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Lebesgue-Stieltjes Measure on Time Scales

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Abstract

The theory of time scales was introduced by Stefan Hilger in his Ph. D. thesis in 1988, supervised by Bernd Auldbach, in order to unify continuous and discrete analysis [5]. Measure theory on time scales was first constructed by Guseinov [4], then further studies were made by Guseinov-Bohner [1], Cabada-Vivero [2] and Rzezuchowski [6]. In this article, we adapt the concept of Lebesgue-Stieltjes measure to time scales. We define Lebesgue-Stieltjes Δ and ∇-measures and by using these measures, we define an integral adapted to time scales, specifically Lebesgue-Stieltjes Δ-integral. We also establish the relation between Lebesgue-Stieltjes measure and Lebesgue-Stieltjes Δ-measure, consequently between Lebesgue-Stieltjes integral and Lebesgue-Stieltjes Δ- integral.

Key Words: Time scales, Lebesgue-Stieltjes Δ-measure, Lebesgue-Stieltjes Δ-integral.

1. Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers. We begin with basic operators on $\mathbb{T}$: the forward jump operator, backward jump operator and graininess function.

Definition 1.1 *Let $\mathbb{T}$ be a time scale. Forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \tag{1}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2}$$

If $\sigma(t) > t$, t is said to be right-scattered, while if $\rho(t) < t$, t is said to be left-scattered. If t is both right-scattered and left-scattered, t is called isolated. Also, if $\sigma(t) = t$, then t is right-dense and if $\rho(t) = t$, then t is left-dense. If t is both right-dense and left-dense, then t is said to be a dense point. For special cases if $t = \max \mathbb{T}$, $\sigma(t) = t$ and if $t = \min \mathbb{T}$, $\rho(t) = t$. The function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t \tag{3}$$

is called graininess function.

2. Measure Theory on Time Scales

Measure theory on time scales was first introduced by Guseinov [4]. The following two theorems give Δ-measures of single point set and different types of intervals respectively, established by Guseinov[4].

Theorem 2.1 *Δ-measure of a single point set $\{t_0\} \subset \mathbb{T} - \{max\mathbb{T}\}$ is given by*

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0), \tag{4}$$

where μ denotes the graininess function.

Theorem 2.2 *If $a, b \in \mathbb{T}$ and $a \leq b$, then*

a) $\mu_\Delta\big([a,b)\big) = b - a$*;*

b) $\mu_\Delta\big((a,b)\big) = b - \sigma(a)$*.*

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ and $a \leq b$, then

c) $\mu_\Delta\big((a,b]\big) = \sigma(b) - \sigma(a)$*;*

d) $\mu_\Delta\big([a,b]\big) = \sigma(b) - a$*.*

Similarly ∇-measures of single point set and any interval are given in [4] as follows

Theorem 2.3 *For each $t_0 \in \mathbb{T} - \{\min \mathbb{T}\}$, the ∇-measure of the single point set $\{t_0\}$ is given by*

$$\mu_\nabla(\{t_o\}) = t_0 - \rho(t_0). \tag{5}$$

Theorem 2.4 *If $c, d \in \mathbb{T}$, then*

a) $\mu_\nabla((c,d]) = c - d$*.*

b) $\mu_\nabla((c,d)) = \rho(d) - c$*.*

If $c, d \in \mathbb{T} - \{\min \mathbb{T}\}$, then

c) $\mu_\nabla([c,d)) = \rho(d) - \rho(c)$*.*

d) $\mu_\nabla([c,d]) = d - \rho(c)$*.*

Remark 2.5 *Measure constructed on time scales is different from the classical Lebesgue measure. In the classical Lebesgue measure, single point set has measure zero. Consequently for $a, b \in \mathbb{R}$ and $a \leq b$, measures of $[a,b]$, $[a,b)$, (a,b), $(a,b]$ are equal, that is, the difference of endpoints, whereas, for measure constructed on a time scale, the single point set may have measure different from zero, depending on the character of the point. As a result, it is natural that different types of intervals with the same endpoints may have different measures.*

Finally, there is a relation between the classical Lebesgue integral and the Lebesgue integral on time scales. We refer to reader to [2] for further information.

3. Lebesgue-Stieltjes Δ and ∇-Measures

The original Lebesgue-Stieltjes measure is defined by introducing a pre-measure μ on all intervals of $\mathbb{R}$ as follows [3]:

i) $\mu([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $\mu([a,b]) = \alpha(b^+) - \alpha(a^-)$,

iii) $\mu((a,b]) = \alpha(b^+) - \alpha(a^+)$,

iv) $b > a$, $\mu((a,b)) = \alpha(b^-) - \alpha(a^+)$,

where $\alpha : \mathbb{R} \to \mathbb{R}$ is an nondecreasing function with

$$\alpha(a^-) = \lim_{t \to a^-} \alpha(t) \text{ and } \alpha(a^+) = \lim_{t \to a^+} \alpha(t).$$

We can generalize this measure to time scales. We will begin with defining a pre-measure $m_1^\alpha : \mathbb{T} \to [0, +\infty]$ on $\Im^\alpha$, the family of all intervals of $\mathbb{T}$, by using a monotone increasing function $\alpha : \mathbb{T} \to \mathbb{R}$, taking domain into account, as

i) $m_1^\alpha([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $m_1^\alpha([a,b]) = \alpha(\sigma(b)^+) - \alpha(a^-)$,

iii) $m_1^\alpha((a,b]) = \alpha(\sigma(b)^+) - \alpha(\sigma(a)^+)$,

iv) If $b > \sigma(a)$, $m_1^\alpha((a,b)) = \alpha(b^-) - \alpha(\sigma(a)^+)$.

The open interval $(a, \sigma(a))$ is understood as the empty set: then $m_1^\alpha((a,\sigma(a))) = 0$. Obviously, $[a,a)$ and $(a,a]$ are also empty sets and have pre-measures zero from the definition and need not to be specified separately. The Lebesgue-Stieltjes Δ-outer measure $(m_1^\alpha)^*$ associated with α is the function defined on all $E \subseteq \mathbb{T}$ by

$$(m_1^\alpha)^*(E) = \inf \sum_{i=1}^{\infty} m_1^\alpha(I_n),$$

provided that there exists at least one finite or countable covering system of intervals $I_n \subset \Im^\alpha$ of E as $E \subset \bigcup_{n=1}^{\infty} I_n$. If there is no such covering of E we say $(m_1^\alpha)^*(E) = \infty$. Let $A \subset \mathbb{T}$. If

$$(m_1^\alpha)^*(A) = (m_1^\alpha)^*(A \cap E) + (m_1^\alpha)^*(A \cap E^c)$$

holds, then we say E is $(m_1^\alpha)^*$- (or $\alpha_\Delta-$) measurable.

Lemma 3.1 *If E_1 and E_2 are α_Δ-measurable, so is $E_1 \cup E_2$*

Proof. Let E_1 and E_2 are α_Δ-measurable. Let for any $A \subset \mathbb{T}$. Since E_1 is α_Δ-measurable then we have

$$(m_1^\alpha)^*(A \cap E_2^c) = (m_1^\alpha)^*(A \cap E_2^c \cap E_1) + (m_1^\alpha)^*(A \cap E_2^c \cap E_1^c). \tag{6}$$

And since $A(E_1 \cup E_2) = (A \cap E_2) \cup (A \cap E_1 \cap E_2^c)$, then we have

$$(m_1^\alpha)^*(A \cap (E_1 \cup E_2)) \leq (m_1^\alpha)^*(A \cap E_2) + +(m_1^\alpha)^*(A \cap E_1 \cap E_2^c). \tag{7}$$

Thus by using inequities (6) and (7), and α_Δ-measurability of E_1 and E_2, we have

$$(m_1^\alpha)^*(A \cap (E_1 \cup E_2)) + (m_1^\alpha)^*(A \cap E_1^c \cap E_2^c) \leq$$

$$(m_1^\alpha)^*(A \cap E_2) + (m_1^\alpha)^*(A \cap E_1 \cap E_2^c) + (m_1^\alpha)^*(A \cap E_2^c \cap E_1^c) =$$

$$(m_1^\alpha)^*(A \cap E_2) + (m_1^\alpha)^*(A \cap E_2^c) = (m_1^\alpha)^*(A).$$

So $E_1 \cup E_2$ is α_Δ-measurable. □

Lemma 3.2 *If we set $E = \cup_{i=1}^\infty E_i$ is the union of a countable collection of pairwise disjoint of α_Δ-measurable sets the E is also α_Δ-measurable.*

Proof. Let $F_n = \cup_{i=1}^n E_i$ then F_n is α_Δ-measurable by previous Lemma and $F_n^c \supset E^c$.

Hence

$$(m_1^\alpha)^*(A) = (m_1^\alpha)^*(A \cap F_n) + (m_1^\alpha)^*(A \cap F_n^c) \geq (m_1^\alpha)^*(A \cap F_n) + (m_1^\alpha)^*(A \cap E^c),$$

Since $A \cap [\cup_{i=1}^n E_i] \cap E_n = A \cap E_n$ and $A \cap [\cup_{i=1}^n E_i] \cap E_n^c = A \cap [\cup_{i=1}^{n-1} E_i]$, and by the α_Δ-measurability of E_n, we have

$$(m_1^\alpha)^*(A \cap F_n) = (m_1^\alpha)^*(A \cap [\cup_{i=1}^n E_i]) = (m_1^\alpha)^*(A \cap E_n) + (m_1^\alpha)^*(A \cap [\cup_{i=1}^{n-1} E_i])$$

$$= (m_1^\alpha)^*(A \cap E_n) + \sum_{i=1}^{n-1} (m_1^\alpha)^*(A \cap E_i) = \sum_{i=1}^{n} (m_1^\alpha)^*(A \cap E_i).$$

Thus we have

$$(m_1^\alpha)^*(A \cap F_n) + \sum_{i=1}^{n} (m_1^\alpha)^*(A \cap E_i).$$

Then

$$(m_1^\alpha)^*(A) \geq \sum_{i=1}^{n} (m_1^\alpha)^*(A \cap E_i) + (m_1^\alpha)^*(A \cap E^c).$$

When $n \to \infty$, we have

$$(m_1^\alpha)^*(A) \geq \sum_{i=1}^{\infty} (m_1^\alpha)^*(A \cap E_i) + (m_1^\alpha)^*(A \cap E^c)$$

$$(m_1^\alpha)^*(A) \geq (m_1^\alpha)^*(A \cap E) + (m_1^\alpha)^*(A \cap E^c).$$

So m_1^α is a countably additive measure on $\Im^\alpha$. □

$\mathbf{M}((m_1^\alpha)^*)$, the family of all $(m_1^\alpha)^*$-measurable subset of $\mathbb{T}$, forms a σ-algebra. We restrict $(m_1^\alpha)^*$ to $\mathbf{M}((m_1^\alpha)^*)$ and denote by μ_Δ^α. This is the Lebesgue-Stieltjes Δ-measure generated by α.

All intervals on $\mathbb{T}$ are α_Δ-measurable since any interval can be covered by itself, which is the smallest cover, thus for any interval I, pre-measure $m_1^\alpha(I)$ and α_Δ-measure $\mu_\Delta^\alpha(I)$ coincide. That is,

i) $\mu_\Delta^\alpha([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $\mu_\Delta^\alpha([a,b]) = \alpha(\sigma(b)^+) - \alpha(a^-)$,

iii) $\mu_\Delta^\alpha((a,b]) = \alpha(\sigma(b)^+) - \alpha(\sigma(a)^+)$,

iv) $b > \sigma(a)$, $\mu_\Delta^\alpha((a,b)) = \alpha(b^-) - \alpha(\sigma(a)^+)$.

Proposition 3.3 *Let* $\{c\} \subset \mathbb{T}$. *Then it is* μ_Δ^α*-measurable and*

$$\mu_\Delta^\alpha(\{c\}) = \mu_\Delta^\alpha([c,c]) = \alpha(\sigma(c)^+) - \alpha(c^-). \tag{8}$$

Proof. It is obvious that single a point set is covered by itself as a closed interval, which is the smallest cover. □

Although $[c,c]$ and $[c,\sigma(c))$ has the same Δ-measure, it differs while considering α_Δ-measure, because in α_Δ-measure, we consider one-sided limits of an increasing function α at endpoints of given intervals.

Remark 3.4 *From Proposition 3.3, it is seen why* Δ*-measure of* $\max\mathbb{T}$ *is infinity. The reason is that we are not able to approach* $\max\mathbb{T}$ *from the right hand side. Furthermore, we are not allowed to take the limit of the minimum point of* $\mathbb{T}$ *from the left hand side. Thus, we say* α_Δ*-measures of maximum and minimum points of a time scale and also* α_Δ*-measure of any set containing at least one of them are undefined* (∞)*.*

However, up to now, Δ*-measure of the minimum point of a bounded below time scale has been introduced as an ordinary interior point of the time scale because the formula* $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0$ *stays the same; but extending theory, replacing measure with respect to an increasing function, it is seen that a single point set has* α_Δ*-measure* $\alpha(\sigma(t_0)^+) - \alpha(t_0^-)$*. And because of the fact that the limit from left hand side at* t_0 *is undefined, we finally get this result.*

Let $\mathbb{T} = \mathbb{R}$, then α_Δ and α measures (see [3]) coincide since for all $t \in \mathbb{T}$, $\sigma(t) = t$.

Let $\mathbb{T} = \mathbb{Z}$, then

i) $\mu_\Delta^\alpha([a,b)) = \alpha(b) - \alpha(a)$,

ii) $\mu_\Delta^\alpha([a,b]) = \alpha(b+1) - \alpha(a)$,

iii) $\mu_\Delta^\alpha((a,b]) = \alpha(b+1) - \alpha(a+1)$,

iv) For $b > a+1$, $\mu_\Delta^\alpha((a,b)) = \alpha(b) - \alpha(a+1)$.

Here we are not interested in right-sided and left-sided limits because all functions defined on $\mathbb{Z}$ is continuous.

Let $\alpha : \mathbb{T} \to \mathbb{T}$, $\alpha(t) = t$, then α_{Δ}-measure turns in to Δ-measure introduced by Guseinov [4] as follows:

i) $\mu^{\alpha}_{\Delta}([a,b)) = b - a$,

ii) $\mu^{\alpha}_{\Delta}([a,b]) = \sigma(b) - a$,

iii) $\mu^{\alpha}_{\Delta}((a,b]) = \sigma(b) - \sigma(a)$,

iv) If $b > \sigma(a)$, $\mu^{\alpha}_{\Delta}((a,b)) = b - \sigma(a)$.

Let us introduce the concept of Lebesgue-Stieltjes ∇-measure on time scales. Let $\alpha : \mathbb{T} \to \mathbb{R}$ be a monotone increasing function. We define a set function m^{α}_2 on the family of all intervals of $\mathbb{T}$ denoted by $\Im^{\alpha}$ as follows:

i) $m^{\alpha}_2([a,b)) = \alpha(\rho(b)^-) - \alpha(\rho(a)^-)$,

ii) $m^{\alpha}_1([a,b]) = \alpha(b^+) - \alpha(\rho(a)^-)$,

iii) $m^{\alpha}_1((a,b]) = \alpha(b^+) - \alpha(a^+)$,

iv) $a < \rho(b)$, $m^{\alpha}_1((a,b)) = \alpha(\rho(b)^-) - \alpha(a^+)$,

where The Lebesgue-Stieltjes ∇-outer measure $(m^{\alpha}_2)^*$ of a set E associated with α is the function defined on all subsets of $\mathbb{T}$ by $(m^{\alpha}_2)^*(E) = \inf \sum_{i=1}^{\infty} (m^{\alpha}_2)(I_n)$ provided that there exists at least one finite or countable covering of intervals $I_n \subset \Im^{\alpha}$ of E such that $E \subset \bigcup_{n=1}^{\infty} I_n$. If there is no such covering of E, we say that $(m^{\alpha}_2)^*(E) = \infty$.

By restriction of the outer measure to the family of all α_{∇}-measurable sets, we obtain a countably additive measure denoted by μ^{α}_{∇}. Similarly, any measurable set including maximum or minimum of a time scale has α_{∇}-measure infinity.

Remark 3.5 *All intervals on $\mathbb{T}$ are α_{∇}-measurable since any interval can be covered by itself which is the smallest cover. Thus for any interval I, pre-measure $m^{\alpha}_2(I)$ and α_{∇}-measure $\mu^{\alpha}_{\nabla}(I)$ coincide so*

i) $\mu^{\alpha}_{\nabla}([a,b)) = \alpha(\rho(b)^-) - \alpha(\rho(a)^-)$,

ii) $\mu^{\alpha}_{\nabla}([a,b]) = \alpha(b^+) - \alpha(\rho(a)^-)$,

iii) $\mu^{\alpha}_{\nabla}((a,b]) = \alpha(b^+) - \alpha(a^+)$,

iv) *If* $a < \rho(b)$, $\mu^{\alpha}_{\nabla}((a,b)) = \alpha(\rho(b)^-) - \alpha(a^+)$.

Let $\mathbb{T} = \mathbb{R}$, then α_{∇}-measure and α-measure coincide since for all $t \in \mathbb{T}$, $\rho(t) = t$. Let $\mathbb{T} = \mathbb{Z}$, then

i) $\mu^{\alpha}_{\nabla}([a,b)) = \alpha(b-1) - \alpha(a-1)$,

ii) $\mu_{\nabla}^{\alpha}([a,b]) = \alpha(b) - \alpha(a-1)$,

iii) $\mu_{\nabla}^{\alpha}((a,b]) = \alpha(b) - \alpha(a)$,

iv) For $b > a+1$, $\mu_{\nabla}^{\alpha}((a,b)) = \alpha(b-1) - \alpha(a)$.

Let $\alpha : \mathbb{T} \to \mathbb{T}$, $\alpha(t) = t$, then α_{Δ}-measure turns in to Δ-measure introduced by Guseinov [4]. That is,

i) $\mu_{\nabla}^{\alpha}([a,b)) = \rho(b) - \rho(a)$,

ii) $\mu_{\nabla}^{\alpha}([a,b]) = b - \rho(a)$,

iii) $\mu_{\nabla}^{\alpha}((a,b]) = b - a$,

iv) For $b > \sigma(a)$, $\mu_{\nabla}^{\alpha}((a,b)) = \rho(b) - a$.

Proposition 3.6 *Let $\{c\} \subset \mathbb{T}$. Then it is μ_{∇}^{α}-measurable and*

$$\mu_{\nabla}^{\alpha}(\{c\}) = \mu_{\nabla}^{\alpha}([c,c]) = \alpha(c^{+}) - \alpha(\rho(c)^{-}). \tag{9}$$

Proof. Proof is obvious from the proof of the Proposition 3.3. □

Example 3.7 *Let $\mathbb{T} = [0,3] \cup \{4\} \cup [6,9]$ and*

$$\alpha(t) = \begin{cases} 3 - e^{-t} & if\ 0 \le t \le 1 \\ 4 & if\ 1 < t < 3 \\ 2t+1 & if\ 3 \le t < 7 \\ t^2 & if\ 7 \le t \le 9 \end{cases}$$

Find the α_{Δ}-measure and α_{∇}-measure of the following sets:
$\{4\}$, $[3,6)$, $(8,9]$, $\{3\}$, $\{7\}$, $[0,1)$.

Solution. Let us first consider the α_{Δ}-measures of the sets:

a) $\mu_{\Delta}^{\alpha}(\{4\}) = \mu_{\Delta}^{\alpha}([4,4]) = \alpha(\sigma(4)^{+}) - \alpha(4^{-}) = \alpha(6^{+}) - \alpha(4^{-}) = 4$.

b) $\mu_{\Delta}^{\alpha}([3,6)) = \alpha(6^{-}) - \alpha(3^{-}) = 9$.

c) $\mu_{\Delta}^{\alpha}((8,9]) = \alpha(\sigma(9)^{+}) - \alpha(\sigma(8)^{-}) = \alpha(9^{+}) - \alpha(8^{-}) = \infty$ since limit from right hand side of α at $t = 9$ is not defined.

d) $\mu_{\Delta}^{\alpha}(\{3\}) = \mu_{\Delta}^{\alpha}([3,3]) = \alpha(\sigma(3)^{+}) - \alpha(3^{-}) = \alpha(4^{+}) - \alpha(3^{+}) = 3$.

e) $\mu_{\Delta}^{\alpha}([7,8]) = \alpha(\sigma(8)^{+}) - \alpha(7^{-}) = \alpha(8^{+}) - \alpha(7^{-}) = 49$.

f) $\mu_{\Delta}^{\alpha}([0,1)) = \alpha(1^{-}) - \alpha(0^{-}) = \infty$ since limit from right hand side of α at $t = 0$ is not defined.

Now, let us consider α_{∇}-measures of the given sets:

a) $\mu^{\alpha}_{\nabla}(\{4\}) = \mu^{\alpha}_{\nabla}([4,4]) = \alpha(4^+) - \alpha(\rho(4)^-) = \alpha(4^+) - \alpha(3^-) = 5$.

b) $\mu^{\alpha}_{\nabla}([3,6)) = \alpha(\rho(6)^-) - \alpha(\rho(3)^-) = \alpha(4^-) - \alpha(3^-) = 5$.

c) $\mu^{\alpha}_{\nabla}((8,9]) = \alpha(9^+) - \alpha(8^+) = \infty$ since $\alpha(9^+)$ is not defined.

d) $\mu^{\alpha}_{\nabla}(\{3\}) = \mu^{\alpha}_{\nabla}([3,3]) = \alpha(3^+) - \alpha(\rho(3)^-) = \alpha(3^+) - \alpha(3^-) = 3$.

e) $\mu^{\alpha}_{\nabla}([7,8]) = \alpha(8^+) - \alpha(\rho(7)^-) == \alpha(8^+) - \alpha(7^-) = 39$
since 7 is a left-dense point.

f) $\mu^{\alpha}_{\nabla}([0,1)) = \alpha(\rho(1)^-) - \alpha(\rho(0)^-) = \infty$
since $\alpha(\rho(0)^-)$ is not defined.

4. Relation Between Lebesgue-Stieltjes Measure and Lebesgue-Stieltjes Δ-Measure

In order to compare Lebesgue-Stieltjes measurable sets and Lebesgue-Stieltjes Δ-measurable sets, we need to extend α to the real numbers since α-measure of interval $(t_i, \sigma(t_i))$ for t_i is any right-scattered point is not defined. $\alpha(\sigma(t_i)^-) = \alpha(\sigma(t_i))$ since any function is left continuous at left-scattered point, similarly, $\alpha(t_i^+) = \alpha(t_i)$ since any function is right continuous at right-scattered points, the interval seems to have α-measure $\alpha(\sigma(t_i)) - \alpha(t_i)$. Although it is practically correct, the approach is theoretically wrong because of the fact that for α-measure of $(t_i, \sigma(t_i))$, α has to be defined on a set that includes this interval. Thus, we need to extend α. The extension could be any function whose reduction to $\mathbb{T}$ corresponds α, monotone everywhere and continuous at scattered points.
A proper choice would be as follows:

$$\alpha^{\sim}(t) = \begin{cases} \alpha(t) & \text{if} \quad t \in \mathbb{T} \\ \left(\dfrac{\alpha(\sigma(t_i)) - \alpha(t_i)}{\sigma(t_i) - t_i}\right) t & \text{if} \quad t \in (t_i, \sigma(t_i)) \end{cases} \tag{10}$$

which is a linear increasing function ($\alpha^{\triangle}(\sigma(t_i)) = \left(\dfrac{\alpha(\sigma(t_i)) - \alpha(t_i)}{\sigma(t_i) - t_i}\right) \geq 0$, since given that $\alpha(t)$ is increasing function) at each interval $(t_i, \sigma(t_i))$, where t_i is right-scattered point and continuous not only at right-scattered, but also at left-scattered points, so that we write $\alpha^{\sim}(t_i^+) = \alpha(t_i)$ where t_i is a right-scattered point and $\alpha^{\sim}(t_i^-) = \alpha(t_i)$ where t_i is a left-scattered point. Then it is clear that $\alpha^{\sim}(t)$ is also increasing.

Proposition 4.1 *Let $[a,b)$ be a half closed bounded interval of $\mathbb{T}$ with $a, b \in \mathbb{T} - \{\min \mathbb{T}\}$. Then*

i) $\mu^{\alpha}_{\Delta}([a,b)) = \mu^{\alpha^{\sim}}([a,b)) + \sum_{i \in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i))$.

ii) $\mu^{\alpha}_{\Delta}([a,b)) = \mu^{\alpha^{\sim}}([a,b)^{\sim})$.

where $[a,b)^{\sim}$ is the extension of $[a,b)$, that is obtained by filling the blanks $(t_i, \sigma(t_i))$ of the interval and $\mu^{\alpha^{\sim}}$ is the Lebesgue-Stieltjes measure generated by $\alpha^{\sim}$.

Proof. **i)** Let $\{t_n\} = \{t_1, t_2, ...t, b\}$ be the sequence of right scattered points of $[a,b)$ such that $a \leq t_1 \leq t_2 \leq ... \leq b$. Suppose that $s = \max\{t_n\}$. Then $[a,b)$ can be written as follows:

$$\begin{aligned} [a,b) &= [a,t_1] \cup [\sigma(t_1), t_2] \cup ... \cup [\sigma(s), b), \quad \text{so} \\ \mu^{\alpha^{\sim}}([a,b)) &= \mu^{\alpha^{\sim}}([a,t_1] \cup [\sigma(t_1), t_2] \cup ... \cup [\sigma(s), b)) \\ &= \mu^{\alpha^{\sim}}\big([a,t_1]\big) + \mu^{\alpha^{\sim}}\big([\sigma(t_1), t_2]\big) + ... + \mu^{\alpha^{\sim}}\big([\sigma(s), b)\big) \\ &= \alpha^{\sim}(t_1^{+}) - \alpha^{\sim}(a^{-}) + \alpha^{\sim}(t_2^{+}) - \alpha^{\sim}(\sigma(t_1)^{-}) + ... + \alpha^{\sim}(b^{-}) - \alpha^{\sim}(\sigma(s)^{-}) \\ &= \alpha^{\sim}(b^{-}) - \alpha^{\sim}(a^{-}) - \sum_{i \in I_{[a,b)}} (\alpha^{\sim}(\sigma(t_i)^{-}) - \alpha^{\sim}(t_i^{+})) \\ &= \alpha(b^{-}) - \alpha(a^{-}) - \sum_{i \in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)). \end{aligned}$$

Thus, we obtain

$$\mu^{\alpha^{\sim}}([a,b)) = \mu_{\Delta}^{\alpha}([a,b)) - \sum_{i \in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)).$$

ii) $(\alpha(\sigma(t_i)) - \alpha(t_i)) = (\alpha(\sigma(t_i)^{-}) - \alpha(t_i)^{+}) = \mu^{\alpha^{\sim}}((t_i, \sigma(t_i)))$,
so $\mu^{\alpha^{\sim}}([a,b)) - \sum_{i \in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)) = \mu^{\alpha^{\sim}}([a,b)^{\sim})$, and we get the desired result. □

Remark 4.2 *Obviously we can generalize Proposition 4.1 to any α_{Δ}-measurable set $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ as*

i) $\mu_{\Delta}^{\alpha}(E) = \mu^{\alpha^{\sim}}(E) + \sum_{i \in I_E} (\alpha(\sigma(t_i)) - \alpha(t_i))$.

ii) $\mu_{\Delta}^{\alpha}(E) = \mu^{\alpha^{\sim}}(E^{\sim})$.

where

$$E^{\sim} = E \cup \bigcup_{i \in I_E} (t_i, \sigma(t_i)). \tag{11}$$

and I_E is the indices set of all right scattered points of E.

5. Lebesgue-Stieltjes Δ-Integral

We will begin with considering Lebesgue-Stieltjes Δ-integral of a simple function.

Definition 5.1 *Let $\mathbb{T}$ be a time scale, $\alpha : \mathbb{T} \to \mathbb{R}$ be an increasing function, μ_{Δ}^{α} be α_{Δ}-measure defined on $\mathbb{T}$, $S : \mathbb{T} \to \mathbb{R}$ be a nonnegative α_{Δ}-measurable simple function such that $S(t) = \sum_{i=1}^{n} a_i \chi_{A_i}$ where A_is are pairwise* *disjoint α_Δ-measurable sets with $A_i = \{t : S(t) = a_i\}$. Then we define α_Δ-integral of S on a α_Δ-measurable set E as*

$$\int_E S(s)\Delta\alpha(s) = \sum_{i=1}^{n} a_i \mu_\Delta^\alpha(A_i \cap E). \tag{12}$$

If for some k, $a_k = 0$ and $\mu_\Delta^\alpha(A_k \cap E) = \infty$, we define $a_k \mu_\Delta^\alpha(A_k \cap E) = \infty$.

Example 5.2 *Let α and $\mathbb{T}$ be defined as in Example 3.7. Let*

$$S_1(t) = \begin{cases} 1 & if\ 0 \le t \le 3 \\ 4 & if\ 3 < t \le 9, \end{cases}$$

$$S_2(t) = \begin{cases} 1 & if\ 0 \le t < 3 \\ 4 & if\ 3 \le t \le 9. \end{cases}$$

Evaluate the integral of S_1 and S_2 on $[1,8]$ with respect to α and compare the results.

Solution. $[0,3] \cap [1,8] = [1,3]$ and $(3,9] \cap [1,8] = (3,8]$ and α_Δ-integral of S_1 on $[1,8]$ is

$$\int_{[1,8]} S_1(s)\Delta\alpha(s) = 1.\mu_\Delta^\alpha([1,3]) + 4.\mu_\Delta^\alpha((3,8])$$

where

$$\begin{aligned} \mu_\Delta^\alpha([1,3]) &= \alpha(\sigma(3)^+) - \alpha(1^-) \\ &= \alpha(4^+) - \alpha(1^-) \\ &= 9 - (3 - e^{-1}) \\ &= 6 + e^{-1} \end{aligned}$$

and

$$\begin{aligned} \mu_\Delta^\alpha((3,8]) &= \alpha(\sigma(8)^+) - \alpha(\sigma(3)^+) \\ &= \alpha(8^+) - \alpha(\sigma(3)^+) \\ &= 64 - 9 \\ &= 55. \end{aligned}$$

Thus we have

$$\int_{[1,8]} S_1(s)\Delta\alpha(s) = 1.(6 + e^{-1}) + 4.55 = 226 + e^{-1}.$$

$[0,3) \cap [1,8] = [1,3)$ and $[3,9] \cap [1,8] = [3,8]$ and from the definition, the α_Δ-integral of S_2 on $[1,8]$ is

$$\int_{[1,8]} S_2(s)\Delta\alpha(s) = 1.\mu_\Delta^\alpha([1,3)) + 4.\mu_\Delta^\alpha([3,8]).$$

where

$$\begin{aligned}\mu_{\Delta}^{\alpha}([1,3)) &= \alpha(3^-)-\alpha(1^-)\\ &= 4-(3-e^{-1})\\ &= 1+e^{-1}\end{aligned}$$

and

$$\begin{aligned}\mu_{\Delta}^{\alpha}([3,8]) &= \alpha(\sigma(8)^+)-\alpha(3^-)\\ &= \alpha(8^+)-\alpha(3^-)\\ &= 64-4\\ &= 60.\end{aligned}$$

Thus,

$$\int_{[1,8]} S_1(s)\Delta\alpha(s) = 1.(1+e^{-1})+4.60 = 241+e^{-1}.$$

Although these two simple functions are nearly the same, the reason for the difference of integrals is the behavior of the functions at the discontinuity point.

6. Relation Between Lebesgue-Stieltjes Integral and Lebesgue-Stieltjes Δ-Integral

In order to establish the relation between Lebesgue-Stieltjes measure constructed on time scales and the classical Lebesgue-Stieltjes integral we need to extend function defined on time scale to real numbers as shown in [2] as follows:

$$f^{\sim}(t) = \begin{cases} f(t) & if\ t\in\mathbb{T} \\ f(t_i) & if\ t\in(t_i,\sigma(t_i)). \end{cases} \tag{13}$$

Lemma 6.1 *Let E be an α_Δ-measurable set of $\mathbb{T}-\{\max\mathbb{T},\min\mathbb{T}\}$. Let $S:\mathbb{T}\to\mathbb{R}$ be a simple function with $S(t)=\sum_{i=1}^{n} a_i\chi_{A_i}$ where $A_i s$ are pairwise disjoint α_Δ-measurable sets, with $A_i=\{t: S(t)=a_i\}$ and $S^{\sim}=\sum_{i=1}^{n} a_i\chi_{A_i^{\sim}}$ be the extension of S as Equation 13, and $A_i^{\sim}$ and $E^{\sim}$ be the extensions of A_i and E that are obtained by filling the blanks of corresponding sets as in Equation 11, $\alpha^{\sim}$ be the extension of α as in Equation 10 and corresponding measures be denoted by $\mu^{\alpha^{\sim}}$; and μ_{Δ}^{α} is theusual Lebesgue-Stieltjes measure and Lebesgue-Stieltjes Δ-measure.*
Then

$$\int_E S(s)\Delta\alpha(s) = \int_{E^{\sim}} S^{\sim}(s)d\alpha^{\sim}(s).$$

Proof. We will use the fact that for each c_i, $S(t) = c_i$, $t \in A_i$, then $S^{\sim}(t) = c_i$, $t \in A_i^{\sim}$. Furthermore, $\mu_{\Delta}^{\alpha}(A_i \cap E) = \mu^{\alpha^{\sim}}(A_i^{\sim} \cap E^{\sim}) = \mu^{\alpha^{\sim}}(A_i \cap E)^{\sim}$.
Multiplying by a_i and summing from 1 to n both sides, we have

$$\sum_{i=1}^{n} a_i \mu_{\Delta}^{\alpha}(A_i \cap E) = \sum_{i=1}^{n} a_i \mu^{\alpha^{\sim}}(A_i \cap E)^{\sim}.$$

We get the integral of $S(t)$ on measurable set E with respect to α on the left hand side of the equation and the integral of $S^{\sim}(t)$ on measurable set $E^{\sim}$ with respect to $\alpha^{\sim}$ on the right hand side of the equation. Thus we get

$$\int_{E} S(s)\Delta\alpha(s) = \int_{E^{\sim}} S^{\sim}(s) d\alpha^{\sim}(s).$$

We know that if a set $A \subset \mathbb{T}$ is α_{Δ}-measurable, then A is α-measurable. □

Definition 6.2 *Let $E \subset \mathbb{T}$ be an α_{Δ}-measurable set and let $f : \mathbb{T} \to [0, +\infty]$ be an α_{Δ}-measurable function. The Lebesgue-Stieltjes Δ-integral of f on E is defined as*

$$\int_{E} f(s)\Delta\alpha(s) = \sup_{0 \leq S \leq f} \int_{E} S(s)\Delta\alpha(s)$$

The next lemma exhibits the Lebesgue-Stieltjes integral and Lebesgue-Stieltjes Δ-integral of functions. Since the proof is similar to a comparison of Lebesgue Δ-integral and usual Lebesgue integral in [2], we do not give the proof.

Lemma 6.3 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ be an α_{Δ}-measurable set, $f : \mathbb{T} \to [0, +\infty]$ be an α_{Δ}-measurable function and $f^{\sim}$ be the extension of f as in Equation 13. Then*

$$\int_{E} f(s)\Delta\alpha(s) = \int_{E^{\sim}} f^{\sim}(s) d\alpha(s)$$

where $E^{\sim}$ denotes the set defined in Equation 11.

Definition 6.4 *Let $E \subset \mathbb{T}$ be a α_{Δ}-measurable set and let $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a α_{Δ}-measurable function. We say that f is Lebesgue-Stieltjes Δ-integrable on E if at least one of the elements $\int_{E} f^{+}(s)\Delta\alpha(s)$ or $\int_{E} f^{-}(s)\Delta\alpha(s)$ is finite, where the positive and negative parts of f, f^{+} and f^{-} respectively. In this case, we define the Lebesgue-Stieltjes Δ-integral of f on E as*

$$\int_{E} f(s)\Delta\alpha(s) = \int_{E} f^{+}(s)\Delta\alpha(s) - \int_{E} f^{-}(s)\Delta\alpha(s).$$

Theorem 6.5 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ be an α_{Δ}-measurable set. If $f : \mathbb{T} \to \overline{\mathbb{R}}$ is α_{Δ}-integrable on E, then*

$$\int_{E} f(s)\Delta\alpha(s) = \int_{E} f(s) d\alpha(s) + \sum_{i \in I_E} f(t_i)(\alpha(\sigma(t_i)) - \alpha(t_i)),$$

where I_E denotes the set of indices of the right-scattered points of E.

Proof.

$$\begin{aligned}
\int_{E^{\sim}} f^{\sim}(s)d\alpha^{\sim}(s) &= \int_{E\cup\left(\bigcup(t_i,\sigma(t_i))\right)} f^{\sim}(s)d\alpha^{\sim}(s) \\
&= \int_{E} f^{\sim}(s)d\alpha^{\sim}(s) + \int_{\bigcup(t_i,\sigma(t_i))} f^{\sim}(s)d\alpha^{\sim}(s) \\
&= \int_{E} f(s)d\alpha(s) + \sum_{i\in I_E}\int_{(t_i,\sigma(t_i))} f(t_i)d\alpha^{\sim}(s) \\
&= \int_{E} f(s)d\alpha(s) + \sum_{i\in I_E} f(t_i)(\alpha(\sigma(t_i))-\alpha(t_i)).
\end{aligned}$$

and we conclude by Lemma 6.3 that,

$$\int_E f(s)\Delta\alpha(s) = \int_E f(s)d\alpha(s) + \sum_{i\in I_E} f(t_i)(\alpha(\sigma(t_i))-\alpha(t_i)). \qquad \square$$

Remark 6.6 *Let* $\mathbb{T}=h\mathbb{Z}$, $f:\mathbb{T}\to\mathbb{R}$ *and* α *be an increasing function with* $\alpha:\mathbb{T}\to\mathbb{R}$. *Since* $[a,b]=\bigcup_{k=1}^{n}\{a+(k-1)h\}$,

$$\begin{aligned}
\int_{[a,b]} f(s)\Delta\alpha(s) &= \int_{\bigcup_{k=1}\{a+(k-1)h\}} f(s)\Delta(s) \\
&= \sum_{k=1}^{n}\int_{\{a+(k-1)h\}} f(s)\Delta(s) \\
&= \sum_{k=1}^{n}\int_{\{a+(k-1)h\}} f(a+(k-1)h)\Delta(s) \\
&= \sum_{k=1}^{n} f(a+(k-1)h)\int_{\{a+(k-1)h\}}\Delta(s) \\
&= \sum_{k=1}^{n} f(a+(k-1)h)\mu_{\Delta}^{\alpha}(\{a+(k-1)h\}) \\
&= \sum_{k=1}^{n} f(a+(k-1)h)\big(\alpha(\sigma(a+(k-1)h)^{+})-\alpha((a+(k-1)h)^{-})\big) \\
&= \sum_{k=1}^{n} f(a+(k-1)h)\big(\alpha(a+kh)-\alpha(a+(k-1)h)\big).
\end{aligned}$$

Remark 6.7 *Let* $f:\mathbb{T}\to\mathbb{R}$, $\alpha:\mathbb{T}\to\mathbb{R}$, $\alpha(t)=c$, c *is any constant. Then*

$$\int_a^b f(s)\Delta\alpha(s)=0 \ \textit{ since for any } k,\ \ \Delta\alpha_k(t)=0.$$

Remark 6.8 *Suppose that* $f : [a,b) \to \mathbb{R},\ f(t) = c$*, where* c *is any constant. Then*

$$\begin{aligned} \int_{[a,b]} f(s)\Delta\alpha(s) &= \int_{[a,b]} c\Delta\alpha(s) \\ &= c\int_{[a,b]} \Delta\alpha(s) \\ &= c\ \mu_\Delta^\alpha([a,b]) = c\ (\alpha(b^+) - \alpha(a^-)). \end{aligned}$$

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