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Korovkin Type Error Estimates for Positive Linear Operators Involving Some Special Functions

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Abstract

In the present paper, we introduce a new sequence of linear positive operators with the help of generating functions. We obtain some Korovkin type approximation properties for these operators and compute rates of convergence by means of the first and second order modulus of continuities and Peetre's K-functional. In order to obtain explicit expressions for the first and second moment of our operators, we obtain a functional differential equation including our operators. Furthermore, we deal with a modification of our operators converging to integral of function f on the interval $(0,1)$.

Key Words: Positive linear operators, Korovkin-Bohman theorem, Bernstein power series, generating function, Pochhammer symbol, hypergeometric function, Peetre's K-functional, first and second order modulus of continuities, functional differential equation.

1. Introduction

The study of the Korovkin-Bohman type approximation theory is a well established area of active research (see, e.g., [4, 6, 14]). Especially, it has connections not only with classical approximation theory, but also with other branches of mathematics, such as functional analysis, harmonic analysis, measure theory, probability theory.

Cheney and Sharma [8], first introduced a new linear positive operators with the help of generating function expansion of Laguerre polynomial. Recently, two different generalizations of linear positive operators involving some generating functions have been introduced, and Korovkin type error estimates and their rates of convergences have been obtained (see [3, 4]).

We now turn to introducing our operators used in this paper.

Consider a new sequence of linear positive operators for $x \in [0,a]$, $a < 1$, $t \in [0,b]$, $b \in \mathbb{R}^+$,

$$(L_n f)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} f(\frac{k}{c_n + k - 1}) g_k^{(n)}(t) x^k, \tag{1.1}$$

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where $g_k^{(n)}(t) = \frac{d^n}{dt^n} g_k(t)$ and $\{c_n\}$ is a sequence satisfying $n \le c_n$. Let $\{F_n(x,t)\}$ be a generating function for the sequence of functions $\{g_k^{(n)}(t)\}_{k\in\mathbb{N}_0}$ $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ satisfying the equality

$$F_n(x,t) = (1-x)^{-c_n} \psi_n\left(\frac{-4xt}{(1-x)^2}\right) = \sum_{k=0}^{\infty} g_k^{(n)}(t) x^k, \tag{1.2}$$

where

$$\psi_n(u) = \sum_{k=0}^{\infty} \gamma_{n,k} u^k \ , \quad \gamma_{n,0} \neq 0.$$

We also assume that following conditions are satisfied.

1° $F_{n+1}(x,t) = p(x) F_n(x,t)\,, |\,p(x)| < M,\ x \in [0,a]$,

2° $(c_n + k - 1)\, g_{k-1}^{(n)}(t) - k g_k^{(n)}(t) = -t(g_k^{(n+1)}(t) + g_{k-1}^{(n+1)}(t));\ g_k^{(n)}(t) = 0$ for $k \in \mathbb{Z}^-$,

3° $\psi_n\left(\frac{-4xt}{(1-x)^2}\right) g_k^{(n)}(t) \ge 0$ $(k = 0,1,2,...)$ for all $x \in [0,a]$ and $t \in [0,b]$.

Remarks: If we choose $c_n = n+1,\ t = 0,\ \gamma_{n,0} = 1$ and $g_k^{(n)}(0) = \binom{n+k}{k}$, then we have the Bernstein power series (see [8])

$$(M_n f)(x) = (1-x)^{n+1} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right)\binom{n+v}{v} x^{\nu}.$$

Let $c_n = a_n - 1$ and

$$\psi_n\left(\frac{-4xt}{(1-x)^2}\right) = {}_2F_1\left(\frac{a_n}{2}\,,\ \frac{a_n-1}{2}\,;-;\ \frac{-4xt}{(1-x)^2}\right).$$

Since

$$\frac{{}_2F_1\left(\frac{a_n}{2}\,,\ \frac{a_n-1}{2}\,;-;\ \frac{-4xt}{(1-x)^2}\right)}{(1-x)^{a_n-1}} = \sum_{k=0}^{\infty} \frac{(a_n-1)_k}{k!}\, y_k(-b_n t; a_n, b_n),$$

one can easily get a new operators as follows

$$(Y_n f)(x,t) = \frac{(1-x)^{a_n-1}}{{}_2F_1\left(\frac{a_n}{2}\,,\ \frac{a_n-1}{2}\,;-;\ \frac{-4xt}{(1-x)^2}\right)} \times \sum_{k=0}^{\infty} f\left(\frac{k}{a_n+k-2}\right) \frac{(a_n-1)_k\, y_k(-b_n t; a_n, b_n)}{k!} x^k,$$

where

$$(a)_0 = 1,\ (a)_k = \prod_{i=0}^{k-1}(a+i)\ \ (k \in \mathbb{N},\ a \in \mathbb{R})$$

and

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \qquad (|x| < 1)$$

are known as Pochhammer symbol and hypergeometric function, respectively, and $y_k(-b_n t; a_n, b_n)$ are called as generalized Bessel polynomials satisfying the following recurrence relation (see [11, Chapter 2])

$$\begin{aligned}
&-b_n t\,\frac{(a_n-1)_k}{k!}\,y_k'(-b_n t;a_n,b_n)-k\frac{(a_n-1)_k}{k!}\,y_k(-b_n t;a_n,b_n)\\
&=-(a_n+k-2)\frac{(a_n-1)_{k-1}}{(k-1)!}\,y_{k-1}(-b_n t;a_n,b_n)+b_n t\frac{(a_n-1)_{k-1}}{(k-1)!}y_{k-1}'(-b_n t;a_n,b_n).
\end{aligned}$$

By replacing $-b_n t$ by t and using $(a)_k=(a+k-1)(a)_{k-1}$, after some simple calculations, we have the following recurrence relation (see, for instance, [19, Theorem 48, p.137]):

$$\begin{aligned}
&t\left[(a_n+k-2)y_k'(t;a_n,b_n)+k\,y_{k-1}'(t;a_n,b_n)\right]\\
&=(a_n+k-2)\,k\left[y_k(t;a_n,b_n)-\,y_{k-1}(t;a_n,b_n)\right].
\end{aligned}$$

Now choose $c_n=1+\alpha_n+\beta_n$ and

$$\psi_n(\frac{-4xt}{(1-x)^2})=\,{}_2F_1\left(\frac{1+\alpha_n+\beta_n}{2}\,,\,\frac{2+\alpha_n+\beta_n}{2}\,;\,1+\alpha_n\,;\,\frac{-4xt}{(1-x)^2}\right).$$

Since

$$\begin{aligned}
&\frac{1}{(1-x)^{1+\alpha_n+\beta_n}}\,{}_2F_1\left(\frac{1+\alpha_n+\beta_n}{2}\,,\,\frac{2+\alpha_n+\beta_n}{2}\,;\,1+\alpha_n\,;\,\frac{-4xt}{(1-x)^2}\right)\\
=\;&\sum_{k=0}^{\infty}\frac{(1+\alpha_n+\beta_n)_k}{(1+\alpha_n)_k}P_k^{(\alpha_n,\beta_n)}(1-2t),
\end{aligned}$$

our operators turn out to be the new operators

$$\begin{aligned}
(J_nf)(x,t)\;&=\;\frac{(1-x)^{1+\alpha_n+\beta_n}}{{}_2F_1\left(\frac{1+\alpha_n+\beta_n}{2}\,,\,\frac{2+\alpha_n+\beta_n}{2}\,;\,1+\alpha_n\,;\,\frac{-4xt}{(1-x)^2}\right)}\\
&\quad\times\sum_{k=0}^{\infty}f\left(\frac{k}{\alpha_n+\beta_n+k}\right)\frac{(1+\alpha_n+\beta_n)_k\,P_k^{(\alpha_n,\beta_n)}(1-2t)}{(1+\alpha_n)_k}x^k,
\end{aligned}$$

where $P_k^{(\alpha_n,\beta_n)}(1-2t)$ are known as Jacobi polynomials which satisfy the recurrence relation

$$\begin{aligned}
&-2t\,\frac{(1+\alpha_n+\beta_n)_k}{(1+\alpha_n)_k}\frac{d}{dt}P_k^{(\alpha_n,\beta_n)}(1-2t)-k\frac{(1+\alpha_n+\beta_n)_k}{(1+\alpha_n)_k}P_k^{(\alpha_n,\beta_n)}(1-2t)\\
&=-(1+\alpha_n+\beta_n-1)\frac{(1+\alpha_n+\beta_n)_{k-1}}{(1+\alpha_n)_{k-1}}P_{k-1}^{(\alpha_n,\beta_n)}(1-2t)+2t\frac{(1+\alpha_n+\beta_n)_{k-1}}{(1+\alpha_n)_{k-1}}\frac{d}{dt}P_{k-1}^{(\alpha_n,\beta_n)}(1-2t).
\end{aligned}$$

Replacing $\frac{1-t}{2}$ by t, we have the following recurrence relation for Jacobi polynomials (see [19, p. 262])

$$\begin{aligned}
&(t-1)\left[(\alpha_n+\beta_n+k)\frac{d}{dt}P_k^{(\alpha_n,\beta_n)}(t)+(\alpha_n+k)\frac{d}{dt}P_{k-1}^{(\alpha_n,\beta_n)}(t)\right]\\
&=(\alpha_n+\beta_n+k)\left[kP_k^{(\alpha_n,\beta_n)}(t)-(\alpha_n+k)P_{k-1}^{(\alpha_n,\beta_n)}(t)\right].
\end{aligned}$$

We know that if $\alpha_n = \beta_n$, then the Jacobi polynomials $P_k^{(\alpha_n,\alpha_n)}(t)$ are called as ultra spherical polynomials. In this case, our operators are written including ultra spherical polynomials as

$$\begin{aligned}(U_n f)(x,t) &= \frac{(1-x)^{1+2\alpha_n}}{{}_2F_1\left(\dfrac{1+2\alpha_n}{2}\,,\,1+\alpha_n\,;\,1+\alpha_n\,;\,\dfrac{-4xt}{(1-x)^2}\right)} \\ &\quad\times\sum_{k=0}^{\infty} f\left(\frac{k}{2\alpha_n+k}\right)\frac{(1+2\alpha_n)_k\,P_k^{(\alpha_n,\alpha_n)}(1-2t)}{(1+\alpha_n)_k}x^k.\end{aligned}$$

We also know that the Gegenbauer polynomials has the form (see [19, p. 277])

$$C_k^n(t) = \frac{(2n)_k P_k^{(n-\frac{1}{2}\,,\,n-\frac{1}{2})}(t)}{(n+\frac{1}{2})_k},$$

where $P_k^{(n-\frac{1}{2}\,,\,n-\frac{1}{2})}(t)$ are ultra spherical polynomials. Thus, choosing $\alpha_n = \beta_n = n-\frac{1}{2}$, then we obtain another new approximating operators including the Gegenbauer polynomials as follows:

$$(G_n f)(x,t) = \frac{(1-x)^{2n}}{{}_2F_1\left(n\,,n+\dfrac{1}{2}\,;\,n+\dfrac{1}{2}\,;\,\dfrac{-4xt}{(1-x)^2}\right)}\sum_{k=0}^{\infty} f\left(\frac{k}{2n+k-1}\right)C_k^n(1-2t)x^k.$$

In a similar manner, our operators L_n generate many new generalization of positive linear operators but we will omit them. Therefore the approximation properties obtained in the present paper are valid in a large spectrum of these type operators, including the some well-known special functions.

In the second part, using similar techniques given by Müller in [17] (see also [1, 3, 8, 9]), we obtain Korovkin type error estimates for our operators L_n. The third section addresses some problems concerning rates of convergence by means of first and second order modulus of continuity and Peetre's K-functional. In the fourth section, in order to obtain explicit expression for the central moments of our operators L_n, we give a functional differential equation. Finally, in the last part, we make a modification of our operators and investigate the rate of pointwise convergence for this modification.

2. Approximation of L_n

In this section, we approximate to continuous functions by means of the sequence of positive linear operators (1.1). Throughout the paper we use the test functions

$$e_i(x) = x^i, \quad i = 0, 1, 2.$$

Theorem 2.1 *If f is continuous on $[0,a]$, $a < 1$, then $(L_n f)(x,t_0)$ converges to $f(x)$ uniformly on $[0,a]$ for each fixed value of $t_0 \in [0,b]$.*

Proof. Because of the Korovkin-Bohman theorem, it will suffice to prove that $(L_n e_i)(x,t_0)$ tends to $e_i(x)$ as $n \to \infty$ for each $i = 0,1,2$. To check these conditions, we will use the similar technique in [17]. From (1.2) it is clear that $(L_n e_0)(x,t_0) \equiv e_0(x) = 1$. By using 2°, we have

$$\begin{aligned}(L_n e_1)(x,t_0) &= \frac{1}{F_n(x,t_0)} \sum_{k=0}^{\infty} \tfrac{k}{c_n+k-1} g_k^{(n)}(t_0) x^k \\ &= \frac{x}{F_n(x,t_0)} \sum_{k=1}^{\infty} \left\{ g_{k-1}^{(n)}(t_0) + \frac{t_0}{c_n+k-1}(g_k^{(n+1)}(t_0) + g_{k-1}^{(n+1)}(t_0)) \right\} x^{k-1}.\end{aligned} \tag{2.1}$$

Using 3°, we get

$$\frac{xt_0}{F_n(x,t_0)} \sum_{k=0}^{\infty} \frac{1}{c_n+k}(g_{k+1}^{(n+1)}(t_0) + g_k^{(n+1)}(t_0))x^k \geq 0.$$

From (2.1),

$$(L_n e_1)(x,t_0) \geq x \tag{2.2}$$

holds. Since $n \leq c_n$, using $\dfrac{1}{n} \geq \dfrac{1}{c_n+k}$ in (2.1), we have

$$\begin{aligned}(L_n e_1)(x,t_0) &\leq x + \frac{1}{n} \frac{xt_0}{F_n(x,t_0)} \sum_{k=0}^{\infty} (g_{k+1}^{(n+1)}(t_0) + g_k^{(n+1)}(t_0))x^k \\ &= x + \frac{(1+x)p(x)t_0}{n}.\end{aligned}$$

From the above inequality and (2.2), it is obvious that

$$\|(L_n e_1)(\cdot,t_0) - e_1(\cdot)\|_{C[0,a]} \leq \frac{bM(1+a)}{n}. \tag{2.3}$$

Finally, we have

$$(L_n e_2)(x,t_0) = \frac{1}{F_n(x,t_0)} \sum_{k=1}^{\infty} \frac{k^2}{(c_n+k-1)^2} g_k^{(n)}(t_0)x^k.$$

Using the recurrence formula 2° twice, we obtain

$$\begin{aligned}\left(\frac{k}{c_n+k-1}\right)^2 g_k^{(n)}(t_0) &= \frac{c_n+k-2}{c_n+k-1} g_{k-2}^{(n)}(t_0) + \frac{t_0}{c_n+k-1}(g_{k-1}^{(n+1)}(t_0) \\ &\quad + g_{k-2}^{(n+1)}(t_0)) + \frac{1}{c_n+k-1} g_{k-1}^{(n)}(t_0) \\ &\quad + \frac{kt_0}{(c_n+k-1)^2}(g_k^{(n+1)}(t_0) + g_{k-1}^{(n+1)}(t_0)).\end{aligned}$$

So, we may write that

$$\begin{aligned}(L_n e_2)(x,t_0) - e_2(x) &= \left(\frac{1}{F_n(x,t_0)}\sum_{k=2}^{\infty}\frac{c_n+k-2}{c_n+k-1}g_{k-2}^{(n)}(t_0)x^k - x^2\right)\\ &+\frac{t_0}{F_n(x,t_0)}\sum_{k=1}^{\infty}\frac{1}{c_n+k-1}(g_{k-1}^{(n+1)}(t_0)+g_{k-2}^{(n+1)}(t_0))x^k\\ &+\frac{1}{F_n(x,t_0)}\sum_{k=1}^{\infty}\frac{1}{c_n+k-1}g_{k-1}^{(n)}(t_0)x^k\\ &+\frac{t_0}{F_n(x,t_0)}\sum_{k=0}^{\infty}\frac{k}{(c_n+k-1)^2}g_{k}^{(n+1)}(t_0)x^k\\ &+\frac{t_0}{F_n(x,t_0)}\sum_{k=1}^{\infty}\frac{k}{(c_n+k-1)^2}g_{k-1}^{(n+1)}(t_0)x^k,\end{aligned}$$

and hence

$$(L_n e_2)(x,t_0) - e_2(x) := K_1 + K_2 + K_3 + K_4 + K_5. \tag{2.4}$$

By (1.2),

$$K_1 = \frac{x^2}{F_n(x,t_0)}\sum_{k=0}^{\infty}\left(\frac{c_n+k}{c_n+k+1}-1\right)g_k^{(n)}(t_0)x^k.$$

Since $\dfrac{-1}{c_n+k+1} < 0$, we can write

$$K_1 < 0. \tag{2.5}$$

Because of $\dfrac{1}{c_n+k+1} < \dfrac{1}{c_n+k} \le \dfrac{1}{n}$, we have

$$K_2 \le \frac{x(1+x)t_0 p(x)}{n} \tag{2.6}$$

and

$$K_3 \le \frac{x}{n}. \tag{2.7}$$

Finally, using $\dfrac{k}{(c_n+k-1)^2} \le \dfrac{2}{n}$, we get

$$K_4 \le \frac{2t_0 p(x)}{n} \tag{2.8}$$

and

$$K_5 \le \frac{2x t_0 p(x)}{n}. \tag{2.9}$$

Hence, by (2.5) – (2.9) in (2.4), we conclude that

$$(L_n s^2)(x,t_0) - x^2 \le \frac{x + t_0 p(x)(1+x)(2+x)}{n}. \tag{2.10}$$

On the other hand, using the expression

$$e_2(s) = s^2 = (s-x)^2 + 2xs - x^2,$$

we see that

$$(L_n e_2)(x,t_0) - e_2(x) = (L_n(s-x)^2)(x,t_0) + 2x(L_n(s-x))(x,t_0).$$

By (2.2) and positivity of L_n,

$$(L_n e_2)(x,t_0) - e_2(x) \geq 0. \tag{2.11}$$

Because of (2.10) and (2.11), we have

$$\|(L_n e_2)(\cdot,t_0) - e_2(\cdot)\|_{C[0,a]} \leq \frac{1}{n}\,(a + bM(1+a)(2+a)) \tag{2.12}$$

which gives the proof. □

3. Rates of Convergence

In this section, we compute the rates of convergence of the sequence $\{L_n\}$ to the function f by the means of the first and second modulus of continuities and Peetre's K-functional.

The modulus of continuity of f denoted by $\omega(f,\delta)$, is defined to be

$$\omega(f,\delta) = \sup_{|s-x|<\delta;\ s,x\in[0,a]} |f(s) - f(x)|.$$

It is well-known that necessary and sufficient condition for a function $f \in C[0,a]$ is

$$\lim_{\delta\to 0} \omega(f,\delta) = 0.$$

It is also well-known that for any $\delta > 0$ we have

$$|f(s) - f(x)| \leq \omega(f,\delta)\left(\frac{|s-x|}{\delta} + 1\right). \tag{3.1}$$

The next result gives the rate of convergence of the sequence $\{L_n\}$ to the function f (for all $f \in C[0,a]$) by means of the first modulus of continuity.

Theorem 3.1 *If $f \in C[0,a]$, then for all $x \in [0,a]$ and fixed $t_0 \in [0,b]$, we have*

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq (1 + \sqrt{a + (1+a)(2+3a)bM})\,\omega\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.2}$$

Proof. For the proof, we use similar technique of Popoviciu [18]. By linearity and monotonicity of L_n and (3.1), we obtain

$$|(L_n f)(x,t_0) - f(x)| \leq \omega\left(f,\delta_n\right)\left\{1 + \frac{1}{\delta_n}\left[\frac{1}{F_n(x,t_0)}\sum_{k=0}^{\infty}\left|\frac{k}{c_n+k-1} - x\right| g_k^{(n)}(t_0)x^k\right]\right\}.$$

By the Cauchy-Schwarz inequality, we have

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq \omega\left(f,\delta_n\right)\left[1 + \frac{1}{\delta_n}\left(\sup_{x\in[0,a]}\varphi_{n,2}(x,t_0)\right)^{1/2}\right], \tag{3.3}$$

where

$$\varphi_{n,2}(x,t_0) = (L_n(s-x)^2)(x,t_0) \tag{3.4}$$

is the second central moment of L_n. For each $x \in [0,a]$, we can write

$$\varphi_{n,2}(x,t_0) \leq \left|(L_n s^2)(x,t_0) - x^2\right| + 2x\left|(L_n s)(x,t_0) - x\right|.$$

So, by (2.3) and (2.12) we get

$$\|\varphi_{n,2}(.,t_0)\|_{C[0,a]} \leq \frac{a + (1+a)(2+3a)bM}{n}. \tag{3.5}$$

If we use (3.5) in (3.3), we obtain the desired result. □

In order to estimate a rate of convergence via second modulus of continuity, we benefit from the Peetre's K-functional. Let us define the following space and norm:

$C^2[0,a] :=$ The space of the functions f of which $f,\ f',\ f'' \in C[0,a]$.

We define the norm in the space $C^2[0,a]$

$$\|f\|_{C^2[0,a]} := \|f\|_{C[0,a]} + \|f'\|_{C[0,a]} + \|f''\|_{C[0,a]}$$

and the following Peetre's K-functional [6] (see also [7]) is

$$K\left(f,\delta_n\right) = \inf_{h\in C^2[0,a]}\left\{\|f-h\|_{C[0,a]} + \delta_n\|h\|_{C^2[0,a]}\right\}. \tag{3.6}$$

Theorem 3.2 *If* $f \in C[0,a]$, *then for each fixed value of* $t_0 \in [0,b]$, *we have*

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq C\,K(f;\frac{1}{n}), \tag{3.7}$$

where

$$C = \max\left\{2, \frac{1}{2}\left(a + (1+a)(4+3a)bM\right)\right\}. \tag{3.8}$$

Proof. If $h \in C^2[0,a]$, then we have

$$h(s) - h(x) = h'(x)(s-x) + \int_x^s h''(u)(s-u)du. \tag{3.9}$$

Applying the operator L_n to (3.9), we get

$$|(L_n h)(x,t_0) - h(x)| \leq \left[\varphi_{n,1}(x,t_0) + \frac{1}{2}\varphi_{n,2}(x,t_0)\right] \|h\|_{C^2[0,a]}. \tag{3.10}$$

On the other hand, since L_n is a linear operator, we have

$$|(L_n f)(x,t_0) - f(x)| \leq |(L_n(f-h)(x,t_0)| + |f(x) - h(x)| + |(L_n h)(x,t_0) - h(x)|.$$

Thus, by using $L_n(1;x) \equiv 1$ and (3.10), we can write that

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq 2\|f-h\|_{C[0,a]} + \left[\|\varphi_{n,1}(\cdot,t_0)\|_{C[0,a]} + \frac{1}{2}\|\varphi_{n,2}(.,t_0)\|_{C[0,a]}\right] \|h\|_{C^2[0,a]} \tag{3.11}$$

If we use (2.3) and (3.5) in (3.11), we easily see that

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,b]} \leq 2\|f-h\|_{C[0,a]} + \frac{1}{2n}(a + (1+a)(4+3a)bM)\|h\|_{C^2[0,b]}. \tag{3.12}$$

Choosing C as in (3.8) and taking infimum over $g \in C^2\ [0,a]$ from the second hand-side of (3.12), we obtain (3.7). □

Let $f \in C[0,A]$, the second order modulus of continuity of f denoted by $\omega_2(f,\delta)$ is defined as

$$\omega_2(f,\delta) = \sup \left\{ \begin{array}{c} |f(x+h) - 2f(x) + f(x-h)|; \\ (x \mp h) \in [0,1], |h| \leq \delta \end{array} \right\}.$$

This modulus is also known as Zygmund modulus for the function f. The following theorem estimates the rate of convergence of the sequence $\{L_n\}$ to the function f via Zygmund modulus.

Theorem 3.3 *If $f \in C[0,a]$, then for each $0 \leq \delta \leq 1$ and for each fixed value of $t_0 \in [0,b]$, we have*

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq C_f \max\left\{\omega_2\left(f, \frac{1}{\sqrt{n}}\right), \frac{1}{n}\right\}. \tag{3.13}$$

Proof. By using the inequality (see [7, Proposition 3.4.1])

$$K(f,\delta) \leq C_1\left[\omega_2(f,\sqrt{\delta}) + \min\{1,\delta\}\|f\|_{C[0,a]}\right]$$

in (3.6), and choosing $\delta = \frac{1}{n}$, we have

$$\|(L_n f)(\cdot,t_0) - f(\cdot)\|_{C[0,a]} \leq C\,C_1\left[\omega_2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|_{C[0,a]}\right], \tag{3.14}$$

where C is defined in (3.8). If $\frac{1}{n} \le \omega_2\left(f, \frac{1}{\sqrt{n}}\right)$ then from (3.14), we get

$$\|(L_n f)(\cdot, t_0) - f(\cdot)\|_{C[0,a]} \le CC_1 \left[1 + \|f\|_{C[0,a]}\right] \omega_2\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.15}$$

Otherwise, if $\omega_2\left(f, \frac{1}{\sqrt{n}}\right) < \frac{1}{n}$, then, from (3.14),

$$\|(L_n f)(\cdot, t_0) - f(\cdot)\|_{C[0,a]} \le CC_1 \left[1 + \|f\|_{C[0,a]}\right] \frac{1}{n} \tag{3.16}$$

holds. Now, choosing $C\,C_1 \left[1 + \|f\|_{C[0,a]}\right] = C_f$ in (3.15) and (3.16), we immediately obtain (3.13). □

4. An Application to Differential Equations

The explicit expressions of the moments can easily compute for a lot of well-known positive linear operators. Some of them are Bernstein polynomials, Bernstein Chlodowsky polynomials, Szasz-Mirakjan and Baskakov operators and their generalization as Gadjiev-Ibragimov operators [10]. But these computations may be difficult for some operators. For example, the second moment of Meyer-König and Zeller operators [16] was not obtained in an explicit form by Müller [17], Sikkema [20], Lupaş and Müller [15], Becker and Nessel [5] respectively. Similar estimations are given for Bleimann, Butzer and Hahn operators in [6].

Recently, Alkemade [2] obtained a functional differential equation so that Meyer-König and Zeller operators are particular solutions of it. He also obtained an explicit formula for the second moment of Meyer-König and Zeller operators with the help of this functional differential equation. In this part, we will obtain a functional differential equation including our operators.

First, in addition to conditions $1^0 - 3^0$, let us assume that the condition

$$4^\circ \qquad \frac{\partial}{\partial x}\psi_n\left(\frac{-4xt}{(1-x)^2}\right) = K_n(x,t)\psi_n\left(\frac{-4xt}{(1-x)^2}\right)$$

holds.

Theorem 4.1 *Under the conditions $1^0 - 4^0$, $(L_n f)(x, t_0)$ satisfies the differential equation*

$$x\frac{d}{dx}(L_n f)(x,t_0) = -x\left(\frac{c_n}{1-x} + K_n(x,t_0)\right)(L_n f)(x,t_0) + (c_n - 1)(L_n f\, h)(x,t_0). \tag{4.1}$$

for each fixed $t_0 \in [0,b]$ and $f \in C[0,a]$, where $h(s) = \frac{s}{1-s}$.

Proof. From 4^0, we have

$$\frac{d}{dx}F_n(x,t_0) = \left(\frac{c_n}{1-x} + K_n(x,t_0)\right)F_n(x,t_0). \tag{4.2}$$

We can differentiate the power series on the right-hand side of (1.1) term by term in $[0, a]$ since it converges on $[0, a]$. Thus, we have

$$\frac{d}{dx}(L_n f)(x, t_0) = -\left(\frac{c_n}{1-x} + K_n(x, t_0)\right)(L_n f)(x, t_0) + \frac{1}{F_n(x,t)} \sum_{k=1}^{\infty} f\left(\frac{k}{c_n + k - 1}\right) g_k^{(n)}(t) k x^{k-1}.$$

Using $k = (c_n - 1)h\left(\frac{k}{c_n + k - 1}\right)$ in this equation, we obtain the desired result. □

Note that (4.1) is not a differential equation but rather a functional differential equation. In the light of Theorem 4.1, using the similar method given by Alkemade [2], if we have explicit expressions of $\{F_n(x,t)\}$ and $\left\{g_k^{(n)}(t)\right\}$, then we can obtain explicit expressions for first and second moment of L_n.

5. A Modification of L_n

Let us recall the following quadrature formula of the mid-point

$$\int_0^1 f(x)dx = \frac{1}{n} \sum_{i=1}^{n} f((2i-1)/2n) + R_n(f),$$

where $R_n(f)$ is remainder term. Error of this remainder term, obtained by Korneichuk in [13] (see also [12]), is

$$R_n(f) = \int_0^1 \omega(f, \frac{u}{2n})du,$$

where $\omega(f, \frac{u}{2n})$ denotes modulus of continuity of the function f.

Now consider a modification of the sequence $\{L_n\}$ as

$$(L_n^* f)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{c_n + k - 1} \sum_{i=1}^{c_n+k-1} f\left(\frac{2i-1}{2(c_n + k - 1)}\right) g_k^{(n)}(t)\, x^k, \tag{5.1}$$

where $c_n \in \mathbb{N}$.

Suppose that $R_n^*(f)$ is remainder term of the following quadrature formula at the mid-point:

$$\int_0^1 f(x)dx = \frac{1}{c_n + k - 1} \sum_{i=1}^{c_n+k-1} f\left(\frac{2i-1}{2(c_n + k - 1)}\right) + R_n^*(f). \tag{5.2}$$

We first obtain the error estimate of remainder term of the formula (5.2) via the Zygmund modulus.

Lemma 5.1 *If f is integrable function on $(0,1)$, then we have*

$$|R_n^*(f)| \le \frac{1}{n}\omega_2(f, \frac{1}{n}). \tag{5.3}$$

Proof. For the reminder term in (5.2), we can write

$$|R_n^*(f)| = \left|\sum_{i=1}^{c_n+k-1}\int_{\frac{i-1}{c_n+k-1}}^{\frac{i}{c_n+k-1}}\left[f(x)-f\left(\frac{2i-1}{2(c_n+k-1)}\right)\right]dx\right| = \left|\sum_{i=1}^{c_n+k-1}\left[\int_{\frac{i-1}{c_n+k-1}}^{\frac{2i-1}{2(c_n+k-1)}}\left[f(x)-f\left(\frac{2i-1}{2(c_n+k-1)}\right)\right]dx\right.\right.$$

$$\left.\left.+\int_{\frac{2i-1}{2(c_n+k-1)}}^{\frac{i}{c_n+k-1}}\left[f(x)-f\left(\frac{2i-1}{2(c_n+k-1)}\right)\right]dx\right]\right| = \left|\sum_{i=1}^{c_n+k-1}(I_1+I_2)\right|.$$

With the substitution $x=\frac{2i-s-1}{2(c_n+k-1)}$ in the integral I_1 and $x=\frac{2i+s-1}{2(c_n+k-1)}$ in the integral I_2, we have

$$|R_n^*(f)| \le \frac{1}{2(c_n+k-1)}\sum_{i=1}^{c_n+k-1}\int_0^1\left|f\left(\frac{2i-s-1}{2(c_n+k-1)}\right)-2f\left(\frac{2i-1}{2(c_n+k-1)}\right)+f\left(\frac{2i+s-1}{2(c_n+k-1)}\right)\right|ds. \quad (5.4)$$

By using $\omega_2(f,\delta_n)$ for $\delta_n=\dfrac{s}{2(c_n+k-1)}$ in (5.4), we obtain

$$|R_n^*(f)| \le \frac{1}{2(c_n+k-1)}\int_0^1 \omega_2(f,\frac{s}{2(c_n+k-1)})ds.$$

Since $2(c_n+k-1)\ge n$, the proof is completed. □

As a result of this lemma, we have this corollary:

Corollary 5.2 *If f is integrable function on $(0,1)$, then we have*

$$\left|(L_n^*f)(x_0,t_0)-\int_0^1 f(u)du\right| \le \frac{1}{n}\omega_2\left(f,\frac{1}{n}\right)$$

for each fixed $x_0\in(0,1)$ and $t_0\in[0,b]$.

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