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On ϕ-Recurrent Kenmotsu Manifolds

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Abstract

The object of this paper is to study ϕ-recurrent Kenmotsu manifolds. Also three-dimensional locally ϕ-recurrent Kenmotsu manifolds have been considered. Among others it is proved that a locally ϕ-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime. Finally we give a concrete example of a three-dimensional Kenmotsu manifold.

Key Words: Kenmotsu manifolds, ϕ-recurrent Kenmotsu manifolds, locally ϕ-recurrent Kenmotsu manifolds.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakend by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [16] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, one of the authors, De, [7] introduced the notion of ϕ-recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Buecken and Vanhecke [3] with several examples.

On the other hand Kenmotsu [11] defined a type of contact metric manifold which is nowadays called Kenmotsu manifold. It may be mentioned that a Kenmotsu manifold is not a Sasakian manifold. Also, a Kenmotsu manifold is not compact because of div$\xi = 2n$. In [11], Kenmotsu showed that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kahler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant.

The present paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we prove that a ϕ-recurrent Kenmotsu manifold is an Einstein manifold and a locally ϕ-recurrent Kenmotsu manifold is locally a hyperbolic space. In the next section, it is proved that a three-dimensional locally ϕ-recurrent Kenmotsu manifold is a manifold of constant curvature. In section 5, we prove that a locally ϕ-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime. In the last section, we construct an example of a three-dimensional Kenmotsu manifold.

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2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost contact Riemannian manifold, where ϕ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that (ϕ,ξ,η,g) satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X,\xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X and Y on M [1], [2].

If, moreover,

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X,\phi Y)\xi, \quad X, Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [11].

Kenmotsu manifolds have been studied by many authors such as Binh, Tamassy, De and Tarafdar [4], Pitiş [15], De and Pathak [5], Jun, De and Pathak [10], Ozgür [13], Ozgür and De [14], Dileo and Pastore [8] and many others.

In a Kenmotsu manifold the following relations hold: [11] .

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.8}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.10}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.11}$$

$$(\nabla_Z R)(X,Y)\xi = g(X,Z)Y - g(Z,Y)X - R(X,Y)Z, \tag{2.12}$$

for any vector fields X, Y, Z, where R is the Riemannian curvature tensor and S is the Ricci tensor.

Definition 1 *A Kenmotsu manifold is said to be a locally ϕ-symmetric manifold if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.13}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

This notion was introduced for Sasakian manifolds by Takahashi [16].

Definition 2 *A Kenmotsu manifold is said to be a ϕ-recurrent manifold if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{2.14}$$

for arbitrary vector fields X, Y, Z, W.

If X, Y, Z, W are orthogonal to ξ, then the manifold is called locally ϕ-recurrent manifold.

If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold.

3. ϕ-Recurrent Kenmotsu Manifolds

To prove the main theorem of the paper we first prove the following lemma.

Lemma 1 *In a ϕ-recurrent Kenmotsu manifold (M^{2n+1}, g), $n > 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by*

$$A(W) = \eta(\rho)\eta(W).$$

Proof. Two vector fields P and Q are said to be co-directional if $P = fQ$ where f is a non-zero scalar. That is,

$$g(P,X) = fg(Q,X) \quad \text{for all } X. \tag{3.15}$$

Let us consider a ϕ-recurrent Kenmotsu manifold. Then by virtue of (2.2) and (2.14), we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{3.16}$$

From (3.16) and the Bianchi identity, we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{3.17}$$

Let $\{e_i\}$, $i = 1, 2, 3, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $Y = Z = e_i$ in (3.17) and taking summation over i, $1 \leq i \leq 2n+1$, we get by virtue of (2.8)

$$A(W)\eta(X) = A(X)\eta(W), \tag{3.18}$$

for all vector fields X, W. Replacing X by ξ in (3.18), it follows that

$$A(W) = \eta(\rho)\eta(W), \tag{3.19}$$

where $A(X) = g(X, \rho)$ and ρ is the vector field associated to the 1-form A. From (3.15) and (3.19) it is clear that ξ and ρ are co-directional. □

Theorem 1 *A ϕ-recurrent Kenmotsu manifold is an Einstein manifold.*

Proof. From (3.16), we have

$$-g(\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U). \tag{3.20}$$

Putting $X = U = e_i$ in (3.20) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z). \tag{3.21}$$

The second term of (3.21) by putting $Z = \xi$ takes the form

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi), \tag{3.22}$$

which is denoted by E. In this case E vanishes. Namely, we have

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &\quad - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned} \tag{3.23}$$

at $p \in M$. In local coordinates $\nabla_X e_i = X^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla_X e_i = 0$. Also we have

$$g(R(e_i,\nabla_W Y)\xi,\xi) = 0, \tag{3.24}$$

since R is skew-symmetric. Using (3.24) and $\nabla_X e_i = 0$ in (3.23), we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

By virtue of $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0$, we have

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0, \tag{3.25}$$

which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

Since R is skew-symmetric

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{3.26}$$

Using (3.26) from (3.21), we get

$$(\nabla_W S)(Y,\xi) = -A(W)S(Y,\xi). \tag{3.27}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Again using (2.6), (2.7) and (2.11), we get

$$(\nabla_W S)(Y,\xi) = -2ng(Y,W) - S(Y,W). \tag{3.28}$$

Now using (3.28) in (3.27), we obtain

$$S(Y,W) = -2nA(W)\eta(Y) - 2ng(Y,W). \tag{3.29}$$

Applying Lemma 1, equation (3.29) reduces to

$$S(Y,W) = -2ng(Y,W) - 2n\eta(\rho)\eta(Y)\eta(W),$$

which implies that the manifold is an η-Einstein manifold. □

In Corollary 9 of Proposition 8 of [11], it is proved that if a Kenmotsu manifold is an η-Einstein manifold of type $S = ag + b\eta \otimes \eta$ and if $b =$constant (or $a =$constant) then M is an Einstein manifold. Hence by the above result a ϕ-recurrent Kenmotsu manifold is an Einstein manifold.

Theorem 2 *A locally ϕ-recurrent Kenmotsu manifold (M^{2n+1}, g), $n > 1$, is a manifold of constant curvature -1, i.e., it is locally a hyperbolic space.*

Proof. From (2.12), we have

$$(\nabla_W R)(X,Y)\xi = g(W,X)Y - g(W,Y)X - R(X,Y)W. \tag{3.30}$$

By virtue of (2.8), it follows from (3.30) that

$$\eta((\nabla_W R)(X,Y)\xi) = 0. \tag{3.31}$$

In view of (3.30) and (3.31), we obtain from (3.16)

$$-(\nabla_W R)(X,Y)\xi = A(W)R(X,Y)\xi, \tag{3.32}$$

from which by using (2.12), it follows that

$$-g(X,W)Y + g(Y,W)X + R(X,Y)W = A(W)R(X,Y)\xi.$$

Hence if X and Y are orthogonal to ξ, then we get from (2.9)

$$R(X,Y)\xi = 0.$$

Thus, we obtain

$$R(X,Y)W = -[g(Y,W)X - g(X,W)Y],$$

for all X, Y, W. □

Remark. It may be mentioned that a semi-symmetric $(R(X,Y).R = 0)$ Kenmotsu manifold and a conformally flat Kenmotsu manifold of dimension > 3 are of constant sectional curvature [11]. Also De and Pathak [5] proved that three dimensional Ricci semi-symmetric $(R(X,Y) \cdot S = 0)$ Kenmotsu manifold is of constant sectional curvature.

4. Three-Dimensional Kenmotsu Manifolds

It is known that in a three-dimensional Kenmotsu manifold the curvature tensor has the following form [5]

$$\begin{aligned} R(X,Y)Z &= (\frac{r+4}{2})[g(Y,Z)X - g(X,Z)Y] \\ &\quad -(\frac{r+6}{2})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.33}$$

Taking the covariant differentiation of the equation (4.33), we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] - \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] - (\frac{r+6}{2})[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)\nabla_W\xi \\ &-g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)\nabla_W\xi + (\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X \\ &-(\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y]\end{aligned} \tag{4.34}$$

Now applying ϕ^2 to the both sides of (4.34), we obtain

$$\begin{aligned}\phi^2(\nabla_W R)(X,Y)Z &= \frac{-dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X] + (\frac{r+6}{2})[(\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X - (\nabla_W\eta)(X)\eta(Z)Y \\ &-\eta(X)(\nabla_W\eta)(Z)Y - (\nabla_W\eta)(Y)\eta(Z)\eta(X)\xi + (\nabla_W\eta)(X)\eta(Z)\eta(Y)\xi].\end{aligned} \tag{4.35}$$

Taking X, Y, Z, W orthogonal to ξ and using (2.14), we finally get from (4.35)

$$A(W)R(X,Y)Z = \frac{-dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.36}$$

Putting $W = \{e_i\}$ in (4.36), where $\{e_i\},\ i = 1,2,3,$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i,\ 1 \le i \le 3$, we obtain

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

where $\lambda = \frac{-dr(e_i)}{2A(e_i)}$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, M^3 is of constant curvature λ. Thus we get the following theorem.

Theorem 3 *A three-dimensional locally ϕ-recurrent Kenmotsu manifold is of constant curvature.*

5. Locally ϕ-Recurrent Kenmotsu Spacetime

In this section we consider locally ϕ-recurrent Kenmotsu spacetime. By a spacetime, we mean a 4-dimensional semi-Riemannian manifold endowed with Lorentzain metric of signature $(-+++)$. In a recent paper one of the authors De and Pathak [6] prove that the characteristic vector field ξ in a Kenmotsu manifold is a concircular vector field [18]. Also from Theorem 2, we can easily prove that a locally ϕ-recurrent Kenmotsu manifold is conformally flat. Hence div$C = 0$, where C denotes the conformal curvature tensor and "div" denotes divergence.

Hence, we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{5.37}$$

Yano [17], prove that, in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta}\ dx^\alpha dx^\beta,$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^r)$ are the functions of x^r only $(\alpha, \beta, r = 2, 3, ..., n)$ and $q = q(x') \neq$ constant is a function of x^1 only. In the semi-Riemannian space, we can prove that

$$ds^2 = -(dx^1)^2 + e^q g^*_{\alpha\beta}\ dx^\alpha dx^\beta.$$

Thus a Kenmotsu spacetime can be expressed as a warped product $-I \times_{e^q} M^*$, where M^* is a three-dimensional Riemannian manifold. But Gebarowski [9] prove that warped product $-I \times_{e^q} M^*$ satisfies (5.37) if and only if M^* is an Einstein manifold. Thus a locally ϕ-recurrent Kenmotsu spacetime must be warped product $-I \times_{e^q} M^*$, where M^* is an Einstein manifold. Since we consider a 4-dimensional manifold, M^* is a three-dimensional Einstein manifold. It is known that a three– dimensional Einstein manifold is a manifold of constant curvature. Hence a locally ϕ-recurrent Kenmotsu spacetime is the warped product $-I \times_{e^q} M^*$, where M^* is a manifold of constant curvature. But such a warped product is the Robertson-Walker spacetime [12].

Thus we have the following theorem.

Theorem 4 *A locally ϕ-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime.*

6. Example of a Three-Dimensional Kenmotsu Manifold

We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, $z \neq 0$ where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z},$$

are linearly indepent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1. \end{aligned}$$

That is, the form of the metric becomes

$$g = \frac{(dx^2 + dy^2 + dz^2)}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\begin{aligned} \eta(e_3) &= 1, \\ \phi^2 Z &= -Z + \eta(Z)e_3, \\ g(\phi Z, \phi W) &= g(Z, W) - \eta(Z)\eta(W), \end{aligned}$$

for any $Z, W \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. Using this formula we obtain

$$\begin{aligned}
\nabla_{e_1} e_3 &= e_1, & \nabla_{e_2} e_3 = e_2,\\
\nabla_{e_1} e_2 &= 0, & \nabla_{e_2} e_2 = -e_3,\\
\nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_2 = -e_3,\\
\nabla_{e_1} e_1 &= -e_3, & \nabla_{e_2} e_1 = 0,\\
\nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 = 0,\\
\nabla_{e_3} e_3 &= 0.
\end{aligned}$$

Thus (2.6) is satisfied. It is straightforward computation to verify that the manifold under consideration is a three-dimensional Kenmotsu manifold.

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