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Multiple Positive Solutions for Nonlinear Third-Order Boundary Value Problems in Banach Spaces

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Abstract

This paper deals with the positive solutions of nonlinear boundary value problems in Banach spaces. By using fixed point index theory, some sufficient conditions for the existence of at least one or two positive solutions to boundary value problems in Banach spaces are obtained. An example illustrating the main results is given.

Key Words: Positive solutions; boundary value problem; Banach spaces; fixed point index.

1. Introduction

In this paper, we consider the following boundary value problem (BVP) for third-order differential equations in a Banach space E:

$$u'''(t) + f(t, u(t)) = \theta, \qquad 0 \leq t \leq 1 \tag{1.1}$$

subject to the boundary conditions

$$u(0) = u'(0) = \theta, \ u'(1) = \alpha u'(\eta), \tag{1.2}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$, $f \in C[[0,1] \times P, P]$; P is a cone of Banach space E, θ is the zero element of E.

Boundary value problems arise from applied mathematics and physics, and they have received a great deal of attention in the literature. Problems of the form (1.1) subject to (1.2), for example, are used to model such phenomena as the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1]. Third-order boundary value problems have been studied widely in the literature (see [1-12] and references therein). However, all of the above-mentioned references consider (1.1) only in scalar space. On the other hand, the theory of ordinary differential equations (ODE) in abstract spaces is becoming an important branch of mathematics in last thirty years because of its application in partial differential equations and ODE's in appropriately infinite dimensional spaces (see, for

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example [13,14]). As a result the goal of this paper is to fill up the gap in this area, that is, to investigate the existence of multiple positive solutions of (1.1) with (1.2) in a Banach space E.

This paper is organized as follows. Section 2 gives some preliminaries and some lemmas. Section 3 is devoted to the proof of the main results. Finally, in Section 4, one example is worked out to illustrate our main results.

2. Preliminaries and Lemmas

In this paper, we suppose throughout that E is a real Banach space. A nonempty closed convex subset P in E is said to be a cone which defines a partial ordering in E by $x \leq y$ if and only if $y-x \in P$, P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P (it is clear, $N \geq 1$). For details on cone theory, see [15]. Let S be a bounded subset of a Banach space. $\alpha(S)$ denotes the Kuratowski's measure of noncompactness of S. In this paper, $\alpha(\cdot)$ denotes the Kuratowski's measure of noncompactness of a bounded subset of both E and $C[[0,1],E]$. Let

$$C[[0,1],E] = \{u : [0,1] \to E \mid u(t) \text{ is continuous on } [0,1]\},$$

$$C^3[[0,1],E] = \{u : [0,1] \to E \mid u(t) \text{ is third order continuously differentiable in } [0,1]\}.$$

For $u = u(t) \in C[[0,1],E]$, let $\|u\|_C = \max_{0\leq t\leq 1} \|u(t)\|$, then $C[[0,1],E]$ becomes a Banach space. Let $P = \{u \in C[[0,1],E] \mid u(t) \geq \theta, t \in [0,1]\}$, then P is a cone in $C[[0,1],E]$. An operator $u(t) \in C[[0,1],E] \cap C^3[[0,1],E]$ is called a positive solution of the BVP (1.1)-(1.2) if $u(t)$ satisfies (1.1)-(1.2) and $u \in P$, $u(t) \not\equiv \theta$, $t \in [0,1]$.

Lemma 2.1 *([12]) Let $\alpha\eta \neq 1$. Then for $y \in C[[0,1],E]$, the BVP*

$$\begin{aligned} &u''' + y(t) = \theta, \quad t \in [0,1], \\ &u(0) = u'(0) = \theta, \ u'(1) = \alpha u'(\eta), \end{aligned}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha-1), & t \leq s \leq \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\ t^2(1-s), & \max\{\eta, t\} \leq s \end{cases} \tag{2.1}$$

is called the Green's function.

Lemma 2.2 *([12]) Let $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. Then for $G(t,s)$ defined in (2.1), we have estimates*

(i) for any $(t,s) \in [0,1] \times [0,1], 0 \leq G(t,s) \leq \Phi(s)$, where

$$\Phi(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s).$$

(ii) for any $(t,s)\in[\frac{\eta}{\alpha},\eta]\times[0,1],\ G(t,s)\ge\gamma\Phi(s)$, *where*

$$0<\gamma=\frac{\eta^2}{2\alpha^2(1+\alpha)}\min\{\alpha-1,1\}<1.$$

Let us list some conditions.

(H_1) $f\in C[[0,1]\times P,P]$, $f(t,u)$ is bounded and uniformly continuous in t on $[0,1]\times P_r$, where $P_r=\{u\in P|\|u\|\le r\}$, and there exists $0\le L<\frac{1}{2M_1}$ such that for $t\in[0,1]$ and bounded $D\subset P$,

$$\alpha(f(t,D))\le L\alpha(D) \tag{2.2}$$

holds, where $M_1=\max\limits_{s\in[0,1]}\Phi(s)$.

(H_2) $f\in C[[0,1]\times P,P]$, $f(t,u)$ is bounded and uniformly continuous in t on $[0,1]\times P_r$ for any $r>0$ and there exists a $h\in C[[0,1],\mathbb{R}_+]$ with $\int_0^1 h(t)dt<\frac{1}{2M_1}$ such that for $t\in[0,1]$ and bounded $D\subset P$,

$$\alpha(f(t,D))\le h(t)\alpha(D) \tag{2.3}$$

holds, where $M_1=\max\limits_{s\in[0,1]}\Phi(s)$.

(H_3) $\frac{\|f(t,u)\|}{\|u\|}\to 0$ as $u\in P$ and $\|u\|\to 0$ uniformly in $t\in[0,1]$.

(H_4) $\frac{\|f(t,u)\|}{\|u\|}\to 0$ as $u\in P$ and $\|u\|\to\infty$ uniformly in $t\in[0,1]$.

(H_5) There exist $u_0\in\mathrm{int}(P)$, and $k\in C[[\frac{\eta}{\alpha},\eta],\mathbb{R}]$ with

$$\int_{\frac{\eta}{\alpha}}^{\eta}k(t)\Phi(t)dt>(\gamma)^{-1}$$

such that $f(t,u)\ge k(t)u_0$ for $t\in[\frac{\eta}{\alpha},\eta]$ and $u\ge u_0$.

(H_6) There exist $u_0\in P\backslash\{\theta\}$, and $k\in C[[\frac{\eta}{\alpha},\eta],\mathbb{R}]$ with

$$\int_{\frac{\eta}{\alpha}}^{\eta}k(t)\Phi(t)dt\ge(\gamma)^{-1}$$

such that $f(t,u)\ge k(t)u_0$ for $t\in[\frac{\eta}{\alpha},\eta]$ and $u\ge u_0$.

Remark 2.1 It is clear that (H_2) is weaker than (H_1), and (H_6) is weaker than (H_5). Also, condition (H_1) is satisfied automatically when E is finite dimensional.

Now we define

$$(Au)(t)=\int_0^1 G(t,s)f(s,u(s))ds,\ \forall\ u\in P. \tag{2.4}$$

Then $(Au)(t)\ge\theta,\ t\in[0,1]$, and using the Lebesgue dominated convergence theorem we know that $(Au)(t)$ is continuous on $[0,1]$, hence the integral operator $A:P\to P$. Further, we can easily show that

(i) If $u\in P$, then $(Au)'''(t)=-f(t,u),\ t\in[0,1]$, hence $Au\in P\cap C^3[[0,1],E]$.

(ii) If $u \in P$ satisfies $Au = u$, then u is a solution of the BVP $(1.1)-(1.2)$.

Therefore, the BVP $(1.1)-(1.2)$ is equivalent to the operator equation $Au = u,\ u \in P$.

Lemma 2.3 *([15,16]) Let D be a bounded set of E and $f : [0,1] \times D \to E$ be bounded. Assume that $f(t,u)$ is uniformly continuous with respect to t. Then, we have*

$$\alpha(f([0,1] \times S)) = \max_{t\in[0,1]} \alpha(f(t,S)), \quad S \subset D.$$

Lemma 2.4 *([15,16]) If $H \subset C[[0,1],E]$ is bounded and equicontinuous, then*

$$\alpha(H([0,1])) = \max_{t\in[0,1]} \alpha(H(t)),$$

where $H([0,1]) = \{u(t) : u \in H,\ t \in [0,1]\}$.

Lemma 2.5 *([15,16]) If $H \subset C[[0,1],E]$ is countable and is bounded, then $\alpha(H(t)) \in L[[0,1],\mathbb{R}_+]$ and*

$$\alpha\Big(\Big\{\int_0^1 u(t)dt | u \in H\Big\}\Big) \le 2\int_0^1 \alpha(H(t))dt.$$

Lemma 2.6 *Assume that (H_1) holds. Then for any $r > 0$, operator A is a strict set contraction on $D \subset P_r$.*

Proof. Since f is uniformly continuous and bounded on P_r, we see from (2.4) that A is continuous and bounded on P_r. Since $f(t,u)$ is bounded and uniformly continuous in t on $[0,1] \times P_r$ for any $r > 0$, it follows from Lemma 2.3 and (2.2) that

$$\alpha(f([0,1] \times D)) = \max_{t\in[0,1]} \alpha(f(t,D)) \le L\alpha(D),\ D \subset P_r. \tag{2.5}$$

Let $S \subset C[[0,1],E]$ be bounded. We know that $A(S) \subset C[[0,1],E]$ is bounded and equicontinuous, so, by Lemma 2.4,

$$\alpha(A(S)) = \max_{t\in[0,1]} \alpha(A(S(t))), \tag{2.6}$$

where $A(S(t)) = \{(Au)(t) | u \in S\} \subset E$ (t is fixed). Using the formula

$$\int_0^1 y(t)dt \in \overline{co}\{y(t)|t \in [0,1]\} \text{ for } y \in C[[0,1],E],$$

and observing Lemma 2.2 and (2.5), we find

$$\begin{aligned} \alpha(A(S(t))) &= \alpha(\{\int_0^1 G(t,s)f(s,u(s))ds)|u \in S\}) \\ &\le \alpha(\overline{co}\{G(t,s)f(s,u(s))|s \in [0,1],\ u \in S\}) \\ &\le M_1\alpha(\{f(s,u(s))|s \in [0,1],\ u \in S\}) \\ &\le M_1\alpha(f([0,1] \times B) \le M_1 L\alpha(B),\ t \in [0,1], \end{aligned} \tag{2.7}$$

where $B = \{u(s)|s \in [0,1],\ u \in S\}$. For any given $\varepsilon > 0$, there is a partition $S = \bigcup_{j=1}^{n} S_j$ such that

$$\mathrm{diam}(S_j) < \alpha(S) + \frac{\varepsilon}{3}, \quad j = 1, 2, \cdots, n. \tag{2.8}$$

Choose $u_j \in S_j (j = 1, s, \cdots, n)$ and a partition of $J : a = t_0 < t_1 < \cdots < t_m = b$ such that

$$\|u_j(t) - u_j(s)\| < \frac{\varepsilon}{3}, \quad \forall\ j = 1, s, \cdots, n;\ t, s \in [t_{i-1}, t_i],\ i = 1, 2, \cdots, m. \tag{2.9}$$

Obviously, $B = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} B_{ij}$, where $B_{ij} = \{u(s)|s \in [t_{i-1}, t_i],\ u \in S_j\}$. For any $u(t),\ \overline{u}(\overline{t}) \in B_{ij}(t, \overline{t} \in [t_{i-1}, t_i],\ u, \overline{u} \in S_j)$. It follows from (2.8), (2.9) that

$$\begin{aligned}\|u(t) - \overline{u}(\overline{t})\| &\leq \|u(t) - u_j(t)\| + \|u_j(t) - u_j(\overline{t})\| + \|u_j(\overline{t}) - \overline{u}(\overline{t})\| \\ &\leq \|u - u_j\|_C + \tfrac{\varepsilon}{3} + \|u_j - \overline{u}\|_C \leq 2\mathrm{diam}(S_j) + \tfrac{\varepsilon}{3} < 2\alpha(S) + \varepsilon.\end{aligned}$$

Consequently,

$$\mathrm{diam}(B_{ij}) \leq 2\alpha(S) + \varepsilon,\ \forall\ i = 1, 2, \cdots, m, j = 1, s, \cdots, n,$$

and so $\alpha(B) \leq 2\alpha(S) + \varepsilon$, which implies, since ε is arbitrary,

$$\alpha(B) \leq 2\alpha(S). \tag{2.10}$$

It follows then from (2.6), (2.7), (2.10) that $\alpha(A(S)) \leq 2M_1 L\alpha(S)$, $\quad S \subset P_r$ with $2M_1 L < 1$, and Lemma 2.6 is proved. □

Lemma 2.7 *Assume that* (H_2) *holds. Then* A *defined by (2.4) is a bounded and continuous operator from* $C[[0,1], P]$ *into C[[0,1],P]; moreover, for any bounded and countable set* $S \subset C[[0,1], E]$, *we have* $\alpha(A(S)) \leq 2kM_1\alpha(S)$, *where*

$$k = \max_{t \in [0,1]} \int_0^1 h(t)dt. \tag{2.11}$$

Proof. Since f is uniform continuous and bounded on P_r, we see from (2.4) that A is continuous and bounded on P_r. Let $S \subset C[J, E]$ be bounded and countable. As in the proof of Lemma 2.6, (2.6) holds. By Lemma 2.5 and (2.3), we have

$$\begin{aligned}\alpha(A(S(t))) &= \alpha(\{\int_0^1 G(t,s)f(s,u(s))ds)|u \in S\}) \\ &\leq 2\int_0^1 G(t,s)\alpha(\{f(s,u(s))|s \in [0,1],\ u \in S\})ds \\ &\leq 2M_1 \int_0^1 h(s)ds\alpha(S),\ \forall\ t \in [0,1].\end{aligned} \tag{2.12}$$

It follows from (2.6) and (2.12) that $\alpha(A(S)) \leq 2kM_1\alpha(S)$. □

We will apply the following fixed point index to obtain solutions of the BVP (1.1) − (1.2).

Lemma 2.8 *([16]) Let X be a nonempty closed convex subset of E and U be a nonempty bounded open convex subset of X. Assume that $A : \overline{U} \to X$ is a strict set contraction such that $A(\overline{U}) \subset U$. Then*

$$i(A, U, X) = 1.$$

Lemma 2.9 *([16]) Let D be a bounded, closed and convex subset of E. Assume that the continuous operator $A : D \to D$ has the property:*

$$C \subset D \text{ countable, } \quad \text{not relatively compact} \quad \Rightarrow \alpha(A(C)) < \alpha(C).$$

Then A has a fixed point in D.

3. Main Results

Theorem 3.1 *Let P be a normal and solid cone in E. Suppose that conditions $(H_1), (H_3)$-(H_5) are satisfied. Then boundary value problems (1.1) − (1.2) has at least two positive solutions $u_1,\ u_2 \in C^3[[0, 1], E] \cap C[[0, 1], P]$ such that $u_1(t) \gg u_0$ for $t \in [\frac{\eta}{\alpha}, \eta]$.*

Proof. Conditions (H_3) and (H_4) imply that we can find two numbers r_1 and r_2 such that

$$0 < r_1 < \frac{\|u_0\|}{N} < r_2 \tag{3.1}$$

and

$$\|f(t, u)\| \le \frac{\|u\|}{2M_1}, \quad \forall\ t, s \in [0, 1],\ u \in P,\ \|u\| \le r_1 \text{ and } \|u\| \ge r_2, \tag{3.2}$$

where u_0 is the element in condition (H_5) and N denotes the normal constant of P. Consequently,

$$\|f(t, u)\| \le \frac{\|u\|}{2M_1} + M, \quad \forall\ t, s \in [0, 1],\ u \in P, \tag{3.3}$$

where

$$M = \sup\{\|f(t, u)\| | t, s \in [0, 1],\ u \in P,\ \|u\| \le r_2\}.$$

Choose

$$r_3 > \max\{2MM_1, r_2\} \tag{3.4}$$

and set $U_1 = \{u \in C[[0, 1], P] | \|u\|_C < r_1\}$, $U_3 = \{u \in C[[0, 1], P] | \|u\|_C < r_3\}$ and $U_2 = \{u \in C[[0, 1], P] | \|u\| < r_3 \text{ and } u(t) \gg u_0 \text{ for } t \in [\frac{\eta}{\alpha}, \eta]\}$. Obviously, U_1 and U_2 are bounded open convex sets of $C[[0, 1], P]$. Moreover, from (3.1) and (3.4) we find

$$U_1 \subset U_3,\ U_2 \subset U_3,\ U_1 \cap U_2 = \emptyset. \tag{3.5}$$

On the other hand, it is clear that $\overline{U}_1 = \{u \in C[[0,1],P] | \|u\|_C \leq r_1\}$, $\overline{U}_3 = \{u \in C[[0,1],P] | \|u\|_C \leq r_3\}$ and $\overline{U}_2 \subset \{u \in C[[0,1],P] | \|u\| \leq r_3 \text{ and } u(t) \geq u_0 \text{ for } t \in [\frac{\eta}{\alpha},\eta]\}$. Now, (3.2), (3.3) and (3.4) imply that

$$u \in \overline{U}_1 \Rightarrow \|Au\|_C \leq \frac{1}{2}\int_0^1 \|u(s)\| ds \leq \frac{1}{2}\|u\|_C < r_1$$

and

$$u \in \overline{U}_3 \Rightarrow \|Au\|_C \leq \int_0^1 (\frac{1}{2}\|u(s)\| + M) ds \leq \frac{r_3}{2} + M < r_3,$$

hence

$$A(\overline{U}_1) \subset U_1, \quad A(\overline{U}_3) \subset U_3. \tag{3.6}$$

For $u \in \overline{U}_2$, we have $\|u\|_C \leq r_3$ and $u(t) \geq u_0$ for $t \in [\frac{\eta}{\alpha},\eta]$, and so $\|Au\|_C < r_3$, and by (H_5),

$$t \in [\frac{\eta}{\alpha},\eta] \Rightarrow (Au)(t) \geq \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s) f(s,u(s)) ds \geq \gamma \int_{\frac{\eta}{\alpha}}^{\eta} k(s)\Phi(s) ds \; u_0 > u_0,$$

which implies $(Au)(t) \gg u_0$ for $t \in [\frac{\eta}{\alpha},\eta]$, and consequently,

$$A(\overline{U}_2) \subset U_2. \tag{3.7}$$

It follows from (3.6), (3.7) and Lemma 2.8 that the fixed point index

$$i(A, U_j, C[[0,1],P]) = 1 \;\; (j = 1,2,3). \tag{3.8}$$

Hence, A has a fixed point $u_1 \in U_2$ which satisfies $u_1(t) \gg u_0$ for $t \in [\frac{\eta}{\alpha},\eta]$. On the other hand, (3.8) implies

$$i(A, U_3 \backslash (\overline{U}_1 \cup \overline{U}_2), C[[0,1],P]) = i(A, U_3, C[[0,1],P]) - i(A, U_1, C[[0,1],P]) - i(A, U_2, C[[0,1],P]) = -1 \neq 0,$$

so A has a fixed point $u_2 \in U_3 \backslash (\overline{U}_1 \cup \overline{U}_2)$, and the theorem is proved. □

Remark 3.1 Condition (H_3) and the continuity of f imply that $f(t,\theta) = \theta$ for $t \in [0,1]$. Hence, under conditions of Theorem 3.1, BVP(1.1) − (1.2) has the trivial solution $u(t) \equiv \theta$ besides two positive solutions u_1 and u_2.

Theorem 3.2 *Suppose that the conditions $(H_2), (H_4)$ and (H_6) are satisfied. Then boundary value problems (1.1) − (1.2) has at least one positive solution $u \in C[[0,1],P]$ such that $u(t) \geq u_0$ for $t \in [\frac{\eta}{\alpha},\eta]$.*

Proof. By virtue of condition (H_2) and Lemma 2.7, the operator A defined by (2.4) is a bounded and continuous operator from $C[[0,1],P]$ into $C[[0,1],P]$. And for any bounded and countable set $S \subset C[[0,1],P]$, $\alpha(A(S)) \leq 2kM_1\alpha(S)$, where $2kM_1 < 1$. On the other hand, as in the proof of Theorem 3.1, (3.2) holds for some $r_2 > \|u_0\|$, and so, (3.3) holds. Choosing r_3 such that (3.4) is satisfied and letting $D = \{u \in C[[0,1],P] | \|u\| \leq r_3 \text{ and } u(t) \geq u_0 \text{ for } t \in [\frac{\eta}{\alpha},\eta]\}$, we see clearly that D is bounded closed convex set of $C[[0,1],E]$ and $D \neq \emptyset$ since $u^* \in D$, where $u^*(t) \equiv u_0$ for $t \in [0,1]$. In a similar way as for the proof of (3.7), we can show $A(D) \subset D$. Hence, Lemma 2.9 implies that A has a fixed point in D. The theorem is proved. □

Remark 3.2 One difference between Theorems 3.1 and 3.2 is that Theorem 3.1 requires the cone P to be both normal and solid, while Theorem 3.2, by contrast, does not.

4. One Example

In this section, in order to illustrate our results, we consider an example.

Example 4.1 Consider the following one dimensional BVP

$$\begin{cases} -u'''(t) = e^t \sin^2 u + 12\sqrt{tu}\ln(1+tu), & 0 \le t \le 1, \\ u(0) = u'(0) = 0, \ u'(1) = \frac{3}{2}u'(\frac{1}{2}). \end{cases} \tag{4.1}$$

Conclusion *BVP (4.1) has two positive solutions* $u_1, \ u_2 \in C^3[[0,1],\mathbb{R}]$ *such that* $u_1(t) > 0, \ u_2(t) > 0$ *for* $0 < t \le 1$ *and* $u_1(t) > 1$ *for* $\frac{1}{3} \le t \le \frac{1}{2}$.

Proof. Let $E = \mathbb{R}^1, \ P = [0,\infty)$, then P is a normal solid cone in E and (4.1) can be regarded as a BVP of the form (1.1), where

$$f(t,u) = e^t \sin^2 u + 12\sqrt{tu}\ln(1+tu),$$

$f : [0,1] \times P \to P$ is continuous. Taking $\eta = \frac{1}{2}, \ \alpha = \frac{3}{2}$, the Green's function satisfies $G(t,s) \le \Phi(s) = 10s(1-s), \ (t,s) \in [0,1] \times [0,1]$.

It is clear that $(H_1), (H_3), (H_4)$ are satisfied and it is easy to see that condition (H_5) of Theorem 3.1 is satisfied for $u_0 = 1$ and $k(t) = 12(ln\frac{3}{2})\sqrt{t}$. Observing $f(t,u) > 0$ for $t > 0$ and $u > 0$, our conclusion follows from Theorem 3.1. □

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