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Stability of an Euler–Lagrange type Cubic Functional Equation

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Abstract

In this paper, we will find out the general solution and investigate the generalized Hyers–Ulam–Rassias stability problem for an Euler–Lagrange type cubic functional equation

$$2mf(x+my)+2f(mx-y)=(m^3+m)[f(x+y)+f(x-y)]+2(m^4-1)f(y)$$

in Banach spaces and in left Banach modules over a unital Banach $*$-algebra for a fixed integer m with $m\neq 0,\pm 1$.

Key Words: Hyers–Ulam–Rassias stability, cubic functional equation.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [14] concerning the stability of group homomorphisms: *Let* $(G_1,*)$ *be a group and let* $(G_2,\diamond,d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon>0$, *does there exist a* $\delta(\epsilon)>0$ *such that if a mapping* $h:G_1\to G_2$ *satisfies the inequality*

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, *then there is a homomorphism* $H:G_1\to G_2$ *with*

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [11]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 6, 9, 13]. The terminology *'generalized Hyers–Ulam–Rassias stability'* originates from

2000 *AMS Mathematics Subject Classification:* Primary 39B52, 46L05, 47B48 these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [3, 4, 12].

Jun and Kim [5] introduced the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.1}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.1).

It is easy to see that the function $f(x)=cx^3$ is a solution of the above functional equation (1.1). Thus, it is natural that (1.1) is called a *cubic functional equation* and every solution of the cubic functional equation (1.1) is said to be a *cubic mapping*.

Jun *et al.* [7] introduced the *Euler–Lagrange type cubic functional equation*

$$f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2a(a^2-1)f(x) \tag{1.2}$$

for a fixed integer a with $a\neq 0,\pm 1$, and they showed that the functional equation (1.1) is equivalent to the functional equation (1.2) (also see [8]).

The first author and C. Park [10] introduced the cubic functional equation

$$2f(x+2y)+f(2x-y)=5f(x+y)+5f(x-y)+15f(y) \tag{1.3}$$

and established the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.3).

In this paper, we deal with the following Euler–Lagrange type cubic functional equation

$$2mf(x+my)+2f(mx-y)=(m^3+m)[f(x+y)+f(x-y)]+2(m^4-1)f(y) \tag{1.4}$$

for a fixed integer m with $m\neq 0,\pm 1$, and we establish the general solution and the generalized Hyers–Ulam–Rassias stability problem for the Euler–Lagrange type cubic functional equation (1.4).

Every solution of the functional equations (1.2) and (1.4) is said to be an *Euler–Lagrange type cubic mapping*.

2. Solution of Eq. (1.4)

Let both E_1 and E_2 be real vector spaces. We here present the general solution of (1.4).

Theorem 2.1 *[7, 10] Let $f: E_1 \to E_2$ be a mapping. The following statements are equivalent:*

- **(i)** *f satisfies the functional equation (1.1);*
- **(ii)** *f satisfies the functional equation (1.2);*
- **(iii)** *f satisfies the functional equation (1.3);*
- **(iv)** *there exists a function $B: E_1\times E_1\times E_1 \to E_2$ such that $f(x)=B(x,x,x)$ for all $x\in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables.*

Theorem 2.2 *Let $f : E_1 \to E_2$ be a mapping. The following statements are equivalent:*

(**i**) *f satisfies the functional equation (1.1);*

(**ii**) *f satisfies the functional equation (1.2);*

(**iii**) *f satisfies the functional equation (1.3);*

(**iv**) *f satisfies the functional equation (1.4);*

(**v**) *there exists a function $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables.*

Proof. We show that (**v**) $\Rightarrow$ (**iv**) $\Rightarrow$ (**ii**), and it proves the theorem.

If we assume that (**v**) holds, by a simple computation we get (**iv**).

Now, we assume that (**iv**) holds. Since $m \neq 1$, by putting $x = y = 0$ in (1.4), we get that $f(0) = 0$. Letting $y = 0$ and $x = 0$ in (1.4), respectively, we get that $f(mx) = m^3 f(x)$ and $f(-y) = -f(y)$, respectively, for all $x, y \in E_1$. So the mapping f is odd. Replacing x and y by $-y$ and x in (1.4), respectively, and using the oddness of f, we get

$$2mf(mx - y) - 2f(x + my) = (m^3 + m)[f(x - y) - f(x + y)] + 2(m^4 - 1)f(x) \tag{2.1}$$

for all $x, y \in E_1$. Multiplying both sides of (2.1) to m, and adding the obtained functional equation to (1.4), we get

$$\begin{aligned} 2(m^2 + 1)f(mx - y) &= (m^3 + m)(m + 1)f(x - y) + (m^3 + m)(1 - m)f(x + y) \\ &\quad + 2m(m^4 - 1)f(x) + 2(m^4 - 1)f(y) \end{aligned} \tag{2.2}$$

for all $x, y \in E_1$. Replacing y by $-y$ in (2.2) and using the oddness of f, we get

$$\begin{aligned} 2(m^2 + 1)f(mx + y) &= (m^3 + m)(m + 1)f(x + y) + (m^3 + m)(1 - m)f(x - y) \\ &\quad + 2m(m^4 - 1)f(x) - 2(m^4 - 1)f(y) \end{aligned} \tag{2.3}$$

for all $x, y \in E_1$. Adding (2.2) to (2.3), we infer

$$(m^2 + 1)[f(mx + y) + f(mx - y)] = (m^3 + m)[f(x + y) + f(x - y)] + 2m(m^4 - 1)f(x) \tag{2.4}$$

for all $x, y \in E_1$. Dividing both sides of (2.4) by $m^2 + 1$, we get (1.2). Therefore (**ii**) holds and the theorem is proved. □

3. Generalized Hyers–Ulam–Rassias stability of Eq. (1.4)

From now on, let X and Y be a real normed space with norm $\|\cdot\|_X$ and a real Banach space with norm $\|\cdot\|_Y$, respectively. In this section, using an idea of Găvruta [1], we prove the stability of Eq. (1.4) in the spirit of Hyers, Ulam and Th.M. Rassias. Thus we find some conditions that there exists a true cubic mapping near a approximately cubic mapping. Throughout this paper, m is an integer with $m \neq 0, \pm 1$. For convenience, we use the following abbreviation:

$$\begin{aligned} D_m f(x,y) :&= 2mf(x+my) + 2f(mx-y) \\ &\quad - (m^3+m)[f(x+y)+f(x-y)] - 2(m^4-1)f(y) \end{aligned}$$

for all $x, y \in X$.

Theorem 3.1 *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{|m|^{3n}} \varphi(m^n x, 0) < \infty, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{|m|^{3n}} \varphi(m^n x, m^n y) = 0 \tag{3.2}$$

for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|D_m f(x,y)\|_Y \leq \varphi(x,y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T_m : X \to Y$ which satisfies the inequality

$$\Big\| T_m(x) - f(x) - \frac{m^4-1}{m^3-1} f(0) \Big\|_Y \leq \frac{1}{2|m|^3} \widetilde{\varphi}(x) \tag{3.4}$$

for all $x \in X$. The mapping $T_m : X \to Y$ is given by

$$T_m(x) = \lim_{n\to\infty} \frac{1}{m^{3n}} f(m^n x) \tag{3.5}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (3.3) and dividing both sides of (3.3) by $2|m|^3$, we have

$$\Big\| \frac{f(mx)}{m^3} - f(x) - \frac{m^4-1}{m^3} f(0) \Big\|_Y \leq \frac{1}{2|m|^3} \varphi(x,0) \tag{3.6}$$

for all $x \in X$. Replacing x by $m^n x$ in (3.6) and dividing both sides of (3.6) by $|m|^{3n}$, we get

$$\Big\| \frac{f(m^{n+1}x)}{m^{3(n+1)}} - \frac{f(m^n x)}{m^{3n}} - \frac{m^4-1}{m^{3(n+1)}} f(0) \Big\|_Y \leq \frac{1}{2|m|^{3(n+1)}} \varphi(m^n x, 0) \tag{3.7}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \Big\| \sum_{i=k}^{n} \Big[\frac{f(m^{i+1}x)}{m^{3(i+1)}} - \frac{f(m^i x)}{m^{3i}} - \frac{m^4-1}{m^{3(i+1)}} f(0) \Big] \Big\|_Y &\leq \sum_{i=k}^{n} \Big\| \frac{f(m^{i+1}x)}{m^{3(i+1)}} - \frac{f(m^i x)}{m^{3i}} - \frac{m^4-1}{m^{3(i+1)}} f(0) \Big\|_Y \\ &\leq \frac{1}{2|m|^3} \sum_{i=k}^{n} \frac{1}{|m|^{3i}} \varphi(m^i x, 0) \end{aligned}$$

for all $x \in X$ and all integers $n \geq k \geq 0$. Hence

$$\Big\|\frac{f(m^{n+1}x)}{m^{3(n+1)}} - \frac{f(m^k x)}{m^{3k}} - \sum_{i=k}^{n} \frac{m^4-1}{m^{3(i+1)}} f(0)\Big\|_Y \leq \frac{1}{2|m|^3} \sum_{i=k}^{n} \frac{1}{|m|^{3i}} \varphi(m^i x, 0) \tag{3.8}$$

for all $x \in X$ and all integers $n \geq k \geq 0$. Since the series $\sum_{i=0}^{\infty} \frac{1}{m^{3i}}$ is convergent, it follows from (3.1) and (3.8) that the sequence $\{\frac{f(m^n x)}{m^{3n}}\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is a Banach space, it follows that the sequence $\{\frac{f(m^n x)}{m^{3n}}\}$ converges. We define $T_m : X \to Y$ by (3.5). It follows from (3.2) and (3.3) that

$$\|D_m T_m(x,y)\|_Y = \lim_{n\to\infty} \frac{1}{|m|^{3n}} \big\|D_m f(m^n x, m^n y)\big\|_Y \leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \varphi(m^n x, m^n y) = 0$$

for all $x, y \in X$. So T_m satisfies Eq. (1.4). Hence by Theorem 2.2, T_m is cubic.

Putting $k = 0$ and letting $n \to \infty$ in (3.8), we get (3.4).

It remains to show that T_m is unique. Suppose that there exists another cubic mapping $Q : X \to Y$ which satisfies (3.4). Since $Q(m^n x) = m^{3n} Q(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, we conclude from (3.1) and (3.4) that

$$\begin{aligned}
\|Q(x) - T_m(x)\|_Y &= \lim_{n\to\infty} \frac{1}{|m|^{3n}} \big\|Q(m^n x) - f(m^n x)\big\|_Y \\
&\leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \Big\|Q(m^n x) - f(m^n x) - \frac{m^4-1}{m^3-1} f(0)\Big\|_Y \\
&\leq \lim_{n\to\infty} \frac{1}{2|m|^{3n+3}} \widetilde{\varphi}(m^n x) = 0
\end{aligned}$$

for all $x \in X$. Hence we have $Q(x) = T_m(x)$ for all $x \in X$, which gives the conclusion. □

Corollary 3.2 *Let $\theta, \delta, \epsilon, p, q$ be non-negative real numbers such that $0 < p, q < 3$. Suppose that a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|D_m f(x,y)\|_Y \leq \theta + \epsilon \|x\|_X^p + \delta \|y\|_X^q$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T_m : X \to Y$ which satisfies the inequality

$$\|f(x) - T_m(x)\|_Y \leq \frac{1}{2}\Big(\frac{\theta}{|m|^3 - 1} + \frac{\epsilon}{|m|^3 - |m|^p} \|x\|_X^p\Big)$$

for all $x \in X$. Also, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T_m(tx) = t^3 T_m(x)$ for all $t \in \mathbb{R}$.

Proof. Define $\varphi : X \times X \to [0, \infty)$ by $\varphi(x,y) = \theta + \epsilon \|x\|_X^p + \delta \|y\|_X^q$ for all $x, y \in X$. So the result follows from Theorem 3.1. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, by the same reasoning as in the proof of [11], the cubic mapping $T_m : X \to Y$ satisfies $T_m(tx) = t^3 T_m(x)$ for all $t \in \mathbb{R}$. □

Theorem 3.3 *Let* $\varphi : X \times X \to [0, \infty)$ *be a function such that*

$$\widetilde{\varphi}(x) := \sum_{n=1}^{\infty} |m|^{3n} \varphi(\frac{x}{m^n}, 0) < \infty, \tag{3.9}$$

$$\lim_{n\to\infty} |m|^{3n} \varphi(\frac{x}{m^n}, \frac{y}{m^n}) = 0 \tag{3.10}$$

for all $x, y \in X$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|D_m f(x, y)\|_Y \le \varphi(x, y) \tag{3.11}$$

for all $x, y \in X$. *Then there exists a unique cubic mapping* $T_m : X \to Y$ *which satisfies the inequality*

$$\|f(x) - T_m(x)\|_Y \le \frac{1}{2|m|^3} \widetilde{\varphi}(x) \tag{3.12}$$

for all $x \in X$. *The mapping* $T_m : X \to Y$ *is given by*

$$T_m(x) = \lim_{n\to\infty} m^{3n} f(\frac{x}{m^n}) \tag{3.13}$$

for all $x \in X$.

Proof. It follows from (3.9) that $\varphi(0, 0) = 0$, and therefore (3.11) implies that $f(0) = 0$.

Putting $y = 0$ in (3.11), we have

$$\|f(mx) - m^3 f(x)\|_Y \le \frac{1}{2} \varphi(x, 0) \tag{3.14}$$

for all $x \in X$. Replacing x by $\frac{x}{m^{n+1}}$ in (3.14) and multiplying both sides of (3.14) to $|m|^{3n}$, we get

$$\Big\|m^{3(n+1)} f(\frac{x}{m^{n+1}}) - m^{3n} f(\frac{x}{m^n})\Big\|_Y \le \frac{1}{2} |m|^{3n} \varphi(\frac{x}{m^{n+1}}, 0) \tag{3.15}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore we have

$$\begin{aligned} \Big\|m^{3(n+1)} f(\frac{x}{m^{n+1}}) - m^{3k} f(\frac{x}{m^k})\Big\|_Y &= \Big\|\sum_{i=k}^{n} \big[m^{3(i+1)} f(\frac{x}{m^{i+1}}) - m^{3i} f(\frac{x}{m^i})\big]\Big\|_Y \\ &\le \sum_{i=k}^{n} \Big\|m^{3(i+1)} f(\frac{x}{m^{i+1}}) - m^{3i} f(\frac{x}{m^i})\Big\|_Y \\ &\le \frac{1}{2|m|^3} \sum_{i=k+1}^{n} |m|^{3i} \varphi(\frac{x}{m^i}, 0) \end{aligned} \tag{3.16}$$

for all $x \in X$ and all integers $n \ge k \ge 0$. It follows from (3.9) and (3.16) that the sequence $\{m^{3n} f(\frac{x}{m^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{m^{3n} f(\frac{x}{m^n})\}$ converges in Y for all $x \in X$. So one can define the mapping $T_m : X \to Y$ by (3.13).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4 *Let δ, ϵ, p, q be non-negative real numbers such that $p, q > 3$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|D_m f(x, y)\|_Y \le \epsilon \|x\|_X^p + \delta \|y\|_X^q$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T_m : X \to Y$ which satisfies the inequality

$$\|f(x) - T_m(x)\|_Y \le \frac{\epsilon}{2(|m|^p - |m|^3)} \|x\|_X^p$$

for all $x \in X$. Also, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T_m(tx) = t^3 T_m(x)$ for all $t \in \mathbb{R}$.

Proof. Define $\varphi : X \times X \to [0, \infty)$ by $\varphi(x, y) = \epsilon \|x\|_X^p + \delta \|y\|_X^q$ for all $x, y \in X$. It is clear that $f(0) = 0$. So the result follows from Theorem 3.3. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, by the same reasoning as in the proof of [11], the cubic mapping $T_m : X \to Y$ satisfies $T_m(tx) = t^3 T_m(x)$ for all $t \in \mathbb{R}$. □

4. Results in Banach modules over a unital Banach $*$-algebra

In this section, let B be a unital Banach $*$-algebra with norm $\|\cdot\|_B$, and let $\mathbb{X}$ and $\mathbb{Y}$ be left Banach B-modules with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, respectively. For $a \in B$, let $b = a^3, aa^*a, a^*aa^*$ or $(aa^*a + a^*aa^*)/2$. A cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ is called B-cubic if $T(ax) = bT(x)$ for all $a \in B$ and all $x \in \mathbb{X}$.

Theorem 4.1 *Let $\varphi : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be a function satisfying (3.1) and (3.2) (respectively, (3.9) and (3.10)) for all $x, y \in X$. Suppose that a mapping $f : \mathbb{X} \to \mathbb{Y}$ satisfies*

$$\|2mf(ax + may) + 2f(max - ay) - (m^3 + m)[f(ax + ay) - f(ax - ay)] - 2(m^4 - 1)bf(y)\|_{\mathbb{Y}} \le \varphi(x, y) \quad (4.1)$$

for all $a \in B\,(\|a\|_B = 1)$ and all $x, y \in \mathbb{X}$, and that for each fixed $x \in \mathbb{X}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathbb{Y}$ is continuous. Then there exists a unique B-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ which satisfies the inequality (3.4) (respectively, (3.12)) for all $x \in \mathbb{X}$.

Proof. By Theorem 3.1 (respectively, Theorem 3.3), it follows from the inequality (4.1) for $a = 1$ that there exists a unique cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ satisfying the inequality (3.4) (respectively, (3.12)) for all $x \in \mathbb{X}$. The mapping $T : \mathbb{X} \to \mathbb{Y}$ is given by (3.5) (respectively, (3.13)) for all $x \in \mathbb{X}$. Therefore, it follows from (4.1) that

$$2mT(ax + may) + 2T(max - ay) = (m^3 + m)[T(ax + ay) + T(ax - ay)] + 2(m^4 - 1)bT(y) \quad (4.2)$$

for all $x, y \in \mathbb{X}$ and all $a \in B\,(\|a\|_B = 1)$. Since T is cubic, by setting $x = 0$ in (4.2), we get

$$T(ay) = bT(y) \quad (4.3)$$

for all $y \in \mathbb{X}$ and all $a \in B\,(\|a\|_B = 1)$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, by the same reasoning as in the proof of [11], the cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ satisfies $T(tx) = t^3 T(x)$ for all $t \in \mathbb{R}$. That is, T is $\mathbb{R}$-cubic. It is clear that (4.3) is also true for $a = 0$. For each element $a \in B\,(a \neq 0)$, $a = \|a\|_B \cdot \frac{a}{\|a\|_B}$. Since T is $\mathbb{R}$-cubic and $T(ax) = bT(x)$ for all $x \in \mathbb{X}$ and all $a \in B\,(\|a\|_B = 1)$, we have

$$T(ax) = T\Big(\|a\|_B \cdot \frac{a}{\|a\|_B}x\Big) = \|a\|_B^3 T\Big(\frac{a}{\|a\|_B}x\Big) = \|a\|_B^3 \cdot \frac{b}{\|a\|_B^3} \cdot T(x) = bT(x)$$

for all $x \in \mathbb{X}$ and all $a \in B\,(a \neq 0)$. So the unique $\mathbb{R}$-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ is also B-cubic. This completes the proof. □

Corollary 4.2 *Let $\theta, \delta, \epsilon, p, q$ be non-negative real numbers such that $0 < p, q < 3$. Suppose that a mapping $f : \mathbb{X} \to \mathbb{Y}$ satisfies $f(0) = 0$ and*

$$\|2mf(ax + may) + 2f(max - ay) - (m^3 + m)[f(ax + ay) - f(ax - ay)] \\ - 2(m^4 - 1)bf(y)\|_{\mathbb{Y}} \leq \theta + \epsilon\|x\|_{\mathbb{X}}^p + \delta\|y\|_{\mathbb{X}}^q$$

for all $a \in B\,(\|a\|_B = 1)$ and all $x, y \in \mathbb{X}$, and that for each fixed $x \in \mathbb{X}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathbb{Y}$ is continuous. Then there exists a unique B-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ which satisfies the inequality

$$\|f(x) - T(x)\|_{\mathbb{Y}} \leq \frac{1}{2}\Big(\frac{\theta}{|m|^3 - 1} + \frac{\epsilon}{|m|^3 - |m|^p}\|x\|_{\mathbb{X}}^p\Big)$$

for all $x \in \mathbb{X}$.

Corollary 4.3 *Let δ, ϵ, p, q be non-negative real numbers such that $p, q > 3$. Suppose that a mapping $f : \mathbb{X} \to \mathbb{Y}$ satisfies*

$$\|2mf(ax + may) + 2f(max - ay) - (m^3 + m)[f(ax + ay) - f(ax - ay)] \\ - 2(m^4 - 1)bf(y)\|_{\mathbb{Y}} \leq \epsilon\|x\|_{\mathbb{X}}^p + \delta\|y\|_{\mathbb{X}}^q$$

for all $a \in B\,(\|a\|_B = 1)$ and all $x, y \in \mathbb{X}$, and that for each fixed $x \in \mathbb{X}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathbb{Y}$ is continuous. Then there exists a unique B-cubic mapping $T : \mathbb{X} \to \mathbb{Y}$ which satisfies the inequality

$$\|f(x) - T(x)\|_{\mathbb{Y}} \leq \frac{\epsilon}{2(|m|^p - |m|^3)}\|x\|_{\mathbb{X}}^p$$

for all $x \in \mathbb{X}$.

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