A New Approach on Constant Angle Surfaces in $\mathbb{E}^3$

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Abstract

In this paper we study constant angle surfaces in Euclidean 3–space. Even that the result is a consequence of some classical results involving the Gauss map (of the surface), we give another approach to classify all surfaces for which the unit normal makes a constant angle with a fixed direction.

Key Words: Constant angle surfaces, Euclidean space.

1. Introduction

Recently, constant angle surfaces were studied in product spaces $\mathbb{S}^2 \times \mathbb{R}$ in [2] or $\mathbb{H}^2 \times \mathbb{R}$ in [3, 4], where $\mathbb{S}^2$ and $\mathbb{H}^2$ represent the unit 2-sphere and the hyperbolic plane, respectively. The angle was considered between the unit normal of the surface M and the tangent direction to $\mathbb{R}$. The idea of studying surfaces with different geometric properties in product spaces was initiated by H. Rosenberg and W. Meeks in [6] and [10], where they have considered the general case of a surface $\mathbb{M}^2$ and they have looked for minimal surfaces properties in the product space $\mathbb{M}^2 \times \mathbb{R}$.

In this article we study the problem of constant angle surfaces in Euclidean 3-space. So, we want to find a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to $\mathbb{R}$.

The applications of constant angle surfaces in the theory of liquid crystals and of layered fluids were considered by P. Cermelli and A. J. Di Scala in [1], but they used for their study of surfaces another method different from ours, the Hamilton-Jacobi equation, correlating the surface and the direction field. In [5], R. Howard explains how shadow boundaries are formed when the light source is situated at an infinite distance from the surface M using the geometric model of constant angle surfaces.

2. Preliminaries

Let $\langle\ ,\ \rangle$ be the standard flat metric in $\mathbb{E}^3$ and $\widetilde{\nabla}$ be its Levi Civita connection. We will consider an orientation of $\mathbb{E}^3$ and denote by k the fixed direction. Let M be a surface isometrically immersed in $\mathbb{E}^3$ and denote by N the unit normal of the surface. Denote by $\theta := \widehat{(N, k)}$, where $\theta \in [0, \pi)$, the angle function between the unit normal and the fixed direction. A vector is tangent to M if it is orthogonal to the normal N.

Recall the Gauss and Weingarten formulas

(G) $\qquad \widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$

(W) $\qquad \widetilde{\nabla}_X N = -AX,$

for every X and Y tangent to M. Here ∇ is the Levi Civita connection on M, h is a symmetric $(1, 2)$-tensor field taking values in the normal bundle and called the second fundamental form of M and A is the shape operator. We have

$$\langle h(X, Y), N\rangle = g(X, AY)$$

for all X, Y tangent to M, where g is the restriction of the scalar product $\langle\ ,\ \rangle$ to M.

Decompose k into the tangent and normal part respectively:

$$\vec{k} = \vec{U} + \cos\theta\vec{N}\ ,\ \text{where U is tangent to M.} \tag{1}$$

It follows $\quad \|\vec{k}\|^2 = \|\vec{U}\|^2 + \cos^2\theta\|\vec{N}\|^2$ and hence $\|\vec{U}\| = \sin\theta$.

For $\theta \neq 0$, we can define a unit vector field on M, namely $e_1 := \frac{U}{\|U\|}$. Let e_2 be an unitary vector field on M and orthogonal to e_1. Thus we obtain an orthonormal basis $\{e_1, e_2\}$ defined in every point of M. From now on we suppose that θ is constant.

Proposition 1 *In these hypothesis, we have:* $[e_1, e_2] \parallel e_2$.

Proof. First we calculate $[e_1, e_2]$ and we will notice that it can be written depending only on e_2.

We will use the following relation:

$$[e_1, e_2] = \widetilde{\nabla}_{e_1} e_2 - \widetilde{\nabla}_{e_2} e_1. \tag{2}$$

From (1) (and the definition of e_1) we have $k = \sin\theta e_1 + \cos\theta N$ and applying $\widetilde{\nabla}_{e_2}$ one gets:

$$0 = \widetilde{\nabla}_{e_2} k = \sin\theta\widetilde{\nabla}_{e_2} e_1 + \cos\theta\widetilde{\nabla}_{e_2} N. \tag{3}$$

Derivating $\langle N, e_1\rangle = 0$ with respect to e_2, we have the following relation:

$$\langle\widetilde{\nabla}_{e_2} N, e_1\rangle + \langle\widetilde{\nabla}_{e_2} e_1, N\rangle = 0. \tag{4}$$

Weingarten formula yields:

$$\widetilde{\nabla}_{e_2} N = -\rho e_1 - \lambda e_2, \text{with } \rho, \lambda \in C^\infty(M). \tag{5}$$

From (3) and (5) it follows

$$\widetilde{\nabla}_{e_2} e_1 = \cot \theta \, (\rho e_1 + \lambda e_2). \tag{6}$$

At this point we consider $\theta \neq \frac{\pi}{2}$ (i.e. $\cot \theta \neq 0$). The particular case $\theta = \frac{\pi}{2}$ will be treated separately.

Combining (4), (5) and (6) we find $\rho = 0$ and hence:

$$\widetilde{\nabla}_{e_2} e_1 = \lambda \cot \theta e_2. \tag{7}$$

Again, by using the Weingarten formula we have

$$\widetilde{\nabla}_{e_1} N = -\alpha e_1 - \gamma e_2, \text{ with } \alpha, \gamma \in \mathrm{C}^\infty(\mathrm{M}). \tag{8}$$

By the same method, applying $\widetilde{\nabla}_{e_1}$ to (1) and using (8) we obtain

$$\widetilde{\nabla}_{e_1} e_1 = \cot \theta \, (\alpha e_1 + \gamma e_2).$$

Since e_1 is unitary it follows that α vanishes. Moreover, due to the symmetry of the shape operator, i.e. $\langle Ae_1, e_2 \rangle = \langle e_1, Ae_2 \rangle$, one immediately gets that γ vanishes too. Hence $Ae_1 = 0$ and

$$\widetilde{\nabla}_{e_1} e_1 = 0. \tag{9}$$

Derivating $\langle e_1, e_2 \rangle = 0$ with respect to e_1 and using (9) we get

$$\langle \widetilde{\nabla}_{e_1} e_2, e_1 \rangle = 0. \tag{10}$$

Using the Gauss formula one can write

$$0 = \langle Ae_1, e_2 \rangle = \langle h(e_1, e_2), N \rangle = \langle \widetilde{\nabla}_{e_1} e_2, N \rangle.$$

It follows

$$\widetilde{\nabla}_{e_1} e_2 = 0. \tag{11}$$

From (2), (7) and (11) we get the following relation for the Lie brackets:

$$[e_1, e_2] = -\lambda \cot \theta e_2 \text{ , equivalently, } [e_1, e_2] \parallel e_2. \tag{12}$$

$\square$

We conclude this section with the following proposition

Proposition 2 *The Levi Civita connection ∇ of M is given by relations*

$$\nabla_{e_1} e_1 = 0, \; \nabla_{e_1} e_2 = 0, \; \nabla_{e_2} e_1 = \lambda \cot \theta \, e_2, \; \nabla_{e_2} e_2 = -\lambda \cot \theta \, e_1. \tag{13}$$

Proof. The expressions can be obtained by straightforward computations. See also [2] and [3, 4]. $\square$

3. The Characterization of Constant Angle Surfaces

Due to Proposition 1 one can choose now a local coordinate system in each point of the surface M, namely a parametrization:

$$r = r(u, v) = (x(u, v), y(u, v), z(u, v))$$

such that the tangent vectors are: $r_u = e_1$ and $r_v \| e_2$. Let $r_v := \beta(u, v)e_2$, where β is a smooth function on M. Hence, the metric on M can be written as

$$g = du^2 + \beta^2(u, v)dv^2. \tag{1}$$

Remark 3 *The coefficients of the first fundamental form are* $E = 1$, $F = 0$, $G = \beta^2(u, v)$.

From Proposition 2 one can write now the Levi Civita connection of M in terms of the coordinates u and v. It follows that the parametrization r satisfies the following PDE's:

$$r_{uu} = 0 \tag{2}$$

$$r_{uv} = \frac{\beta_u}{\beta} \, r_v \tag{3}$$

where β satisfies the following PDE

$$\beta_u - \beta \lambda \cot \theta = 0 \tag{4}$$

and finally,

$$r_{vv} = \frac{\beta_v}{\beta} \, r_v - \beta^2 \lambda \cot \theta r_u + \beta^2 \lambda N. \tag{5}$$

Using the Schwartz identity $\widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{\partial_v} N = \widetilde{\nabla}_{\partial_v} \widetilde{\nabla}_{\partial_u} N$, and the expressions of the partial derivatives of the unit normal of the surface M: $N_u = 0$ and $N_v = -\lambda r_v$, we have that λ satisfies the following PDE:

$$\lambda_u + \lambda^2 \cot \theta = 0. \tag{6}$$

Now we have to find the functions λ and β in order to write the parametrization r of the surface M.

Remark 4 *Since* $N_u = 0$ *it follows that two coefficients of the second fundamental form:* $e = f = 0$. *This fact implies that the Gaussian curvature of* M *vanishes,* $K = 0$. *So, the surface* M *is locally flat.*

Remark 5 *In terms of the Gauss map of the surface we can say that it makes a constant angle with a fixed direction, which is equivalent to the fact that the Gauss map lies on a circle in the sphere* $\mathbb{S}^2$. *Since it has no interior points in* $\mathbb{S}^2$, *it follows that the Gaussian curvature of the surfaces vanishes identically.*

Proposition 6 *The functions* λ *and* β *are given by the expressions*

$$\lambda(u, v) = \frac{\tan \theta}{u + \alpha(v)} \tag{7}$$

$$\beta(u, v) = \varphi(v)(u + \alpha(v)), \tag{8}$$

where α and φ are smooth functions on M or,

$$\lambda(u, v) = 0 \tag{9}$$

$$\beta(u, v) = \beta(v). \tag{10}$$

Proof. First we solve (6) and we find λ and then we substitute it in (4), obtaining β. □

Theorem 7 (of characterization) *A surface M in $\mathbb{E}^3$ is a constant angle surface if and only if it is locally isometric to one of the following surfaces:*

(i) *either a surface given by*

$$r : M \to \mathbb{E}^3, \ (u, v) \mapsto (u \cos\theta(\cos v, \sin v) + \gamma(v), u \sin\theta) \tag{11}$$

with

$$\gamma(v) = \cos\theta \ \Big(- \int\limits_0^v \alpha(\tau) \sin\tau d\tau, \int\limits_0^v \alpha(\tau) \cos\tau d\tau \Big) \tag{12}$$

for α a smooth function on an interval I,

(ii) *or an open part of the plane $x \sin\theta - z \cos\theta = 0$,*

(iii) *or an open part of the cylinder $\gamma \times \mathbb{R}$, where γ is a smooth curve in $\mathbb{R}^2$.*

Here θ is a real constant.

Proof. First we prove that all these surfaces define indeed a constant angle surface in $\mathbb{E}^3$. Item (ii) is obvious and item (iii) corresponds to $\theta = \frac{\pi}{2}$. For item (i) we have the tangent vectors

$$r_u = \big(\cos\theta \cos v, \cos\theta \sin v, \sin\theta \big)$$
$$r_v = \big((u + \alpha(v)) \cos\theta(-\sin v, \cos v), 0\big).$$

Thus, the unit normal is $N = \big(-\sin\theta(\cos v, \sin v), \cos\theta \big)$ and hence, the angle between N and and the fixed direction k is the constant θ.

Conversely, we have to prove that a constant angle surface in $\mathbb{E}^3$ is as in the statement of the theorem. Since $e_1 = r_u$, from (1) we get

$$k = \sin\theta \ r_u + \cos\theta \ N.$$

Using Remark 3 and from the previous relation it easily follows that $\langle r_u, k \rangle = \sin\theta$ and $\langle r_v, k \rangle = 0$. Hence the third component of $r(u, v)$ is $z(u, v) = u \sin\theta$.

At this point, the parametrization of M becomes

$$r(u, v) = (h(u, v), u \sin\theta), \tag{13}$$

where $h(u, v) = (x(u, v), y(u, v)) \in \mathbb{R}^2$.

We analyze the two cases for λ and β furnished by the Proposition 6.

Case I. Since $r_{uu} = 0$ we have $h_{uu} = 0$. On the other hand, $e_1 = r_u = (h_u, \sin\theta)$ is a unit vector, which means that $|h_u| = \cos\theta$. Hence $h_u = \cos\theta f(v)$, where $f(v) \in \mathbb{R}^2$ and $|f(v)| = 1$ for any v, i.e. f is a parametrization of the circle $\mathbb{S}^1$. By integration we obtain

$$h(u, v) = u\cos\theta f(v) + \gamma(v)$$

where γ is a smooth curve in $\mathbb{R}^2$.

It follows that $r_v = (u\cos\theta f'(v) + \gamma'(v), 0)$. Since $r_{uv} = \frac{\beta_u}{\beta} r_v$ we get $\gamma'(v) = \cos\theta\alpha(v)f'(v)$.

Without loss of the generality we can suppose that f is the natural parametrization for $\mathbb{S}^1$, i.e. $f(v) = (\cos v, \sin v)$ (this corresponds to a change of the parameter v). Thus, the function φ which appears in (8) is constnat, namely $\varphi = \cos\theta$.

One obtains the parametrization for M

$$r(u, v) = \big(u\cos\theta(\cos v, \sin v) + \gamma(v), u\sin\theta\big),$$

where γ is given by (12).

Case II. Due to $r_{uu} = 0$ and $r_{uv} = 0$, it follows that $h_{uu} = 0$ and $h_{uv} = 0$, which imply that h_u is a constant vector in $\mathbb{R}^2$ of length $\cos\theta$, i.e. $h_u = \cos\theta(\cos\mu, \sin\mu)$, $\mu \in \mathbb{R}$. Hence

$$h(u, v) = u\cos\theta(\cos\mu, \sin\mu) + \gamma(v),$$

where γ is a smooth curve in $\mathbb{R}^2$.

Recall that r_u and r_v are orthogonal. Consequently,

$$\gamma(v) = \alpha(v)(-\sin\mu, \cos\mu), \ \alpha \in C^\infty(I). \qquad (*)$$

The parametrization of M can be written as

$$r(u, v) = \big(u\cos\theta(\cos\mu, \sin\mu) + \gamma(v), u\sin\theta\big)$$

with γ given by $(*)$.

A rotation of angle μ in the (x, y)-plane yields the following parametrization for M:

$$r(u, v) = (u\cos\theta, \alpha(v), u\sin\theta), \qquad (14)$$

which parameterizes the plane $x\sin\theta - z\cos\theta = 0$.

Particular cases for the constant angle θ:

• $\theta = 0$: the normal N coincides with the direction k. Since r_u and r_v are tangent to M it follows $\langle r_u, k\rangle = 0$ and $\langle r_v, k\rangle = 0$ and thus $\langle r, k\rangle = \text{constant}$. This is the equation of a plane parallel to $(x, y)-$plane.

It can be parameterized as $r(u, v) = (u, v, 0)$.

• $\theta = \frac{\pi}{2}$: k is tangent to the surface. In this case M is the product of a curve in $\mathbb{R}^2$ and $\mathbb{R}$ (cylindrical surface), which can be parameterized as in (11) by: $r(u, v) = (\gamma(v), u)$, where $\gamma(v) \in \mathbb{R}^2$.

Now the theorem is completely proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We give some examples of constant angle surfaces, parameterized by (11) for different functions α in (12).

All pictures are realized by using Matlab.

Example 8 In all the following four examples we consider $\theta = \frac{\pi}{4}$.

1. $\alpha(v) = 1$:
$$r(u, v) = \tfrac{1}{\sqrt{2}} \left((1 + u) \cos v - 1, (1 + u) \sin v, u \right)$$

2. $\alpha(v) = v$:
$$r(u, v) = \tfrac{1}{\sqrt{2}} \left((u + v) \cos v - \sin v, (u + v) \sin v + \cos v - 1, u \right)$$

3. $\alpha(v) = \cos v$:
$$r(u, v) = \tfrac{1}{\sqrt{2}} \left(u \cos v - \tfrac{\sin^2 v}{2}, u \sin v + \tfrac{v + \sin v \cos v}{2}, u \right)$$

4. $\alpha(v) = 2 \sin v$:
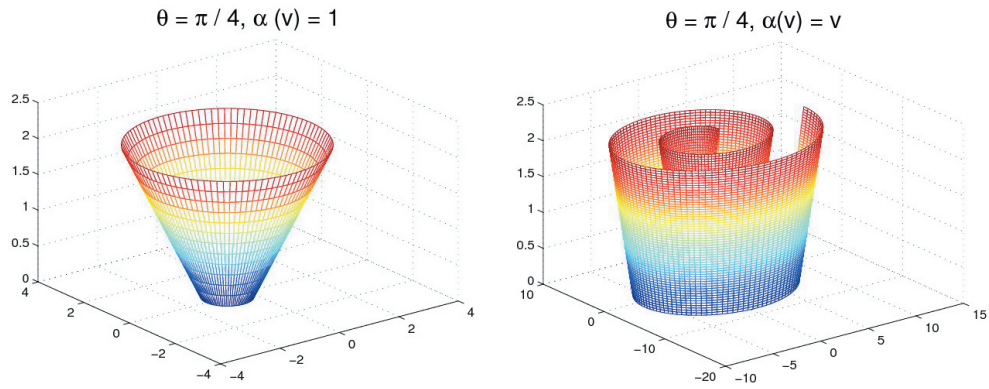$$r(u, v) = \tfrac{1}{\sqrt{2}} \left(u \cos v - v + \cos v \sin v, u \sin v + \sin^2 v, u \right).$$

Figure 1. Example 8: cases 1 and 2.
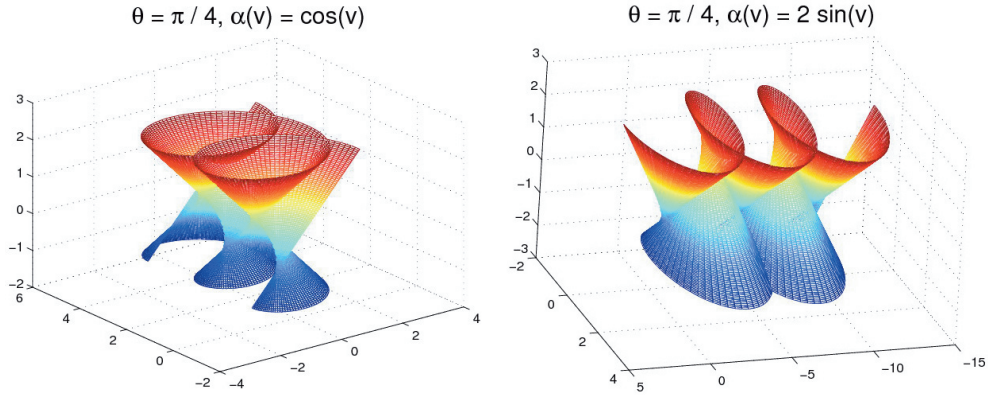
We give now the following result.

θ = π / 4, α(v) = cos(v)   θ = π / 4, α(v) = 2 sin(v)

Figure 2. Example 8: cases 3 and 4.

Proposition 9

1. *The only minimal constant angle surfaces in Euclidean 3-space are the planes which make the angle θ with the fixed direction k.*

2. *The constant angle surfaces in Euclidean 3-space with non-zero constant mean curvature are the cylindrical surfaces.*

Proof. Recall the formula $H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}$. Using Remark 3 and Remark 4 we get that

$$H = \frac{1}{2} \frac{g}{\beta^2(u, v)} \ . \tag{15}$$

Looking for all minimal surfaces (i.e. $H = 0$) we should have $g = 0$. Now we refind here the case $\lambda = 0$ corresponding to the planes which make the angle θ with the fixed direction k.

For the second statement (M is CMC), (15) implies that $\lambda = \text{constant}$. But λ satisfies (6) so we must have $\theta = \frac{\pi}{2}$. In this particular case we found the cylindrical surfaces $\gamma \times \mathbb{R}$, γ smooth curve in $\mathbb{R}^2$. □

4. Conclusions

We can compare now all three results obtained for different ambient spaces, namely for $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{E}^3$, respectively. Thus we have: *M is a constant angle surface if and only if it is given by an immersion r of the following form:*

1. $r : M \to \mathbb{S}^2 \times \mathbb{R}$,

   $(u, v) \mapsto (\cos(u \cos \theta) f(v) + \sin(u \cos \theta) f(v) \times f'(v), \ u \sin \theta)$ where $f : I \to \mathbb{S}^2$ is an unit speed curve in $\mathbb{S}^2$ - the unit 2-sphere and "$\times$" is the vector cross product in $\mathbb{R}^3$;

2. $r : M \to \mathcal{H} \times \mathbb{R}$,

   $(u, v) \mapsto (\cosh(u \cos \theta) f(v) + \sinh(u \cos \theta) f(v) \boxtimes f'(v), \ u \sin \theta)$ where $f : I \to \mathcal{H}$ is an unit speed curve on $\mathcal{H}$ - the hyperboloid model of $\mathbb{H}^2$ and "$\boxtimes$" is the Lorentzian cross product in $\mathbb{R}_1^3$, namely Lorentzian 3–space;

3. $r : M \to \mathbb{E}^3$, $(u, v) \mapsto (u \cos \theta f(v) + \gamma(v), \; u \sin \theta)$

where $f : I \to \mathbb{R}^2$ is a parametrization of the unit circle $\mathbb{S}^1$, or f is a unit constant vector and $\gamma'(v) \perp f(v)$.

Remark 10 The third component (along $\mathbb{R}$) in all of these cases is the same: $z(u, v) = u \sin \theta$.

Remark 11 In $\mathbb{S}^2 \times \mathbb{R}$ the surface M has the constant Gaussian curvature $K = \cos^2 \theta > 0$, in $\mathcal{H} \times \mathbb{R}$ one gets $K = -\cos^2 \theta < 0$ while in $\mathbb{E}^3$ it vanishes ($K = 0$).

5. Appendix

Applications to the theory of liquid crystals. In terms of differential geometry, we studied the constant angle surfaces in $\mathbb{E}^3$ whose unit normal forms a constant angle with an assigned direction field. From the point of view of physics, this geometric condition is equivalent to an Hamilton-Jacobi equation correlating the surface and the direction field.

In the physics of interfaces in liquid crystals and of layered fluids, these surfaces are studied when the direction field, in our case k, is singular along a line or a point. We can see in [1] how constant angle surfaces may be used to describe interfaces occurring in special equilibrium configurations of *nematic* (an ordered fluid whose constituents are macromolecules which tend to align parallel to each other) and *smectic C liquid crystals* (ordered fluids characterized by a layered structure) and to determine the shape of disclination cores in nematics. The last aspect, applications of constant angle surfaces in nematics was developed by E.G. Virga in [11], and more recently, for example, by P. Prinsen and P. van der Schoot in [7], [8], [9].

Acknowledgement

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