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Sufficient Conditions for the Lp-Equivalence Between two Nonlinear Impulse Differential Equations

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Abstract

Sufficient conditions for the Lp-equivalence between two nonlinear impulse differential equations with unbounded linear parts and possibly unbounded nonlinearity parts are given. An example of two nonlinear impulse differential parabolic equations is considered.

Key words and phrases: Impulse differential equations, Lp-equivalence, Partial impulse differential equations of parabolic type.

1. Introduction

We consider the Lp-equivalence between two nonlinear impulse differential equations with unbounded linear parts and possibly unbounded nonlinearity parts. This means that every solution of the one equation which lies in a closed, convex set induces a solution of another equation which lies in a possibly another closed and convex set and vice versa. Moreover the difference of the two solutions lies in the space Lp $(1 < p < \infty)$.

With the help of a transformation every parabolic impulse differential equation can be reduced to an ordinary impulse diferential equation with a unbounded operator. We give an example for this situation and show the Lp-equivalence between the ordinary equations.

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2. Statement of the Problem

Let X be a Banach space with norm $\|.\|$, identity I and $\mathbb{R}_+ = [0, \infty)$.

By $D(T) \subset X$ we will denote the domain of the operator $T : D(T) \to X$.

Consider the impulse diferential equations

$$\frac{du_i}{dt} = A_i(t)u_i + f_i(t, u_i) \quad \text{for } t \neq t_n \tag{1}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) + h_n^i(u_i(t_n)) \quad \text{for } n = 1, 2, ..., \tag{2}$$

where $A_i(t) : D(A_i(t)) \to X$ $(t \in \mathbb{R}_+)$ and $Q_n^i : D(Q_n^i) \to D(A_i(t_n))$ $(i = 1, 2)$ are linear possibly unbounded operators and $f_i(t, .) : \mathbb{R}_+ \times X \to X$ are possibly unbounded for any fixed $t \in \mathbb{R}_+$ functions and $h_n^i : X \to X$ are possibly unbounded functions. The sets $D(A_i(t))$ and $D(Q_n^i)$ $(i = 1, 2; n = 1, 2, ...; t \in \mathbb{R}_+)$ lie dense in X. The points of jump t_n satisfy the conditions $0 = t_0 < t_1 < t_2 < ... < t_n < ...,$ $\lim_{n\to\infty} t_n = \infty$.

Set $Q_0^i = I$, $h_0^i(u) = 0$ $(i = 1, 2;\ u \in X)$.

Furthermore, we assume that all considered functions are left continuous and that there exist the Cauchy operators $U_i(t, s)$ $(i = 1, 2)$ of the equations

$$\frac{du_i}{dt} = A_i(t)u_i \tag{3}$$

Remark 1 Sufficient conditions for the existence of $U_i(t, s)$ can be found in [5]

We introduce the following condition.

$(H1)$ [5] There exist constants $c_i > 0$ $(i = 1, 2)$ such that for $t \in \mathbb{R}_+$

$$\|(\lambda I + A_i(t))^{-1}\| \leq \frac{c_i}{1 + |\lambda|} \tag{4}$$

for each λ which $Re\lambda \leq 0$ hold.

From (4) it follows that

$$\|A_i^{-\alpha}(t)\| \leq c_i \tag{5}$$

for $\alpha \in (0, 1)$ and $t \in \mathbb{R}_+$ $(i = 1, 2)$.

It is not hard to check that

$$V_i(t, s) = U_i(t, t_n)Q_n^i U_i(t_n, t_{n-1})Q_{n-1}^i ... Q_k^i U_i(t_k, s) \tag{6}$$

$(0 \le s \le t_k < t_n < t;\ i = 1, 2)$ are the Cauchy operators of the linear part of the impulse equations $(1), (2)$.

Lemma 1 *Let condition* $(H1)$ *holds and integral equations*

$$\begin{aligned} w_i(t) \quad &= V_i(t,0)A_i^{\alpha}(0)\xi_i + \int_0^t A_i^{\alpha}(0)V_i(t,s)f_i(s, A_i^{-\alpha}(0)w_i(s))ds + \\ &+ \sum_{0<t_n<t} A_i^{\alpha}(0)V_i(t,t_n^+)h_n^i(A_i^{-\alpha}(0)w_i(t_n)) \end{aligned} \tag{7}$$

for $0 \le s \le t,\ \ \xi_i \in D(A_i^{\alpha}(0)),\ \ u_i(0) = \xi_i$ *have solutions* $(i = 1, 2)$.

Then the functions

$$u_i(t) = A_i^{-\alpha}(0)w_i(t) \tag{8}$$

are, for any $\alpha \in (0, 1)$*, solutions of the equations* $(1), (2)$ $(i = 1, 2)$.

Lemma 1 can be proved by help of the proof of Theorem 23.6 (*pp.*473) [5] and straightforward verification.

Remark 2 In [5] are given sufficient conditions for the solvability of the ordinary case i.e. without of impulses of (7). If these conditions are fulfilled and h_n^i are of Lipschitz type, then equations (7) have unique solutions for any $\xi_i \in D(A_i^{\alpha}(0))$.

By $L_p(X)$, $1 \le p < \infty$ we denote the space of all functions $u : \mathbb{R}_+ \to X$ for which $\int_0^\infty \|u(t)\|^p dt < \infty$ with norm $\|u\|_p = (\int_0^\infty \|u\|^p dt)^{\frac{1}{p}}$.

Definition 1 *Eqution* $(1), (2),\ \ i = 2$*, is called* L_p*-equialent to equation* $(1), (2),\ \ i = 1$*, in the unempty, closed and convex subset* B *of* X *if there exists convex and closed subset* D *of* X *such that for any solution* $u_1(t)$ *of* $(1), (2),\ \ i = 1$*, lying in the set* B *there exists a solution* $u_2(t)$ *of* $(1), (2),\ \ i = 2$*, lying in the set* $B \cup D$ *and satisfying the relation* $u_2(t) - u_1(t) \in L_p(X)$*. If equation* $(1), (2),\ \ i = 2$*, is* L_P*-equivalent to equation* $(1), (2),\ \ i = 1$*, in set* B*, and vice versa, we shall say that equations* $(1), (2)$ $i = 1$ *and* $(1), (2),\ \ i = 2$*, are* L_p*-equivalent in set* B*.*

We introduce the condition $(H2)$ $A_1(0) = A_2(0)$. Set $A = A_1(0)$.

The paper aims at finding of sufficient conditions for the existence of L_p-equivalence between impulse equations $(1), (2),\ \ i = 1, 2$.

3. Main Results

Set

$$u(t) = u_2(t) - u_1(t), \tag{9}$$

where $u_i(t)\ \ (i = 1, 2)$ are defined by (8).

Then the function $u(t)$ is a solution of integral equation

$$u(t) = T(A^{-\alpha}w_1, u)(t), \tag{10}$$

where $w_1(t) = A^\alpha u_1(t)$ and

$$\begin{aligned} T(u_1, u)(t) &= V_2(t,0)(u_1(0) + u(0)) - V_1(t,0)u_1(0) + \\ &+ \int_0^t \{V_2(t,s)f_2(s, u_1(s) + u(s)) - V_1(t,s)f_1(s, u_1(s))\}ds + \\ &+ \sum_{0<t_n<t} \{V_2(t, t_n^+)h_n^2(u_1(t_n) + u(t_n)) - V_1(t, t_n^+)h_n^1(u_1(t_n))\}. \end{aligned} \tag{11}$$

By $S(\mathbb{R}_+, X)$ we denote the linear set of all functions which are continuous for $t \neq t_n \quad (n = 1, 2, ...)$, have at points t_n limits on the left and right and are left continuous. The set $S(\mathbb{R}_+, X)$ is a localy convex space with respect to the metric

$$\rho(u, v) = \sup_{0<T<\infty} (1 + T)^{-1} \frac{\max\limits_{0\le t\le T} \|u(t) - v(t)\|}{1 + \max\limits_{0\le t\le T} \|u(t) - v(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

Lemma 2 [1] *The set $H \subset S(\mathbb{R}_+, X)$ is relatively compact if and only if the intersections $H(t) = \{h(t) : h \in H\}$ are relatively compact for $t \in \mathbb{R}_+$ and H is equicontinuous on each interval $(t_n, t_{n+1}] \quad (n = 0, 1, 2...)$.*

Proof. We apply the theorem of Arzella-Ascoli on each intervals $(t_n, t_{n+1}] \quad (n = 0, 1, 2, ...)$ and constitute a diagonal line sequence. □

Let C be an unempty subset of X. Set $\tilde{C} = \{u \in S(\mathbb{R}_+, X) : u(t) \in C,\ \ t \in \mathbb{R}_+\}$. We now have the following lemma.

Lemma 3 *Let C be a non empty, convex and closed subset of X. Suppose an operator F transforms $\tilde{C}$ in the itself and is continuous and compact.*

Then F has a fixed point in $\tilde{C}$.

Proof. The proof of Lemma 3 follows from the fixed point principle of Schauder-Tychonoff. □

Theorem 1 *The following conditions are fulfilled.*

1. *Let conditions $(H1)$ and $(H2)$ hold.*

2. *There exists an unempty, convex, closed subset D of X such that $T(A^{-\alpha}w_1, u)(t) \in D$ for each u with $u(t) \in D$ $(t \in \mathbb{R}_+)$.*

3. *There exist positive functions $q_i(t,s)$ $(i = 1, 2)$ such that*

$$\|V_1(t,s)\xi\| \leq q_1(t,s)\|\xi\|,$$

$$\|(V_2(t,s) - V_1(t,s))\eta\| \leq q_2(t,s)\|\eta\|,$$

where $\xi \in D(A_1(s))$, $\eta \in D(A_1(s)) \cap D(A_2(s))$, $0 \leq s \leq t$ and functions $q_i(t,s)$, $f_i(t,v)$ and $h_n^i(v)$ $i = 1, 2$, satisfy the following conditions.

3.1 *The functions $q_i(.,0)$ belong to $L_p(\mathbb{R}_+)$ $(i = 1, 2)$.*

3.2

$$\sup_{\substack{v \in \tilde{B} \\ w \in \tilde{B} \cup \tilde{D}}} \int_0^t q_1(t,s)\|f_2(s,w) - f_1(s,v)\| ds + \sup_{w \in \tilde{B} \cup \tilde{D}} \int_0^t q_2(t,s)\|f_2(s,w)\| ds \leq \psi(t),$$

where the function $\psi(t)$ is continuous and $\psi \in L_p(\mathbb{R}_+)$.

3.3 *For any fixed $u_1 \in \tilde{B}$ the following inclusions hold:*

$$\int_0^t V_2(t,s) f_2(s, u_1(s) + u(s)) ds \in K^{u_1}(t), \quad (u \in \tilde{D}),$$

where $K^{u_1}(t)$ is for any fixed $t \in \mathbb{R}_+$ a compact subset of X.

3.4

$$\sup_{\substack{v \in \tilde{B} \\ w \in \tilde{B} \cup \tilde{D}}} \sum_{0 < t_n < t} q_1(t, t_n^+)\|h_n^2(w) - h_n^1(v)\| + \sup_{w \in \tilde{B} \cup \tilde{D}} \sum_{0 < t_n < t} q_2(t, t_n^+)\|h_n^2(w)\| \leq \varphi(t),$$

where $\varphi \in L_p(\mathbb{R}_+)$.

3.5 *For any fixed* $u_1 \in \tilde{B}$ *the following inclusion holds:*

$$\sum_{0<t_n<t} V_2(t, t_n^+) h_n^2(u_1(t_n) + u(t_n)) \in K_n^{u_1}, \quad (u \in \tilde{D})$$

where $K_n^{u_1}$ *is for any fixed* $n = 1, 2, ...$ *a compact subset of* X.

3.6

$\int\limits_0^t q_1(t,s) \sup\limits_{w \in \tilde{B} \cup \tilde{D}} \|f_2(s,w)\| ds < \infty$ *and* $\int\limits_0^t q_2(t,s) \sup\limits_{w \in \tilde{B} \cup \tilde{D}} \|f_2(s,w)\| ds < \infty$ *for any fixed* $t \in \mathbb{R}_+$.

Then the equation $(1), (2)$ $(i = 2)$ *is* L_p*-equivalent to the equation* $(1), (2)$ $(i = 1)$ *in the set* B.

Proof. We will prove that for each solution $u_1(t)$ of equation $(1), (2)$ $(i = 1)$ lying in the set B the operator $T(A^{-\alpha}w_1, u)$ has a fixed point $u(t)$ such that $u_1 + u \in \tilde{B} \cup \tilde{D}$ and which lies in $L_p(X)$.

From condition 2 of Theorem 1 it follows that the operator $T(A^{-\alpha}w_1, u)$ defined by (11) maps the set

$$\tilde{D} = \{u \in S(\mathbb{R}_+, X) : u(t) \in D, \; t \in \mathbb{R}_+\}$$

into itself for $A^{-\alpha}w_1 \in \tilde{B}$.

Let be $H = \{h(t) = T(A^{-\alpha}w_1, u)(t) : u \in \tilde{D}, \; t \in \mathbb{R}_+\}$.

We will show the equicontinuity of the functions of the set H. Let $t' > t''$ and

$t', t'' \in (t_n, t_{n+1}]$. It can be verified that

$$\begin{aligned}
&\|h(t') - h(t'')\| \leq \\
&\leq \|V_2(t',0)u_2(0) - V_2(t'',0)u_2(0)\| + \|V_1(t',0)u_1(0) - V_1(t'',0)u_1(0)\| + \\
&+ \sup_{w\in\tilde{B}\cup\tilde{D}} \int_0^{t''} \|V_2(t',s)f_2(s,w) - V_2(t'',s)f_2(s,w)\|ds + \\
&+ \sup_{v\in\tilde{B}} \int_0^{t''} \|V_1(t',s)f_1(s,v) - V_1(t'',s)f_1(s,v)\|ds + \\
&+ \sup_{w\in\tilde{B}\cup\tilde{D}} \int_{t''}^{t'} q_2(t',s)\|f_2(s,w)\|ds + \sup_{\substack{v\in\tilde{B}\\ w\in\tilde{B}\cup\tilde{D}}} \int_{t''}^{t'} q_1(t',s)\|f_2(s,w) - f_1(s,v)\|ds + \\
&+ \sup_{w\in\tilde{B}\cup\tilde{D}} \sum_{0<t_n<t''} \|V_2(t',t_n^+)h_n^2(w) - V_2(t'',t_n^+)h_n^2(w)\| + \\
&+ \sup_{v\in\tilde{B}} \sum_{0<t_n<t''} \|V_1(t',t_n^+)h_n^1(v) - V_1(t'',t_n^+)h_n^1(v)\|.
\end{aligned}$$

From (6) and condition 3.2 of Theorem 1, it follows equicontinuity of the set H.

From conditions 3.3, 3.5 and (11) it follows the compactness of the intersections $H(t) = \{h(t) : h \in H\}$ for $t \in \mathbb{R}_+$. Consequently from Lemma 2 it follows the compactness of the set H.

We will show that the operator $T(A^{-\alpha}w_1, u)$ is continuous in $S(\mathbb{R}_+, X)$.

Let the sequence $\{\tilde{u}_k\} \subset \tilde{D}$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ to the function $\tilde{u} \in \tilde{D}$. Then for $t \in \mathbb{R}_+$ the sequence $f_2(t, A^{-\alpha}w_1(t) + \tilde{u}_k(t))$ convergence to $f_2(t, A^{-\alpha}w_1(t) + \tilde{u}(t))$. From condition 3 of Theorem 1 we obtain

$$\begin{aligned}
&\|V_2(t,s)f_2(s,w)\| - \|V_1(t,s)f_2(s,w)\| \leq \\
&\leq \|(V_2(t,s) - V_1(t,s))f_2(t,w)\| \leq q_2(t,s)\|f_2(s,w)\|.
\end{aligned}$$

Hence

$$\|V_2(t,s)f_2(s,w)\| \leq (q_1(t,s) + q_2(t,s)) \sup_{w\in\tilde{B}+\tilde{D}} \|f_2(s,w)\|. \tag{12}$$

From condition 3.6 of Theorem 1 and (12) it follows that the convergent sequence of functions $V_2(t,s)f_2(s, A^{-\alpha}w_1(s) + \tilde{u}_k(s))$ is majorized by a integrable function. Hence $T(A^{-\alpha}w_1, \tilde{u}_k)(t)$ tends to $T(A^{-\alpha}w_1, \tilde{u})(t)$ for $t \in \mathbb{R}_+$.

From Lemma 3 it follows that for any $u_1 = A^{-\alpha}w_1 \in \tilde{B}$ the operator $T(A^{-\alpha}w_1, u)$ has a fixed point u in $\tilde{D}$ i.e. $u = T(A^{-\alpha}w_1, u)$.

We will show that this fixed point u lies in $Lp(X)$.

$$\|u(t)\| \leq q_2(t,0)\|u_1(0)+u(0)\| + q_1(t,0)\|u(0)\| +$$

$$+ \sup_{w\in\tilde{B}\cup\tilde{D}} \int_0^t q_2(t,s)\|f_2(s,w)\|ds + \sup_{\substack{v\in\tilde{B}\\ w\in\tilde{B}\cup\tilde{D}}} \int_0^t q_1(t,s)\|f_2(s,w)-f_1(s,v)\|ds +$$

$$+ \sup_{w\in\tilde{B}\cup\tilde{D}} \sum_{0<t_n<t} q_2(t,t_n^+)\|h_n^2(w)\| + \sup_{\substack{v\in\tilde{B}\\ w\in\tilde{B}\cup\tilde{D}}} \sum_{0<t_n<t} q_1(t,t_n^+)\|h_n^2(w)-h_n^1(v)\| \leq$$

$$\leq q_2(t,0)\|u_1(0)+u(0)\| + q_1(t,0)\|u(0)\| + \psi(t) + \varphi(t),$$

hence

$$\|u\|_p \leq \|u_1(0)+u(0)\|\|q_2\|_p + \|u(0)\|\|q_1\|_p + \|\psi\|_p + \|\varphi\|_p.$$

Hence this fixed point belongs to the space $L_p(X)$ i.e. equation $(1),(2), i=2$, is L_p-equivalent to the equation $(1),(2), i=1$ in the set B. □

We shall illustrate Theorem 1 with an example of the qualitative theory of the nonlinear partial impulse differential equations. We shall begin with the following lemma.

Lemma 4 *Let $A_i(t)$, Q_n^i $(i=1,2)$ be unbounded operators and $W_i(t,s)(i=1,2)$ are the Cauchy operators of the corresponding linear impulse equations.*

Let the following conditions hold.

1. *The operators $A_2(t)-A_1(t)$ are bounded for any $t\in\mathbb{R}_+$.*
2. *The operators $Q_n^2-Q_n^1$ are bounded for any $n=1,2,...$*
3. *There exist constants $N>0$ and $\nu\in\mathbb{R}$ such that*

$$\|W_1(t,s)\| \leq Ne^{-\nu(t-s)}$$

for any $0<s,t<\infty$.

Then the following estimates hold

$$\|W_2(t,s)\| \leq Ne^{-\nu(t-s)} e^{N\int_s^t \|A_2(\tau)-A_1(\tau)\|d\tau} (1+\prod_{s<t_j<t}\|Q_j^2-Q_j^1\|) \tag{13}$$

and

$$\begin{aligned} \|W_2(t,s) - W_1(t,s)\| \le N e^{-\nu(t-s)} e^{N \int_s^t \|A_2(\tau)-A_1(\tau)\| d\tau} \cdot \\ (1 + \prod_{s<t_j<t} \|Q_j^2 - Q_j^1\|)(1 + N \sum_{s<t_j<t} \|Q_j^2 - Q_j^1\|). \end{aligned} \quad (14)$$

Proof. The function $V_2(t) = W_2(t,0)$ is a solution of the initial value problem.

$$\begin{aligned} &\tfrac{dV_2}{dt} - A_1 W_2 = (A_2 - A_1) W_2 \quad (t \neq t_n) \\ &V_2(t_n^+) = Q_n^1 V_2(t_n) + (Q_n^2 - Q_n^1) V_2(t_n) \\ &V_2(0) = I. \end{aligned}$$

The solution $X(t)$ of the impulse equation

$$\begin{aligned} &\tfrac{dX}{dt} - A_1 X = (A_2(t) - A_1(t)) V_2(t) \quad (t \neq t_n) \\ &X(t_n^+) = Q_n^1 X(t_n) + (Q_n^2 - Q_n^1) V_2(t_n) \\ &X(0) = I \end{aligned}$$

has the representation

$$\begin{aligned} X(t) \quad &= W_1(t,0) + \int_0^t W_1(t,\tau)(A_2(\tau) - A_1(\tau)) V_2(\tau) d\tau + \\ &+ \sum_{0<t_j<t} W_1(t,t_j^+)(Q_j^2 - Q_j^1) V_2(t_j) \end{aligned}$$

i.e.

$$\begin{aligned} V_2(t) \quad &= W_1(t,0) + \int_0^t W_1(t,\tau)(A_2(\tau) - A_1(\tau)) V_2(\tau) d\tau + \\ &+ \sum_{0<t_j<t} W_1(t,t_j^+)(Q_j^2 - Q_j^1) V_2(t_j). \end{aligned}$$

With $\varphi(t) = \|V_2(t)\|$, $p(t) = \|A_2(t) - A_1(t)\|$, $q_j = \|Q_j^2 - Q_j^2\|$ we obtain

$$\varphi(t) \le N e^{-\nu t} + N \int_0^t e^{-\nu(t-\tau)} p(\tau) \varphi(\tau) d\tau + N \sum_{0<t_j<t} e^{-\nu(t-t_j)} q_j \varphi(t_j). \quad (15)$$

From [6] and (15) it follows the estimate (13). Inequality (14) follows from (13). □

Example In our example we shall consider two nonlinear partial impulse differential equations. We transform this equations to ordinary impulse differential equations and show that they satisfy the conditions of Theorem 1. The following short introduction in the qualitative theory of nonlinear parabolic impulse differential equations is taken from [2].

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $Q = (0, \infty) \times \Omega$ and $\Gamma = (0, \infty) \times \partial\Omega$.

We denote

$$P_n = \{(t_n, x) : x \in \Omega\}, \quad P = \bigcup_{n=1}^{\infty} P_n,$$

$$\Lambda_n = \{(t_n, x) : x \in \partial\Omega\}, \quad \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n.$$

Consider the impulse nonlinear parabolic initial value problems

$$\begin{aligned}
&\tfrac{\partial u_i}{\partial t} = \tilde{A}_i(t, x, D)u_i + \tilde{f}_i(t, x, u_i), \quad (t, x) \in Q \setminus P \\
&D^\alpha u_i(t, x) = 0, \quad |\alpha| < m, \quad (t, x) \in \Gamma \setminus \Lambda \\
&u_i(0, x) = v_i(x), \quad x \in \Omega \\
&u_i(t_n^+, x) = Q_n^i(u_i(t_n, x)) + h_n^i(u_i(t_n, x)), \quad x \in \overline{\Omega}, \ \ n = 1, 2, ...,
\end{aligned}$$

where

$$\tilde{A}_i(t, x, D)u_i = \sum_{|\alpha| \le 2m} a_\alpha(t, x) D^\alpha u_i + k_i u_i,$$

$\tilde{Q}_n^i : D(Q_n^i) \to D(\tilde{A}_i(t, x, D))$ $(n = 1, 2, ...; \ i = 1, 2)$ are linear operators and $\tilde{f}_i(.,.,.) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $h_n^i : \mathbb{R} \to \mathbb{R}$.

Let $X = L_p(\Omega, \mathbb{R})$, $(1 < p < \infty)$, where

$$L_p(\Omega, \mathbb{R}) = \{v : \Omega \to \mathbb{R}, \ \int_\Omega |v(x)|^p dx < \infty\}$$

with norm $|v|_p = (\int_\Omega |v(x)|^p dx)^{\frac{1}{p}}$.

With the family $\tilde{A}_i(t, x, D)$, $t \in \mathbb{R}_+$, $i = 1, 2$, of strongly elliptic operators we associate a family of linear operators $A_i(t)$, $t \in \mathbb{R}_+$, $n = 1, 2$, acting in X by

$$A_i(t)u_i = \tilde{A}_i(t, x, D)u_i, \text{ for } u_i \in D.$$

This is done with $D = D(A_i(t)) = W^{2m,p}(\Omega) \bigcap W_0^{m,p}(\Omega)$, $(i = 1, 2;\ t \in \mathbb{R}_+)$.

The real constans $k_i > 0$ $(i = 1, 2)$ are chosen such that condition $(H1)$ holds.

Let $v_i \in X$. We set

$$f_i(t, u_i)(x) = \tilde{f}_i(t, x, u_i(t, x)), \ \ u_i \in X,\ t \in \mathbb{R}_+,\ x \in \overline{\Omega}\ (i = 1, 2),$$
$$Q_n^i(u_i(t_n))(x) = \tilde{Q}_n^i(u_i(t_n, x)), \ \ h_n^i(u_i(t_n))(x) = \tilde{h}_n^i(u_i(t_n, x)),$$

where $Q_n^i : D(Q_n^i) \to D$ $(D(Q_n^i) \subset X$ lie dense in X $(i = 1, 2))$ are linear operators, $f_n^i : \mathbb{R}_+ \times X \to X$ and $h_n^i : X \to X$.

Let $U_i(t, s)$ $i = 1, 2$, be the Cauchy operators of the equations

$$\frac{du_i}{dt} = A_i(t)u_i$$

In [4] are given sufficient conditions for the validaty of the estimates

$$|U_i(t, s)|_{p \to p} \le M_i \ \ (0 \le s \le t;\ M_i > 0 \text{ constants}, i = 1, 2).$$

We shall consider the concrete case when $t_n = n$ $(n = 1, 2, ...)$,

$$\tilde{Q}_n^1 \xi = \frac{e^{-2\kappa_1}}{12(M_1^2 + 1)}\xi, \ \ \tilde{Q}_n^2 \xi = \frac{12(M_1^2 + 1) + 2^n e^{-2\kappa_1}}{12.2^n(M_1^2 + 1)}\xi, \ \ (\xi \in \mathbb{R})$$

where $\kappa_1 > \frac{1}{12}|k_2 - k_1|$.

We denote

$$Q_n^1 \eta = \frac{e^{-2\kappa_1}}{12(M_1^2 + 1)}\eta, \ \ Q_n^2 \eta = \frac{12(M_1^2 + 1) + 2^n e^{-2\kappa_1}}{12.2^n(M_1^2 + 1)}\eta \ \ (\eta \in X).$$

Let $V_i(t, s)$ $(i = 1, 2;\ 0 \le s \le t)$ are the Cauchy operators of the linear impulse equations

$$\frac{du_i}{dt} = A_i(t)u_i \ \text{ for } t \ne t_n$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) \ \text{ for } n = 1, 2, ...$$

Then for $0 < s \leq k < n < t,\ \xi \in D$ the following estimates are valid

$$|V_1(t,s)\xi|_p = |U_1(t,t_n)Q_n^1 ... Q_k^1 U_1(t_k,s)\xi|_p \leq$$

$$\leq M_1 \tfrac{e^{-2\kappa_1}}{12(M_1^2+1)} ... M_1 \tfrac{e^{-2\kappa_1}}{12(M_1^2+1)} M_1 |\xi|_p \leq$$

$$\leq \tfrac{1}{12} e^{-\kappa_1(n-k+2)} \tfrac{e^{-\kappa_1(n-k)}}{12^{(n-k)}} |\xi|_p \leq \tfrac{1}{12} e^{-\kappa_1(t-s)} |\xi|_p.$$

Let

$$q_1(t,s) = \frac{1}{12} e^{-\kappa_1(t-s)}.$$

From Lemma 4 we obtain the estimate

$$|V_2(t,s)\xi - V_1(t,s)\xi|_p \leq$$

$$\leq \tfrac{1}{12} e^{-\kappa_1(t-s)} e^{\frac{1}{12}\int_s^t |k_2-k_1| d\tau} (1 + \prod_{s<j<t} \tfrac{1}{2^j})(1 + \tfrac{1}{12} \sum_{s<j<t} \tfrac{1}{2^j})|\xi|_p \leq$$

$$\leq \tfrac{1}{12} e^{(\frac{1}{12}|k_2-k_1| - \kappa_1)(t-s)} 2(1 + \tfrac{1}{12} 1)|\xi|_p = \tfrac{13}{72} e^{(\frac{1}{12}|k_2-k_1| - \kappa_1)(t-s)} |\xi|_p.$$

Let

$$q_2(t,s) = \frac{13}{72} e^{-\kappa_2(t-s)},$$

where $\kappa_2 = \kappa_1 - \frac{1}{12}|k_2 - k_1|$. We take

$$\tilde{f}_1(t,x,u_1) = e^{\gamma t}(\ln(1 + 2^{-|u_1(t,x)|}) - u_1(t,x)),$$
$$\tilde{f}_2(t,x,u_2) = e^{\gamma t} \ln(2 + |u_2(t,x)|),$$
$$\tilde{h}_n^1(u_1(t_n,x)) = e^{\alpha t_n}(\sin u_1(t_n,x) - u_1(t_n,x)),$$
$$\tilde{h}_n^2(u_2(t_n,x)) = e^{\alpha t_n}(\sin \tfrac{1}{1+|u_2(t_n,x)|} + u_2(t_n,x)),$$

where $\gamma + \kappa_i < -1$ and $\alpha + \gamma_i < \ln \frac{1}{2}$ $(i = 1, 2)$. Then

$$f_1(t,u_1) = e^{\gamma t}(\ln(1 + 2^{-|u_1(t)|}) - u_1(t)),$$
$$f_2(t,u_2) = e^{\gamma t} \ln(2 + |u_2(t)|),$$
$$h_n^1(u_1(t_n)) = e^{\alpha t_n}(\sin u_1(t_n) - u_1(t_n)),$$
$$h_n^2(u_2(t_n)) = e^{\alpha t_n}(\sin \tfrac{1}{1+|u_2(t_n)|} + u_2(t_n)).$$

Let $r > 0$ and

$$\rho > \frac{21}{5}(r + (\mu(\Omega))^{\frac{1}{p}}). \tag{16}$$

We shall show that the conditions of Theorem 1 are fulfilled.

Set $B_\sigma = \{u \in X : |u|_p \leq \sigma\}$. For any $\xi \in B_\rho$, $\eta \in B_{r+\rho}$, $t \in \mathbb{R}_+$ we obtain

$$q_1(t,0)|\xi|_p + q_2(t,0)|\eta|_p \leq \frac{1}{12}e^{-\kappa_1 t}\rho + \frac{13}{72}e^{-\kappa_2 t}(r+\rho).$$

Let us set

$$\chi_{r,\rho}(t) = \frac{1}{12}e^{-\kappa_1 t}\rho + \frac{13}{72}e^{-\kappa_2 t}(r+\rho).$$

We shall show the validaty of the condition 3.2 of Theorem 1.

$$\begin{aligned}
&\sup_{\substack{|v|_p\leq r\\ |w|_p\leq r+\rho}} \int_0^t q_1(t,s)|f_2(s,w)-f_1(s,v)|_p ds + \sup_{|w|_p\leq r+\rho}\int_0^t q_2(t,s)|f_2(s,w)|_p ds = \\
&= \sup_{\substack{|v|_p\leq r\\ |w|_p\leq r+\rho}} \int_0^t \tfrac{1}{12}e^{-\kappa_1(t-s)}e^{\gamma s}|\ln\tfrac{2+|w|}{1+2^{-|v|}} + v|_p ds + \\
&+ \sup_{|w|_p\leq r+\rho}\int_0^t \tfrac{13}{72}e^{-\kappa_2(t-s)}e^{\gamma s}|\ln(2+|w|)|_p ds \leq \\
&\leq \sup_{\substack{|v|_p\leq r\\ |w|_p\leq r+\rho}} \int_0^t \tfrac{1}{12}e^{-\kappa_1(t-s)}e^{\gamma s}(|2+|w||_p + |v|_p) ds + \\
&+ \sup_{|w|_p\leq r+\rho}\int_0^t \tfrac{13}{72}e^{-\kappa_2(t-s)}e^{\gamma s}|2+|w||_p ds \leq \\
&\leq \tfrac{1}{12}e^{-\kappa_1 t}\tfrac{2(\mu(\Omega))^{\frac{1}{p}}+2r+\rho}{-(\kappa_1+\gamma)} + \tfrac{13}{72}e^{-\kappa_2 t}\tfrac{2(\mu(\Omega))^{\frac{1}{p}}+r+\rho}{-(\kappa_2+\gamma)}.
\end{aligned}$$

Let us set

$$\psi_{r,\rho}(t) = \frac{1}{12}e^{-\kappa_1 t}\frac{2(\mu(\Omega))^{\frac{1}{p}}+2r+\rho}{-(\kappa_1+\gamma)} + \frac{13}{72}e^{-\kappa_2 t}\frac{2(\mu(\Omega))^{\frac{1}{p}}+r+\rho}{-(\kappa_2+\gamma)}.$$

We prove condition 3.4 of Theorem 1.

$$\sup_{\substack{|v|_p\le r\\ |w|_p\le r+\rho}} \sum_{0<t_n<t} q_1(t,t_n^+)|h_2(w)-h_1(v)|_p + \sup_{|w|_p\le r+\rho} \sum_{0<t_n<t} q_2(t,t_n^+)|h_2(w)|_p =$$
$$= \sup_{\substack{|v|_p\le r\\ |w|_p\le r+\rho}} \sum_{0<n<t} \tfrac{1}{12} e^{-\kappa_1(t-n)} e^{\alpha n} \Big|\sin\tfrac{1}{1+|w|} + w - \sin v + v\Big|_p +$$
$$+ \sup_{|w|_p\le r+\rho} \sum_{0<n<t} \tfrac{13}{72} e^{-\kappa_2(t-n)} e^{\alpha n} \Big|\sin\tfrac{1}{1+|w|} + w\Big|_p \le$$
$$\le \sup_{\substack{|v|_p\le r\\ |w|_p\le r+\rho}} \tfrac{1}{12} e^{-\kappa_1 t} \sum_{0<n<t} e^{(\kappa_1+\alpha)n} (2(\mu(\Omega))^{\frac{1}{p}} + |w|_p + |v|_p) +$$
$$+ \sup_{|w|_p\le r+\rho} \tfrac{13}{72} e^{-\kappa_2 t} \sum_{0<n<t} e^{(\kappa_2+\alpha)n} ((\mu(\Omega))^{\frac{1}{p}} + |w|_p) \le$$
$$\le \tfrac{1}{12} e^{-\kappa_1 t} (2(\mu(\Omega))^{\frac{1}{p}} + 2r + \rho) \tfrac{e^{\kappa_1+\alpha}}{1-e^{\kappa_1+\alpha}} + \tfrac{13}{72} e^{-\kappa_2 t} ((\mu(\Omega))^{\frac{1}{p}} + r + \rho) \tfrac{e^{\kappa_2+\alpha}}{1-e^{\kappa_2+\alpha}}.$$

Set

$$\varphi_{r,\rho}(t) = \tfrac{1}{12} e^{-\kappa_1 t} (2(\mu(\Omega))^{\frac{1}{p}} + 2r + \rho) \tfrac{e^{\kappa_1+\alpha}}{1-e^{\kappa_1+\alpha}} + \tfrac{13}{72} e^{-\kappa_2 t} ((\mu(\Omega))^{\frac{1}{p}} + r + \rho) \tfrac{e^{\kappa_2+\alpha}}{1-e^{\kappa_2+\alpha}}.$$

It is not hard to check that the functions $q_i(.,0)$ $(i = 1,2)$, $\psi_{r,\rho}$ and $\varphi_{r,\rho}$ lie in the space $L_p(\mathbb{R}_+)$.

From condition (16) we obtain

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho \text{ for each } t \in \mathbb{R}_+.$$

By means of a compactness criterion from [3] we prove condition 3.3.

Set

$$M(t) = \{m(t) = \int_0^t V_2(t,s) f_2(s, u_1(s) + u(s)) ds : |u|_p \le \rho\}$$

is a compact subset of X for any fixed $t \in \mathbb{R}_+$.

Indeed,

$$
\begin{aligned}
&|m(t)|_p \le \int_0^t |(V_2(t,s)-V_1(t,s))f_2(s,u_1(s)(x)+u(s)(x))|_p ds+\\
&+\int_0^t |V_1(t,s)f_2(s,u_1(s)(x)+u(s)(x))|_p ds \le\\
&\le \tfrac{13}{72}e^{-\kappa_2 t}\int_0^t e^{(\kappa_2+\gamma)s}|\ln(2+|u_1(s)(x)+u(s)(x)|)|_p ds+\\
&+\tfrac{1}{12}e^{-\kappa_1 t}\int_0^t e^{(\kappa_1+\gamma)s}|\ln(2+|u_1(s)(x)+u(s)(x)|)|_p ds \le\\
&\le \tfrac{13}{72}\tfrac{2(\mu(\Omega))^{\frac{1}{p}}+r+\rho}{-(\kappa_2+\gamma)}+\tfrac{1}{12}\tfrac{2(\mu(\Omega))^{\frac{1}{p}}+r+\rho}{-(\kappa_1+\gamma)}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&|m(t)(x+h)-m(t)(x)|_p \le\\
&\le \tfrac{13}{72}e^{-\kappa_2 t}\int_0^t e^{(\kappa_2+\gamma)s}\left|\ln\tfrac{2+|u_1(s)(x+h)+u(s)(x+h)|}{2+|u_1(s)(x)+u(s)(x)|}\right|_p ds+\\
&+\tfrac{1}{12}e^{-\kappa_1 t}\int_0^t e^{(\kappa_1+\gamma)s}\left|\ln\tfrac{2+|u_1(s)(x+h)+u(s)(x+h)|}{2+|u_1(s)(x)+u(s)(x)|}\right|_p ds.
\end{aligned}
$$

In a similar way we show the validaty of condition 3.5.

Obviously condition 3.6 of Theorem 1 is fulfilled.

The conditions of Theorem 1 are fulfilled and hence the equations (1), (2)($i = 1, 2$) are in B_r, L_p-equivalent.

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