)N_l. \tag{38}$$

Here, since A is self-adjoint, we have

$$g(h(Y, Y), Fe_k) = g(\bar{\nabla}_Y Y, Fe_k) = g(\bar{\nabla}_Y FY, e_k) = -g(A_{FY}Y, e_k)$$

$$= -g(h(e_k, Y), FY) = -g(\bar{\nabla}_{e_k} Y, FY) = -g(\bar{\nabla}_{e_k} FY, Y)$$

$$= g(A_{FY}e_k, Y) = 0,$$

359

that is, $h(Y, Y)$ has no component in $F(TM_\perp)$. Thus we have

$$h(Y, Y) = \sum_{l=1}^{s} g(h(Y, Y), N_l)N_l. \tag{39}$$

Thus from (37) and (39), we have

$$
\begin{aligned}
g(h(X, X), h(Y, Y)) &= \sum_{k=1}^{s}\sum_{j=1}^{s} g(h(X, X), N_j)g(h(Y, Y), N_k)g(N_j, N_k) \\
&= \sum_{k=1}^{s} g(h(X, X), N_k)g(h(Y, Y), N_k) \\
&= \sum_{k=1}^{s} g(A_{N_k}X, X)g(A_{N_k}Y, Y). \tag{40}
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
\|h(X, Y)\|^2 &= \sum_{i=1}^{p} g(h(X, Y), Fe_i)g(h(X, Y), Fe_i) \\
&+ \sum_{k=1}^{s} g(h(X, Y), N_k)g(h(X, Y), N_k) \\
&= \sum_{k=1}^{s} g(h(X, Y), N_k)g(h(X, Y), N_k). \tag{41}
\end{aligned}
$$

Let $X = e^1, e^2, ..., e^q$ and $Y = e_1, e_2, ..., e_p$ be in (32) and taking account of (36), then we infer

$$
\begin{aligned}
\triangle \ln f &= -\frac{1}{p}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}(h_{ij}^k)^2 + \frac{1}{q}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}h_{ii}^k h_{jj}^k \\
&+ \frac{p}{4}(c_1 + c_2) - \frac{q^2}{f^2}\|\mathrm{grad}f\|^2 + q\sum_{i=1}^{p}(e_i \ln f)^2. \tag{42}
\end{aligned}
$$

360

On the other hand, we can easily see that

$$
\begin{aligned}
\sum_{i=1}^{p}(e_i \ln f)^2 &= \sum_{i=1}^{p}(g(\operatorname{grad} \ln f, e_i))^2 = \sum_{i=1}^{p}(g(\frac{\operatorname{grad} f}{f}, e_i))^2 = \frac{1}{f^2}\sum_{i=1}^{p} g(\operatorname{grad} f, e_i)^2 \\
&= \frac{1}{f^2}\|\operatorname{grad} f\|^2
\end{aligned}
$$

and hence (42) becomes

$$
\begin{aligned}
\triangle \ln f &= -\frac{1}{p}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}(h_{ij}^k)^2 + \frac{1}{q}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}h_{ii}^k h_{jj}^k + \frac{p}{4}(c_1+c_2) \\
&+ \frac{(1-q)}{f^2}\|\operatorname{grad} f\|^2.
\end{aligned} \tag{43}
$$

From (11) and (43) we conclude

$$
\int_M \{ -\frac{1}{p}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}(h_{ij}^k)^2 \quad + \quad \frac{1}{q}\sum_{k=1}^{s}\sum_{i=1}^{p}\sum_{j=1}^{q}h_{ii}^k h_{jj}^k + \frac{p}{4}(c_1+c_2)
$$
$$
+ \quad \frac{(1-q)}{f^2}\|\operatorname{grad} f\|^2 \} dV = 0. \tag{44}
$$

Here if (33) is satisfied, (44) implies that $\operatorname{grad} f = 0$ which is equivalent to f is a constant function on $M_\perp$ because $q \neq 1$. Thus M is a Riemannian product.

Conversely, if M is a Riemannian product, then we have

$$
g(\nabla_Z X, Y) = g(\bar{\nabla}_Z FX, FY) = g(h(Z, FX), FY) = 0,
$$

for any $X \in \Gamma(TM_T)$, $Y \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM)$, which implies that $h(Z, FX) \in \Gamma(\nu)$. Then our assertion is valid. $\qquad\square$

Acknowledgment

References

[1] Atceken, M.: Warped Product Semi-Slant Submanifolds in a Locally Riemannian Product Manifold. Bulletin of the Australian Mathematical Society (in press).

[2] Bejancu, A.: Geometry of CR-Submanifolds. Kluwer, Dortrecht (1986).

[3] Chen, B.Y.: CR-Submanifolds of a Kaehler Manifold. J.Differential Geometry, 16, 305-322 (1981).

[4] Chen, B.Y.: CR-Submanifolds of a Kaehler Manifold. J. Differential Geometry, 16, 493-509 (1981).

[5] Chen, B.Y.: Geometry of Warped Product CR-Submanifolds in Kaehler Manifolds. Monatsh. Math.133, 177-195 (2001).

[6] Chen, B.Y.: CR-Warped Products in Complex Projective Spaces with Compact Holomorphic Factor. Monatsh. Math.141,177-184 (2004).

[7] O'Neill, B.: Semi-Riemannian Geometry. Academic Press, New York (1983).

[8] Senlin, X. and Yilong, N.: Submanifolds of Product Riemannian Manifold. Acta Mathematica Scientia, 20(B), 213-218 2000, pp.

[9] Shahid, M.N.: CR-Submanifolds of Kaehlerian product manifold. Indian J.Pure and App. Math. 23(12): 873-879 (1992).

[10] Yano, K. and Kon,M.: Structures on Manifolds. World Scientific (1984).

Mehmet ATÇEKEN   Received 28.05.2007
Gaziosmanpaşa University,
Faculty of Arts and Sciences,
Department of Mathematics,
60256 Tokat-TURKEY
e-mail: matceken@gop.edu.tr