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Local Fourier Bases and Ultramodulation Spaces

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Abstract

It was proved that local Fourier bases are unconditional bases for modulation spaces $M^w_{p.q}$. We prove that the local Fourier bases are unconditional bases for ultramodulation spaces $M^{w_\gamma}_p = M^{w_\gamma}_{p.p}$, where $0 < p < \infty$ and $w_\gamma = e^{s|x|^\gamma}$, $s > 0$, $\gamma \in (0,1)$, $x \in \mathbb{R}$.

1. Introduction

Modulation spaces, denoted by $M^w_{p.q}$, where $0 < p, q \leq \infty$ and w is a weight function, are very interesting spaces in functional analysis. They have so many applications in physics, signal analysis and psuedodifferential operators theory.

These spaces were invented in 1983 by Feichtinger. He developed his theory in terms of the behavior of the short time Fourier transform.

The local Fourier bases are bases of the form

$$\left\{\sqrt{\frac{2}{\Delta_k}} b_k(x) \sin \frac{l\pi}{\Delta_k}(x - \alpha_k)\right\},\ k \in \mathbb{Z}, l = 1, 2, \ldots,$$

where $\alpha_k < \alpha_{k+1} < \cdots, \Delta_k = \alpha_{k+1} - \alpha_k$ is a partition of $\mathbb{R}$ and $b_k(x)$ is a smooth function called a "bell function".

Wilson bases represent a special case of local Fourier bases. They are defined by

$$\psi_{l,k}(x) = \begin{cases} \sqrt{2}b(x - \frac{k}{2}) & \text{if } k \text{ is even and } l = 0; \\ \sqrt{2}b(x - \frac{k}{2}) \cos 2\pi l(x + \frac{1}{4}) & \text{if } k \text{ is even and } l \in \mathbb{N}_0; \\ \sqrt{2}b(x - \frac{k}{2}) \sin 2\pi l(x - \frac{1}{4}) & \text{if } k \text{ is odd and } l \in \mathbb{N}_0. \end{cases}$$

In 1992, Feichtinger, Gröchenig and Walnut [1] proved that Wilson bases of exponential decay are unconditional bases for all modulation spaces.

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In [4], it was proved that the local Fourier bases are unconditional bases for modulation spaces $M^w_{p,q}$, where $0 < p, q \leq \infty$ and $p = q$. This work was extended for $p \neq q$ [9]. This means that the local Fourier bases are unconditional bases for modulation spaces $M^w_{p,q}$ for all $0 < p, q \leq \infty$.

In this paper we prove that the local Fourier bases are unconditional base for ultramodulation spaces ($M^{w_\gamma}_{p,p}$) where $w_\gamma = e^{s|x|^\gamma}$, $s \in \mathbb{R}^+$, $\gamma \in (0, 1)$, $x \in \mathbb{R}$ and $0 < p < \infty$.

2. Tools from Time Frequency Analysis

In this section we give important definitions and lemmas which will be used in the next sections.

Throughout this paper the integrals are taken over $\mathbb{R}$, unless otherwise indicated.

For $f \in L^1(\mathbb{R})$ the Fourier transform is defined by

$$\widehat{f}(w) = \int f(x) e^{-2\pi i w x} dx.$$

The inner product of $f, g \in L^2(\mathbb{R})$ is defined by

$$\langle f, g \rangle = \int f(x) \overline{g(x)} dx.$$

The Schwartz space $\mathcal{S}$ is the space of all smooth functions with rapid decay, and the dual space of $\mathcal{S}$, denoted by $\mathcal{S}'$, can be considered as the space of all functions with slow growth. The elements of $\mathcal{S}'$ are called *tempered distributions.*

For $x, y \in \mathbb{R}$ the *translation* and *modulation* operators are defined respectively by:

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_y f(t) = e^{2\pi i y t} f(t). \tag{1}$$

The *window* function is a non-zero smooth cut-off a function in an interval.

The short time fourier transform (STFT) of $f \in \mathcal{S}'$ with respect to the window $g \in \mathcal{S}$ is defined as

$$S_g f(x, y) = \langle f, M_y T_x g \rangle = \int f(t) \overline{g(t - x)} e^{-2\pi i y t} dt = (f . T_x \overline{g})\widehat{\ }(y), \tag{2}$$

for all $x, y \in \mathbb{R}$.

We need the following definitions and inequalities.

- If $a \geq 0$, and $w_a(x) = (1 + |x|)^a$, $\forall x \in \mathbb{R}$, then a strictly positive and continuous function w on $\mathbb{R}^2$ is called *moderate weight* with respect to w_a if

$$w(x + y) \leq C w_a(x) w(y), \quad x, y \in \mathbb{R}^2, \quad C\text{: constant.}$$

  We say that the weight w is *submultiplicative* if $w(x + y) \leq w(x) w(y)$.

- If $f \in L^1(I)$ for every bounded subset I of a set G, we say that f is *locally integrable* on the set G and we write $f \in L^1_{loc}(G)$.

- *The weighted L^p-space* denoted by L^p_w is the space of all functions f satisfying the relation

$$\{f: \quad ||f||_{L^p_w} = ||fw||_p < \infty\}.$$

- *Modulation Spaces*: Given $0 < p, q \le \infty$, $0 \ne g \in \mathcal{S}(\mathbb{R})$ arbitrary window, and a moderate weight w on $\mathbb{R}^2$, we define the modulation space $M^w_{p,q}$ to be the space of all tempered distribution f for which the norm

$$||f||_{M^w_{p,q}} = \left(\int \left(\int |S_g f(x,y)|^p (w(x,y))^p dx \right)^{q/p} dy \right)^{1/q} \tag{3}$$

is finite. In the case $p = q = \infty$, we use the supremum. If $p = q$, we write M^w_p instead of $M^w_{p,q}$, and if w is constant weight, then we write $M_{p,q}$ instead of $M^w_{p,q}$.

Next, we mention a useful pointwise estimate of STFT. For this we recall the set of functions

$$\mathcal{C} = \mathcal{C}(M,K,N) = \{g \in C^N(\mathbb{R}) : \ supp\, g \subseteq [-K,K], \max_{k=0,1,\dots,N} ||g^{(k)}||_1 \le M\}. \tag{4}$$

Lemma 1 *[7] Let $\varphi \in C^\infty(\mathbb{R})$, $supp\, \varphi \subseteq [-L,L]$, and $C = K + L$. Then*

$$\sup_{g\in\mathcal{C}} |S_\varphi g(x,y)| \le C_0 \frac{1}{(1+|y|)^N} \mathcal{X}_{[-C,C]}(x), \quad \textit{for all } x,y \in \mathbb{R},$$

with a constant $C_0 > 0$ depending on M,K,N.

3. Weights

The weights are strictly positive and continuous functions on $\mathbb{R}^2$, and we denote them by letters: $v, w, \dots$. A weight v is submultiplicative if:

$$v(x+\xi, y+\eta) \le v(x,y)v(\xi,\eta), \ \forall x,y,\xi,\eta \in \mathbb{R}.$$

A weight w is v-moderate if $\exists C > 0$ such that:

$$w(x+\xi, y+\eta) \le Cv(x,y)w(\xi,\eta), \ \forall x,y,\xi,\eta \in \mathbb{R},$$

If v is of the form $(1+|x|+|y|)^s$, $s \ge 0$, then w is called s-moderate weight.
We consider the weight function w satisfying Beurling-Domar's non-quasi analyticity condition:

$$\sum_{n=1}^{\infty} n^{-2} \log w(nx, ny) < \infty. \ x,y \in \mathbb{R}. \tag{5}$$

We exhibit some examples of weight functions satisfying condition (5).

Example 1.

1. $(1+|x|+|y|)^s$ where $x, y \in \mathbb{R},\ s \geq 0$.
2. $e^{s_1|x|^\gamma + s_2|y|^\gamma}$ where $x, y \in \mathbb{R}, , s_1, s_2 \geq 0, \gamma \in (0,1)$.

Proof. 1.

$$\sum_{n=1}^{\infty} n^{-2} \log(1+|nx|+|ny|)^s = \sum_{n=1}^{\infty} sn^{-2} \log(1+n(|x|+|y|)) \leq \sum_{n=1}^{\infty} sn^{-2} \sqrt{n(|x|+|y|)} < \infty,$$

2.

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-2} \log e^{s_1|nx|^\gamma + s_2|ny|^\gamma} &= \sum_{n=1}^{\infty} n^{-2} \frac{\ln(e^{s_1|nx|^\gamma + s_2|ny|^\gamma})}{\ln 10} \\ &= \frac{1}{\ln 10} \sum_{n=1}^{\infty} s_1|x|^\gamma n^{-2+\gamma} + \frac{1}{\ln 10} \sum_{n=1}^{\infty} s_2|y|^\gamma n^{-2+\gamma} \\ &= \frac{1}{\ln 10} (s_1|x|^\gamma + s_2|y|^\gamma) \sum_{n=1}^{\infty} n^{-2+\gamma} < \infty\end{aligned}$$

□

Definition 1 *[6] A strictly positive and continuous function w_γ on $\mathbb{R}\times\mathbb{R}, \gamma \in (0,1)$, is said to be an exponential type (exp-type) weight if there exist $s \in \mathbb{R}$ and $C > 0$ such that:*

$$w_\gamma(x+\xi, y+\eta) \leq Ce^{s(|x|^\gamma + |y|^\gamma)} w_\gamma(\xi, \eta),\ x, y, \xi, \eta \in \mathbb{R},$$

and

$$w_\gamma(x, \epsilon y) = w_\gamma(x, y),\ \epsilon \in \{-1, 1\}.$$

Proposition 1 *The condition of exp-type weights*

$$w_\gamma(x+\xi, y+\eta) \leq Ce^{s(|x|^\gamma + |y|^\gamma)} w_\gamma(\xi, \eta),\ x, y, \xi, \eta, s \in \mathbb{R},$$

is equivalent to:

$$w_\gamma(x+\xi, y+\eta) \leq Ce^{S(|x|^2 + |y|^2)^{\gamma/2}} w_\gamma(\xi, \eta),\ x, y, \xi, \eta, S \in \mathbb{R}.$$

Proof.

$$\begin{aligned}w_\gamma(x+\xi, y+\eta) &\leq Ce^{s(|x|^\gamma + |y|^\gamma)} w_\gamma(\xi, \eta) \\ &\leq Ce^{2s(|x|+|y|)^\gamma} w_\gamma(\xi, \eta) \\ &\leq Ce^{2s\left(2(|x|^2+|y|^2)\right)^{\gamma/2}} w_\gamma(\xi, \eta) \\ &= Ce^{2^{\gamma/2+1} s(|x|^2+|y|^2)^{\gamma/2}} w_\gamma(\xi, \eta) = Ce^{S(|x|^2+|y|^2)^{\gamma/2}} w_\gamma(\xi, \eta),\end{aligned}$$

where $S = 2^{\gamma/2+1}s$. □

From the definition of the weights of exponential type we see that w_γ is a weight moderate with respect to $v(x,y) = e^{s(|x|^\gamma+|y|^\gamma)}$.

Example 2 (Weights of Exp-Type).

1. $w_\gamma(x,y) = e^{s_1|x|^\gamma+s_2|y|^\gamma},\ x,y \in \mathbb{R}, \gamma \in (0,1), s_1, s_2 \geq 0.$

2. $\tilde{w}_\gamma(x,y) = w_\gamma(x,y)e^{-\lambda|x|^\gamma-\tau|y|^\gamma}$, where $w_\gamma(x,y)$ is exp-type weight and $x,y,\lambda,\tau \in \mathbb{R}, \gamma \in (0,1)$.

4. Ultramodulation Spaces

In this section we introduce the class of modulation spaces called ultramodulation spaces defined by the corresponding class of weights.

Definition 2 (Ultramodualtion Spaces) *[10] Modulation spaces $M^w_{p,q}$ defined by an exp-type weight w_γ are called ultramodulation spaces.*

Here we take up a special case: $w_1 \otimes w_2(x,y) = w_1(x)w_2(y)$, where:

$$w_1(x) = e^{s|x|^\gamma},\ w_2(y) = e^{s|y|^\gamma}, x,y \in \mathbb{R},\ \gamma \in (0,1),\ s \geq 0,$$

the corresponding ultramodulation space, denoted by $M^{w_1\otimes w_2}_{p,q}$, is defined by

$$M^{w_1\otimes w_2}_{p,q} = \left\{ f \in \mathcal{S}' :\ \int_{\mathbb{R}} (\int_{\mathbb{R}} |\langle f, M_y T_x g\rangle|^p e^{ps(|x|^\gamma+|y|^\gamma)} dx)^{q/p} dy < \infty \right\}. \tag{6}$$

with norm

$$\|f\|_{M^{w_\gamma}_{p,q}} = \left(\int_{\mathbb{R}} (\int_{\mathbb{R}} |\langle f, M_y T_x g\rangle|^p e^{ps(|x|^\gamma+|y|^\gamma)} dx)^{q/p} dy \right)^{1/p}.$$

Proposition 2 *The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ is an isomorphism between $M^{1\otimes w_2}_{p,q}$ and $M^{w_1\otimes 1}_{p,q}$.*

Proof. In the proof of this proposition we will use the following facts:

- $|\langle f, M_y T_x g\rangle| = |\langle \overline{T_x M_y g}, \overline{f}\rangle|$, where $g \in \mathcal{S}$ and $f \in \mathcal{S}'$.

- $|\langle M_x T_y g, f\rangle| = |\langle T_y M_x g, f\rangle|$, where $g \in \mathcal{S}$ and $f \in \mathcal{S}'$.

- $\langle h, \hat{f}\rangle = \langle \hat{h}, f\rangle$, where $h \in \mathcal{S}$ and $f \in \mathcal{S}'$.

- $(\overline{T_x M_y g})\hat{} = M_{-x} T_{-y} \hat{\overline{g}}$, where $g \in \mathcal{S}$.

Now,

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle\hat{f},M_yT_xg\rangle|^p e^{ps|x|^\gamma}dx\right)^{q/p}dy=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle\overline{T_xM_yg},\hat{\overline{f}}\rangle|^p e^{ps|x|^\gamma}dx\right)^{q/p}dy$$
$$=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle M_{-x}T_{-y}\hat{\overline{g}},\overline{f}\rangle|^p e^{ps|x|^\gamma}dx\right)^{q/p}dy$$
$$=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle T_{-y}M_{-x}\tilde{g},\overline{f}\rangle|^p e^{ps|x|^\gamma}dx\right)^{q/p}dy$$
$$=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\langle T_xM_y\tilde{g},\overline{f}\rangle|^p e^{ps|y|^\gamma}dy\right)^{q/p}dx,$$

where $\tilde{g}=\hat{\overline{g}}$, and we replace $-x$ by y and $-y$ by x in the last equality. Therefore, $\hat{f}\in M^{w_1\otimes 1}_{p,q}\iff f\in M^{1\otimes w_2}_{p,q}$. $\square$

A special case for ultramodulation spaces ($p=q=2$):

$$M^{1\otimes w_2}_{2,2}=\left\{f\in\mathcal{S}':\ ||f||^2_{M^{1\otimes w_2}_{2,2}}=\int_{\mathbb{R}}\int_{\mathbb{R}}|\langle f,M_yT_xg\rangle|^2e^{2s|y|^\gamma}dxdy<\infty\right\}$$
$$=\left\{f\in\mathcal{S}':\ ||f||^2_{M^{1\otimes w_2}_{2,2}}=\int_{\mathbb{R}}\int_{\mathbb{R}}|S_gf(x,y)|^2e^{2s|y|^\gamma}dxdy<\infty\right\},\tag{7}$$

where $0\neq g\in\mathcal{S}$.

5. Unconditional Bases for Ultramodulation Spaces

It was proved that the local Fourier bases are unconditional bases for all modulation spaces defined via weight functions satisfying the condition $w(x+y)\leq C(1+|x|)^a w(y)$, $C,a\in\mathbb{R}^+$ and $x,y\in\mathbb{R}$. In this section we will show that the local Fourier bases are unconditional bases for ultramodulation spaces $M^{w_\gamma}_p=M^{w_\gamma}_{p.p}$ defined via an exp-type weight $w_\gamma=e^{s|x|^\gamma}$, where $\gamma\in(0,1)$, $s>0$ and $x\in\mathbb{R}$.

Theorem 1 *Suppose that* $\{\psi_{kl},\ (k,l)\in\mathbb{Z}\times\mathbb{N}\}\subseteq C^N(\mathbb{R})$ *are the local Fourier bases whenever the underlying partition satisfies* $\frac{1}{A}\leq\alpha_{k+1}-\alpha_k\leq A,\ A>1$, *and* $\inf_k\epsilon_k=\epsilon>0$. *If* $w_\gamma=e^{s|x|^\gamma}$, $\gamma\in(0,1)$ *and* $s>0,x\in\mathbb{R}$ *is a weight of exponential type on* $\mathbb{R}^2$ *for* $N>\max(1,\frac{1}{p})$ *where* $0<p<\infty$ *and* N *was defined in lemma (1), then* $\{\psi_{kl}\}$ *are unconditional bases for* $M^{w_\gamma}_p$. *Every distribution* $f\in M^{w_\gamma}_p$ *has a unique expansion*

$$f=\sum_{(k,l)\in\mathbb{Z}\times\mathbb{N}}\langle f,\psi_{kl}\rangle\psi_{kl},\tag{8}$$

with unconditional convergence in the norm of $M^{w_\gamma}_p$. *Moreover,*

$$\frac{1}{C}||f||_{M^{w_\gamma}_p}\leq\left(\sum_{(k,l)\in(\mathbb{Z}\times\mathbb{N})}|\langle f,\psi_{kl}\rangle|^p w_\gamma(\alpha_k)^p\right)^{1/p}\leq C||f||_{M^{w_\gamma}_p},\tag{9}$$

for some constant $C > 0$.

Since the Wilson bases is a special case of the local Fourier bases, we have the following corollary

Corollary 1 *The Wilson bases of exponential decay are unconditional bases for* $M_p^{w_\gamma}$, $0 < p < \infty$.

We use the same techniques in [7] to prove Theorem 1. First we define the analysis operator $\tau : L^2(\mathbb{R}) \to l^2(\mathbb{R})$,

$$\tau f = \langle f, \psi_{k,l} \rangle_{(k,l)\in I},$$

where I is the index set.

The synthesis operator defined by $\tau^\star : l^2(\mathbb{R}) \to L^2(\mathbb{R})$, is

$$\tau^\star((c_{kl})_{k,l}) = \sum_{(k,l)\in I} c_{kl}\psi_{kl}.$$

We write: $\eta_{kl} = (\alpha_k, \frac{l}{2\Delta_k})$, $(k,l) \in \mathbb{Z}^2$; and for a given weight function w we denote its restriction to the discrete set $\{\eta_{kl}\}$ by $w'(k,l) = w(\eta_{kl})$.

Lemma 2 *[4] Using the notation of Lemma 1, set*

$$G(x,y) = \mathcal{X}_{[-C,C]}(x)\frac{1}{(1+|y|)^N}.$$

If $\{\psi_{kl},\ (k,l) \in \mathbb{Z}\times\mathbb{N}\} \subseteq C^N(\mathbb{R})$ *are the local Fourier bases satisfying the assumptions of Theorem (2), then there exists* $C_1 > 0$, *such that*

$$|S_\psi \psi_{kl}(x,y)| \leq C_1(T_{\eta_{kl}}G(x,y) + T_{\eta_{k,-l}}G(x,y)), \quad \textit{for all } x,y \in \mathbb{R}.$$

Lemma 3 *(Schur)[4] Suppose that* $w_1(i)$, $i \in I$ *and* $w_2(j)$, $j \in J$ *are two weight functions on the index sets* I, J *respectively, and let* $A = (a_{ji})_{j\in J, i\in I}$ *is an infinite matrix such that*

$$\sum_{i\in I} |a_{ji}| w_1(i)^{-1} \leq C_0 w_2(j)^{-1} < \infty \quad \forall j \in J, \tag{10}$$

and

$$\sum_{j\in J} |a_{ji}| w_2(j) \leq C_1 w_1(i) < \infty \quad \forall i \in I. \tag{11}$$

for some constants $C_0, C_1 > 0$. *Then the map* A *is bounded from* $l^p_{w_1}(I)$ *into* $l^p_{w_2}(J)$ *for* $1 \leq p < \infty$.

Lemma 4 *(Schur) Suppose that* $w_1(i)$, $i \in I$ *and* $w_2(j)$, $j \in J$ *are two weight functions on the countable index sets* I, J *respectively, and let* $A = (a_{ji})_{j\in J, i\in I}$ *be an infinite matrix such that*

$$\sum_{j\in J} |a_{ji}|^p w_2(j)^p \leq C_2 w_1(i)^p < \infty \quad \forall i \in I, \tag{12}$$

for some constant $C_2 > 0$. *Then the map* A *is bounded from* $l^p_{w_1}(I)$ *into* $l^p_{w_2}(J)$ *for* $0 < p < 1$.

Proof. Let $c = (c_i)_{i\in I} \in \ell^p_{w_1}(I)$. Then

$$\begin{aligned} \|Ac\|^p_{\ell^p_{w_2}} &= \sum_{j\in J}\left|\sum_{i\in I} a_{ji}c_i\right|^p w_2(j)^p \\ &\le \sum_j\sum_i |a_{ji}|^p|c_i|^p w_2(j)^p \quad \text{(because } p<1) \\ &= \sum_i |c_i|^p \sum_j |a_{ji}|^p w_2(j)^p \\ &= C_0 \sum_i |c_i|^p w_1(i)^p = C_0\, \|c\|^p_{\ell^p_{w_1}}. \end{aligned}$$

□

Theorem 2 *[3] Given $g \in \mathcal{S}$, $0 < p < \infty$, and a moderate weight w. Let $\delta, \beta > 0$ be such that for some integer $M \ge 1$, $\delta\beta \le 1/M$. Suppose that $M_{\delta n}T_{\beta m}g,\ k,n \in \mathbb{Z}$ generates a frame for L^2. Then given any $f \in M^w_p$, we can write*

$$f = \sum_{m,n}\langle f, M_{\delta n}T_{\beta m}S^{-1}g\rangle M_{\delta n}T_{\beta m}g.$$

The sum converges in the norm topology of M^w_p. Moreover, there exists $C = C(\delta,\beta,g) > 0$ such that for all $f \in M^w_p$

$$\frac{1}{C}||f||_{M^w_p} \le \left(\sum_{m,n}|\langle f, M_{\delta n}T_{\beta m}g\rangle|^p w(\beta m, \delta n)^p\right)^{1/p} \le C||f||_{M^w_p},$$

where S is the Gabor frame operator

$$Sf = \sum_{m,n}\langle f, M_{\delta n}T_{\beta m}g\rangle M_{\delta n}T_{\beta m}g.$$

Proposition 3 *Suppose that $w_\gamma(x) = e^{s|x|^\gamma}$ is an exp-type weight, where $\gamma \in (0,1)$ and $x, s \in \mathbb{R}$, then:*

1. *$w_\gamma(x) \le w_\gamma(\beta m)e^{s|x-\beta m|^\gamma}$, where $\beta > 0$, $m \in \mathbb{Z}$.*
2. *$w_\gamma(x+y) \le w_\gamma(x)w_\gamma(y)$, where $x, y \in \mathbb{R}$.*

Proof.

1. $$w_\gamma(\alpha_k) = e^{s|\alpha_k|^\gamma} = e^{s|\alpha_k-\beta m+\beta m|^\gamma} \le e^{s(|\alpha_k-\beta m|+|\beta m|)^\gamma} \le e^{s|\alpha_k-\beta m|^\gamma}e^{s|\beta m|} = w_\gamma(\beta m)e^{s|\alpha_k-\beta m|^\gamma}.$$

2. $$w_\gamma(x+y) = e^{s|x+y|^\gamma} \le e^{s(|x|+|y|)^\gamma} \le e^{s(|x|^\gamma+|y|^\gamma)} = e^{s|x|^\gamma}e^{s|y|^\gamma} = w_\gamma(x).w_\gamma(y).$$

□

Proposition 4 *Let* $0 < p < \infty$ *and let* $\{\psi_{kl},\ (k,l) \in \mathbb{Z} \times \mathbb{N}\} \subseteq C^N(\mathbb{R})$ *be the local Fourier bases whose underlying partition satisfies* $\frac{1}{A} \leq \alpha_{k+1} - \alpha_k \leq A,\ A > 1$, *and* $\inf_k \epsilon_k = \epsilon > 0$. *If* $w_\gamma = e^{s|x|^\gamma},\ \gamma \in (0,1),\ 0s > 0, x \in \mathbb{R}$ *is an exp-type weight, then for* $N > \max(1, \frac{1}{p})$, *then* τ *is a bounded operator from* $M_p^{w_\gamma}$ *into* $l^p_{w'_\gamma}(I)$.

Proof. The proof of this proposition is based on Lemmas 3, 4 and Theorem 2.

For the case $1 \leq p < \infty$ we follow the same steps as in [9, Proposition 1].
We use also Proposition 4 for the proof of this case.

For the case $0 < p < 1$. Since by Theorem 2

$$f \in M_p^w \quad \text{if and only if} \quad f = \sum_{m,n\in\mathbb{Z}} \langle f, M_{\delta n} T_{\beta m}\rangle M_{\delta n} T_{\beta m} g$$

with

$$\frac{1}{C}||f||_{M_p^w} \leq \left(\sum_{m,n\in\mathbb{Z}} |\langle f, M_{\delta n} T_{\beta m}\rangle|^p w(\beta m, \delta n)^p\right)^{1/p} \leq C||f||_{M_p^w}.$$

$$(\tau f)_{kl} = \langle f, \psi_{k,l}\rangle = \sum_{m,n\in\mathbb{Z}} \langle f, M_{\delta n} T_{\beta m}\rangle \langle M_{\delta n} T_{\beta m} g, \psi_{kl}\rangle,$$

then to show $((\tau f)_{kl}) \in l^p_{w'_\gamma}$, it is enough to show that the map

$$A_{(k,l),(m,n)} = \langle M_{\delta n} T_{\beta m} g, \psi_{kl}\rangle,$$

maps the sequence $(\langle f, M_{\delta n} T_{\beta m}\rangle) \in l^p_{w_1}(\mathbb{Z}^2),\ w_1(m) = w_\gamma(\beta m)$ into $l^p_{w'_\gamma}(I)$. For this, it is sufficient to verify Condition (12), i.e.

$$\sum_{k,l} |\langle M_{\delta n} T_{\beta m} g, \psi_{kl}\rangle|^p w_\gamma(\alpha_k)^p \leq C_2 w_\gamma(\beta m)^p < \infty, \qquad \forall m, n \in \mathbb{Z}.$$

By Lemma 2, Condition (12) becomes

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}} |\langle M_{\delta n} T_{\beta m} g, \psi_{kl}\rangle|^p w_\gamma(\alpha k)^p \\
&\quad\leq C_2 w_\gamma(\beta m)^p \sum_{k,l\in\mathbb{Z}} G(\beta m - \alpha_k, \delta n - \frac{\pm l}{2\Delta_k})^p e^{ps|\alpha_k - \beta m|^\gamma} \\
&\quad\leq C_2 w_\gamma(\beta m)^p \sum_{k,l\in\mathbb{Z}} \mathcal{X}_{[-C,C]}(\beta m - \alpha_k) \left(1 + |\frac{\pm l}{2\Delta_k} - \delta n|\right)^{-pN} e^{ps|\alpha_k - \beta m|^\gamma} = (*).
\end{aligned}$$

Since $\frac{1}{A} \leq \alpha_{k+1} - \alpha_k \leq A,\ A > 1$, there are at most $2CA$ terms α_k in every interval of length $2C$.

$$(*) \leq C_2 w_\gamma(\beta m)^p 2CAe^{p.s.C^\gamma} \sup_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \left(1 + |\frac{\pm l}{2\Delta_k} - \delta n|\right)^{-pN}.$$

Since $N > \frac{1}{p}$, the sum is finite with a bound independent of m and n. □

Proposition 5 *$\tau^\star$ is a bounded operator from $l^p_{w'_\gamma}$ into $M_p^{w_\gamma}$ for $0 < p < \infty$.*

Proof. **For the case** $1 \leq p < \infty$ the proof is similar to that in [4, Proposition 2].

For the case $0 < p < 1$ the proof is similar to that in [7, Proposition 3]. □

Proof of Theorem (1).

Proof. Since τ and $\tau^\star$ are bounded operators on $M_p^{w_\gamma}$ and $l^p_{w'_\gamma}$, the identity

$$f = \sum_{k,l} \langle f, \psi_{kl}\rangle \psi_{kl} = \tau^\star \tau f,$$

extends from $L^2(\mathbb{R})$ to $M_p^{w_\gamma}$, $0 < p < \infty$ with unconditional convergence of the series above. Thus $\{\psi_{kl}\}_{kl}$ spans a dense subspace in $M_p^{w_\gamma}$. The norm in the theorem follows from:

$$\begin{aligned}\|f\|_{M_p^{w_\gamma}} &= \|\tau^\star(\langle f, \psi_{kl}\rangle_{(k,l)\in I})\|_{M_p^{w_\gamma}} \\ &\leq \|\tau^\star\|_{op} \|\langle f, \psi_{kl}\rangle_{(k,l)\in I}\|_{l^p_{w'_\gamma}} \\ &= \|\tau^\star\|_{op} \|\|\tau\|_{op} \|f\|_{M_p^{w_\gamma}},\end{aligned}$$

also, since the coefficients in $f = \sum_{k,l} c_{kl}\psi_{kl}$ are uniquely determined by

$$c_{kl} = \langle f, \psi_{kl}\rangle = (\tau f)_{kl},$$

we estimate

$$\begin{aligned}\|\sum_{k,l} \lambda_{kl} c_{kl} \psi_{kl}\|_{M_p^{w_\gamma}} &= \|\tau^\star(\lambda_{kl} c_{kl})_{(k,l)}\|_{M_p^{w_\gamma}} \\ &\leq \|\tau^\star\|_{op} \|(\lambda_{kl} c_{kl})_{(k,l)}\|_{l^p_{w'_\gamma}} \\ &\leq \|\tau^\star\|_{op} \|\lambda\|_\infty \|\tau\|_{op} \|f\|_{M_p^{w_\gamma}},\end{aligned}$$

where $\lambda = (\lambda_{kl})_{(k,l)\in I}$, $c = (c_{kl})_{(k,l)\in I}$. This completes the proof that $\{\psi_{kl},\ (k,l) \in I\}$ is an unconditional bases for $M_p^{w_\gamma}$. □

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