Turk J Math
33 (2009) , 107 – 110.

doi:10.3906/mat-0712-25

On Symmetric Monomial curves in $\mathbb{P}^3$

Mesut Şahin

Abstract

In this paper, we give an elementary proof of the fact that symmetric arithmetically Cohen-Macaulay monomial curves are set-theoretic complete intersections. The proof is constructive and provides the equations of the surfaces cutting out the monomial curve.

Key Words: Set-theoretic complete intersections, monomial curves

1. Introduction

Let K be an algebraically closed field and R be the polynomial ring $K[x_0, \ldots, x_n]$. To any irreducible curve C in $\mathbb{P}^n$, one can associate a prime ideal $I(C) \subset R$ to be the set of all polynomials vanishing on C. The arithmetical rank of C, denoted by $\mu(C)$, is the least positive integer r for which $I(C) = rad(f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r$ or equivalently $C = H_1 \bigcap \cdots \bigcap H_r$, where $H_1, \ldots, H_r$ are the hypersurfaces defined by $f_1 = 0, \cdots, f_r = 0$, respectively. We denote by $\mu(I(C))$ the minimal number r for which $I(C) = (f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r \in R$. These invariants are known to be bounded below by the codimension of the curve (or height of its ideal). So, one has the following relation:

$$n - 1 \leq \mu(C) \leq \mu(I(C))$$

Although $\mu(I(C))$ has no upper bound (see [1], for an example), an upper bound for $\mu(C)$ is provided to be n in [7] via commutative algebraic methods. Later in [2, 22] the equations of these n hypersurfaces that cut out the curve C were given explicitly by using elementary algebraic methods.

The curve C is called a *complete intersection* if $\mu(I(C)) = n - 1$. It is called an *almost complete intersection*, if instead, one has $\mu(I(C)) = n$. When the arithmetical rank of C takes its lower bound, that is $\mu(C) = n - 1$, the curve C is called a *set-theoretic complete intersection*, s.t.c.i. for short. It is clear that complete intersections are set-theoretic complete intersection. The corresponding question for almost complete intersection monomial curves is answered affirmatively in a series of papers by Eto [8, 9, 10].

Determining s.t.c.i. monomial curves is a classical and longstanding problem in algebraic geometry. Even more difficult is to give explicitly the equations of the hypersurfaces involved. It is known that all monomial

2000 *AMS Mathematics Subject Classification:* Primary 14M10; Secondary 14H45.

curves are s.t.c.i. in the projective n-space over a field of positive characteristic [3, 12, 14]. On the other hand, nobody knows whether or not the same question has an affirmative answer in the characteristic zero case. However, there are many partial results in this case [11, 13, 15, 16, 17, 19, 20, 21]. In fact, even the case of *symmetric* monomial curves is still mysterious.

The purpose of this note is to give an alternative proof of the fact that symmetric monomial curves in $\mathbb{P}^3$ which are arithmetically Cohen-Macaulay are s.t.c.i. by elementary algebraic methods inspired by [4]. The proof is constructive and provides the equations of the surfaces cutting out the curve.

2. Symmetric ACM Monomial Curves in $\mathbb{P}^3$

Let $p < q < r$ be some positive integers. A monomial curve $C(p,q,r)$ in $\mathbb{P}^3$ is given parametrically by

$$(w,x,y,z) = (u^r, u^{r-p}v^p, u^{r-q}v^q, v^r),$$

where $(u,v) \in \mathbb{P}^1$.

We say that the monomial curve $C(p,q,r)$ is *symmetric* if $p+q=r$. In this case the parametric representation of the curve $C(p,q,p+q)$ becomes $(u^{p+q}, u^q v^p, u^p v^q, v^{p+q})$.

It is known that all monomial curves are s.t.c.i. in $\mathbb{P}^3$, if the base field K is of positive characteristic [12]. But, no one knows whether even the symmetric monomial curves are s.t.c.i. in $\mathbb{P}^3$ in the characteristic zero case. To address this case, we work with an algebraically closed field K of characteristic zero throughout the paper.

A *minimal* system of generators for the ideal of symmetric monomial curves in $\mathbb{P}^3$ is given in [6] as follows:

$$f = xy - wz \quad \text{and} \quad F_i = w^{q-p-i}y^{p+i} - x^{q-i}z^i, \quad \text{for all} \quad 0 \le i \le q-p.$$

Recall that a monomial curve $C(p,q,r) \subset \mathbb{P}^3$ is called Arithmetically Cohen-Macaulay (ACM) if its projective coordinate ring is Cohen-Macaulay. In the same article [6], it is also proven that a monomial curve in $\mathbb{P}^3$ is ACM if and only if its ideal is generated by at most 3 polynomials. Now, if the ideal of a symmetric monomial curve $C(p,q,p+q)$ is generated by two polynomials it would follow that $p=q$. But, this contradicts with the assumption that $p<q<r$. So, the ideal of an ACM symmetric monomial curve $C(p,q,p+q)$ is generated by three polynomials and hence $p = q-1$, where necessarily $q>1$. Thus, all symmetric ACM monomial curves in $\mathbb{P}^3$ are of the form $C(q-1,q,2q-1)$ and their defining ideals are generated minimally by the following three polynomials:

$$\begin{aligned} f &= xy - zw, \\ g: &= -F_1 = x^{q-1}z - y^q, \\ h: &= -F_0 = x^q - y^{q-1}w. \end{aligned}$$

The fact that $C(q-1,q,2q-1)$ is a s.t.c.i. curve was shown in [17], but the equation of the second surface was not given. Here, we give an alternative proof that constructs the polynomial G such that the symmetric ACM monomial curve is the intersection of the surface $G=0$ and a binomial surface defined by one of f,g

and h. We construct G by adding $x^q g$ to the q-th power of f and dividing the sum by z. Hence we get the following theorem.

Theorem. *Any symmetric Arithmetically Cohen-Macaulay monomial curve in $\mathbb{P}^3$, which is given by $C(q-1, q, 2q-1)$ for some $q > 1$, is a set-theoretic complete intersection of the following two surfaces:*

$$g = x^{q-1}z - y^q = 0 \quad \text{and}$$

$$G = x^{2q-1} + \sum_{k=1}^{q} (-1)^k \frac{q!}{(q-k)!k!} x^{q-k} y^{q-k} z^{k-1} w^k = 0.$$

Proof. Note first that $zG = f^q + x^q g$. Take a point (w_0, x_0, y_0, z_0) from $Z(f, g, h)$. Then, by $z_0 G(w_0, x_0, y_0, z_0)$ $= f^q(w_0, x_0, y_0, z_0) + x_0^q g(w_0, x_0, y_0, z_0) = 0$ we observe that either $G(w_0, x_0, y_0, z_0) = 0$ or $z_0 = 0$.

If $G(w_0, x_0, y_0, z_0) = 0$ then $(w_0, x_0, y_0, z_0) \in Z(g, G)$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $h(w_0, x_0, y_0, z_0) = 0$ we get $x_0 = 0$. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(g, G)$.

Let us now take a point $(w_0, x_0, y_0, z_0) \in Z(g, G)$. Then either $z_0 = 0$ or we can assume $z_0 = 1$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $G(w_0, x_0, y_0, z_0) = 0$ we obtain $x_0 = 0$ in this case. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(f, g, h)$. On the other hand, if $z_0 = 1$ then by $G = f^q + x_0^q g$ we see that $f(w_0, x_0, y_0, z_0) = 0$. Moreover, we have $x_0 y_0 = w_0$ and $x_0^{q-1} = y_0^q$ in this case. Hence we obtain the following $x_0^q = x_0 x_0^{q-1} = x_0 y_0^q = x_0 y_0 y_0^{q-1} = w_0 y_0^{q-1}$, meaning that $h(w_0, x_0, y_0, z_0) = 0$. □

Remark. Note that the symmetric ACM monomial curves above are s.t.c.i. on the binomial surface $g = 0$. This is not true for symmetric non-ACM monomial curves; that is, they can never be a s.t.c.i. on a binomial surface [18, Theorem 5.1]. Thus it is very difficult to construct surfaces on which symmetric non-ACM monomial curves in $\mathbb{P}^3$ are s.t.c.i. with the simplest open case being the Macaulay's quartic curve $C(1, 3, 4)$.

Acknowledgement

The author would like to thank Ali Sinan Sertöz, for his extensive helps and invaluable comments.

References

[1] Arslan, F.: Cohen-Macaulayness of tangent cones, Proc. Amer. Math. Soc. 128, 2243-2251 (2000).

[2] Barile, M. and Morales, M.: On the equations defining projective monomial curves, Comm. Algebra **26**, 1907-1912 (1998).

[3] Barile, M., Morales, M. and Thoma, A.: On simplicial toric varieties which are set-theoretic complete intersections, Journal of Algebra **226**, 880-892 (2000).

[4] Bresinsky, H.: Monomial space curves in $\mathbb{A}^3$ as set-theoretic complete intersection, Proc. Amer. Math. Soc. **75**, 23-24 (1979).

[5] Bresinsky, H.: Monomial Gorenstein curves in $\mathbb{A}^4$ as set-theoretic complete intersection, Manuscripta Math. **27**, 353-358 (1979).

[6] Bresinsky, H., Schenzel, P. and Vogel, W.: On liaison, arithmetical Buchsbaum curves and monomial curves in $\mathbb{P}^3$, Journal of Algebra **86**, 283-301 (1984).

[7] Eisenbud, D. and Evans, E.G.: Every algebraic set in n-space is the intersection of n hypersurfaces, Inventiones Math. **19**, 107-112 (1973).

[8] Eto, K.: Almost complete intersection monomial curves in $\mathbb{A}^4$, Comm. Algebra **22**, 5325-5342 (1994).

[9] Eto, K.: Almost complete intersection monomial curves in $\mathbb{A}^5$, Gakujutsu Kenkyu (Academic Studies) Math. Waseda Univ. **43**, 35-47 (1995).

[10] Eto, K.: Almost Complete Intersection Lattice Ideals, Report of researches **35(2)**, 237-248 (2005/9).

[11] Eto, K.: Set-theoretic complete intersection lattice ideals in monoid rings, Journal of Algebra **299**, 689-706 (2006).

[12] Hartshorne, R.: Complete intersections in characteristic $p > 0$, Amer. J. Math. **101**, 380-383 (1979).

[13] Katsabekis, A.: Projection of cones and the arithmetical rank of toric varieties, J. Pure Appl. Algebra **199**, 133-147 (2005).

[14] Moh, T.T.: Set-theoretic complete intersections, Proc. Amer. Math. Soc. **94**, 217-220 (1985).

[15] Morales, M. and Thoma, A.: Complete intersection lattice ideals, Journal of Algebra **284**, 755-770 (2005).

[16] Robbiano, L. and Valla, G.: On set-theoretic complete intersections in the projective space, Rend. Sem. Mat. Fis. Milano LIII, 333-346 (1983).

[17] Robbiano, L. and Valla, G.: Some curves in $\mathbb{P}^3$ are set-theoretic complete intersections, in: Algebraic Geometry-Open problems, Proceedings Ravello 1982, Lecture Notes in Mathematics, Vol **997** (Springer, New York), 391-346 (1983).

[18] Thoma, A.: On the arithmetically Cohen-Macaulay property for monomial curves, Comm. Algebra **22(3)**, 805-821 (1994).

[19] Thoma, A.: On the set-theoretic complete intersection problem for monomial curves in $\mathbb{A}^n$ and $\mathbb{P}^n$, J. Pure Appl. Algebra **104**, 333-344 (1995).

[20] Thoma, A.: Affine semigroup rings and monomial varieties, Comm. Algebra **24(7)**, 2463-2471 (1996).

[21] Thoma, A.: Construction of set-theoretic complete intersections via semigroup gluing, Contributions to Algebra and Geometry **41(1)**, 195-198 (2000).

[22] Thoma, A.: On the binomial arithmetical rank, Arch. Math. **74**, 22-25 (2000).

Mesut ŞAHİN
Department of Mathematics,
Atılım University,
06836, Ankara-TURKEY
e-mail: mesut@atilim.edu.tr

Received 25.12.2007