

## Criteria of nilpotency and influence of contranormal subgroups on the structure of infinite groups\*

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### Abstract

Following J.S. Rose, a subgroup  $H$  of a group  $G$  is called *contranormal* if  $G = H^G$ . In a certain sense, contranormal subgroups are antipodes to subnormal subgroups. It is well known that a finite group is nilpotent if and only if it has no proper contranormal subgroups. However, for infinite groups this criterion is not valid. There are examples of non-nilpotent infinite groups whose subgroups are subnormal; in particular, these groups have no contranormal subgroups. Nevertheless, for some classes of infinite groups, the absence of contranormal subgroups implies nilpotency of the group. The present article is devoted to the search of such classes. Some new criteria of nilpotency in certain classes of infinite groups have been established.

**Key word and phrases:** Contranormal subgroups, descending subgroups, nilpotent subgroups, minimax groups.

There is a variety of well-known criteria of nilpotency in finite groups. One of them states that a finite group is nilpotent if and only if every maximal subgroup is normal. Since each proper subgroup lies in some maximal subgroup, this criterion can be reformulated: *A finite group  $G$  is nilpotent if and only if for every proper subgroup  $H$  its normal closure  $H^G$  is a proper subgroup.* Following J.S. Rose [10], a subgroup  $H$  of a group  $G$  is called *contranormal* if  $H^G = G$ . Taking this into account, we can reformulate this criterion in the following way: *A finite group  $G$  is nilpotent if and only if  $G$  does not include proper contranormal subgroups.*

It follows from the definition that the contranormal subgroups are in some sense antipodes to subnormal and descendant subgroups. On the other hand, the contranormal subgroups are in close connection with the descendant subgroups.

Let  $G$  be a group,  $H$  be a subgroup of  $G$  and  $X$  be a subset of  $G$ . Put

$$H^X = \langle h^x = x^{-1}hx \mid h \in H, x \in X \rangle.$$

In particular,  $H^G$  (the *normal closure* of  $H$  in  $G$ ) is the smallest normal subgroup of  $G$  containing  $H$ .

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Starting from the normal closure of  $H$ , we can construct the normal closure series of  $H$  in  $G$ ,

$$H^G = H_0 \geq H_1 \geq \dots H_\alpha \geq H_{\alpha+1} \geq \dots H_\gamma,$$

by the following rule:  $H_{\alpha+1} = H^{H_\alpha}$  for every  $\alpha < \gamma$ ,  $H_\lambda = \bigcap_{\mu < \lambda} H_\mu$  for a limit ordinal  $\lambda$ . The term  $H_\alpha$  of this series is called the  $\alpha$ th *normal closure of  $H$  in  $G$*  and will be denoted further by  $H^{G,\alpha}$ . The last term  $H_\gamma$  of this series is called *the lower normal closure of  $H$  in  $G$*  and will be denoted by  $H^{G,\infty}$ .

In Finite Group Theory the subgroup  $H^{G,\infty}$  is called *the subnormal closure of  $H$  in  $G$* . The rationale for this is the following. In a finite group  $G$ , the normal closure series of every subgroup  $H$  is finite, and  $H^{G,\infty}$  is the smallest subnormal subgroup of  $G$  containing  $H$ . The normal closure series play an important role in certain problems of Group Theory. As we can see, every subgroup  $H$  is contranormal in its lower normal closure. In particular, a subgroup  $H$  is contranormal in  $G$  if its lower normal closure  $H^{G,\infty}$  coincides with  $G$ . Conversely, a subgroup  $H$  is *descendant in  $G$*  if  $H$  coincides with its lower normal closure  $H^{G,\infty}$ . Properties of descendant subgroups and their influence on the structure of a group have been studied only superficially. With the exception of subnormal subgroups (an important particular case of descendant subgroups), we have no significant information regarding descending subgroups. A subnormal subgroup is exactly a descending subgroup having finite normal closure series. The subnormal subgroups form one of the most important families of subgroups in a group. These subgroups have been studied very successfully for a very long period of time. Their investigation have brought about many interesting and meaningful results.

As we have seen above, the absence of contranormal subgroups in a finite group implies its nilpotency. The question about an analog of this criterion for infinite groups is very natural. Some classes of infinite groups without proper abnormal subgroups have been considered in a series of articles [5, 9, 7, and 8]. Abnormal subgroups form an important subclass of the class of contranormal subgroups. Recall that a subgroup  $H$  is *abnormal* in a group  $G$  if  $g \in \langle H^g, H \rangle$  for every element  $g \in G$ . In the articles mentioned above, the authors proved that in the considered classes of groups the absence of abnormal subgroups implies local nilpotency of groups. However, it is worth mentioning that there are locally nilpotent groups having proper contranormal subgroups. For example, let  $G = K \rtimes \langle d \rangle$ , where  $K$  is a Prüfer 2-group,  $d$  be an element of order 2 and  $x^d = x^{-1}$  for each  $x \in K$ . The group  $G$  is non-nilpotent but hypercentral, and  $\langle d \rangle$  is a proper contranormal subgroup of  $G$ . This example shows that the absence of contranormal subgroups is a stronger condition than the absence of abnormal subgroups. However, in general, this condition does not imply nilpotency. In fact, there exist non-nilpotent groups all subgroups of which are subnormal. The first such example has been constructed by H. Heineken and I.J. Mohamed [4]. Nevertheless, for some classes of infinite groups the absence of contranormal subgroups does imply nilpotency of a group. The current article is dedicated to the search of such classes. In passing, in certain classes of infinite groups some new criteria of nilpotency were established.

Here are our main results.

**Theorem A** *Let  $G$  be group and  $H$  be a normal soluble-by-finite subgroup such that  $G/H$  is nilpotent. Suppose that  $H$  satisfies Min- $G$ . If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary A1** *Let  $G$  be a soluble-by-finite group satisfying the minimal condition on normal subgroups. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary A2** *Let  $G$  be a group and  $H$  be a normal Chernikov subgroup of  $G$ . Suppose that  $G/H$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

*In particular, a Chernikov group without proper contranormal subgroups is nilpotent.*

**Theorem B** *Let  $G$  be a group and  $C$  be a normal subgroup of  $G$  such that  $G/C$  is nilpotent. Suppose that  $C$  has a finite series of  $G$ -invariant subgroups*

$$\langle 1 \rangle = C_0 \leq C_1 \leq \dots \leq C_n = C,$$

*whose factors  $C_j/C_{j-1}$ ,  $1 \leq j \leq n$ , satisfy one of the following conditions:*

- (i)  $C_j/C_{j-1}$  is finite;*
- (ii)  $C_j/C_{j-1}$  is hyperabelian and minimax;*
- (iii)  $C_j/C_{j-1}$  is hyperabelian and finitely generated;*
- (iv)  $C_j/C_{j-1}$  is abelian and satisfies Min- $G$ .*

*If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary B1** *Let  $G$  be a group that has a finite series of  $G$ -invariant subgroups*

$$\langle 1 \rangle = C_0 \leq C_1 \leq \dots \leq C_n = G$$

*whose factors  $C_j/C_{j-1}$ ,  $1 \leq j \leq n$ , satisfy one of the following conditions:*

- (i)  $C_j/C_{j-1}$  is finite;*
- (ii)  $C_j/C_{j-1}$  is hyperabelian and minimax;*
- (iii)  $C_j/C_{j-1}$  is hyperabelian and finitely generated;*
- (iv)  $C_j/C_{j-1}$  is abelian and satisfies Min- $G$ .*

*If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent and minimax.*

**Corollary B2** *Let  $G$  be a group and let  $C$  be a normal subgroup of  $G$  such that  $G/C$  is nilpotent. Suppose that  $C$  is a hyperabelian minimax subgroup. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent. In particular, if  $G$  is a hyperabelian minimax group without proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary B3** *Let  $G$  be a group and let  $C$  be a normal subgroup of  $G$  such that  $G/C$  is nilpotent. Suppose that  $C$  is a hyperabelian finitely generated subgroup. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent. In particular, if  $G$  is hyperabelian finitely generated group without proper contranormal subgroups, then  $G$  is nilpotent.*

## 1. Groups having a normal subgroup satisfying min- $G$

Let  $G$  be a group. Given  $x, g \in G$ , we put

$$[x, {}_1g] = [x, g] \text{ and } [x, {}_{n+1}g] = [[x, {}_ng], g] \text{ for } n \geq 1.$$

We start with some auxiliary results.

**Lemma 1.1** *Let  $G$  be a group, and let  $A$  be an abelian normal subgroup of  $G$ . If  $C_G(A) \neq gC_G(A) \in \zeta(G/C_G(A))$ , then for any  $x \in G$ ,  $a \in A$  and  $n \in \mathbb{N}$  we have*

$$[xa, {}_n g] = [x, {}_n g][a, {}_n g].$$

**Proof.** Since  $[x, g] \in C_G(A)$ , we have  $[xa, g] = [x, g]^a[a, g] = [x, g][a, g]$ . The case  $n = 1$  is proved. Suppose that  $n > 1$ , and we have already proved that

$$[xa, {}_{n-1} g] = [x, {}_{n-1} g][a, {}_{n-1} g].$$

Applying a fundamental commutator identity, we obtain

$$[xa, {}_n g] = [[x, {}_{n-1} g], g] = [[x, {}_{n-1} g][a, {}_{n-1} g], g] =$$

$$([a, {}_{n-1} g]^{-1}[[x, {}_{n-1} g], g][a, {}_{n-1} g])[a, {}_{n-1} g], g = ([a, {}_{n-1} g]^{-1}[x, {}_n g][a, {}_{n-1} g])[a, {}_n g].$$

Since  $A$  is normal in  $G$ ,  $[a, {}_{n-1} g] \in A$ . Furthermore, the choice of the element  $g$  justifies the inclusion  $[x, g] \in C_G(A)$ . It follows that  $[x, {}_n g] \in C_G(A)$ , so that

$$([a, {}_{n-1} g]^{-1}[x, {}_n g][a, {}_{n-1} g]) = [x, {}_n g],$$

and hence

$$[xa, {}_n g] = [x, {}_n g][a, {}_n g].$$

□

**Lemma 1.2** *Let  $G$  be a group and  $A$  be a normal abelian subgroup of  $G$ . If  $g \in G, a \in A, n \in \mathbb{N}$ , then*

$$[a^{-1}, {}_n g] = [a, {}_n g]^{-1}.$$

**Proof.** We have

$$[a, g]^{-1} = g^{-1}a^{-1}ga = a^{-1}ag^{-1}a^{-1}ga = a^{-1}[a^{-1}, g]a.$$

Since  $A$  is normal in  $G$ ,  $[a^{-1}, g] \in A$ . Then  $a^{-1}[a^{-1}, g]a = [a^{-1}, g]$ , because  $A$  is abelian. Suppose that  $n > 1$  and we have already proved the identity

$$[a^{-1}, {}_{n-1} g] = [a, {}_{n-1} g]^{-1}.$$

Since  $A$  is abelian, we have

$$\begin{aligned} [a^{-1}, {}_n g] &= [[a^{-1}, {}_{n-1} g], g] = \\ &= [[a, {}_{n-1} g]^{-1}, g] = [a^{-1}, {}_{n-1} g]^{-1}[[a, {}_{n-1} g], g]^{-1}[a^{-1}, {}_{n-1} g] = [a, {}_n g]^{-1}. \end{aligned}$$

□

Let  $G$  be a group and  $n$  a positive integer. Define the left  $n$ -Engelizer  $E_{G,n}(g)$  of an element  $g \in G$  by the rule

$$\mathbf{E}_{G,n}(g) = \{x \in G \mid [x, {}_n g] = 1\}.$$

Observe, that in general the subset  $\mathbf{E}_{G,n}(g)$  is not a subgroup.

If  $G$  is a nilpotent group then we denote the nilpotency class of  $G$  by  $ncl(G)$ .

**Lemma 1.3** *Let  $G$  be a group and let  $g$  be an element of  $G$  such that  $[G, g]$  is nilpotent and  $ncl([G, g]) = k$ . Then for each  $n \in \mathbb{N}$  there exists a number  $m = m(n, k)$ , depending only on  $k$  and  $n$ , such that  $\langle \mathbf{E}_{G,n}(g) \rangle \subseteq \mathbf{E}_{G,m}(g)$ .*

**Proof.** We will apply the arguments of the proof of Corollary 3\* of [11]. We will provide some specific concretization of the situation. Let  $x, y \in \mathbf{E}_{G,n}(g)$ , so that  $[x, {}_n g] = 1 = [y, {}_n g]$ , and put  $L = \langle x, y, g \rangle$  and  $D = [L, g]$ . It follows that  $D$  is nilpotent of class at most  $k$ . If  $h \in L$ , then Lemma 4 of [11] implies  $[h, {}_n g] \in [D, D]$ . Therefore  $\langle g \rangle^L / [D, D]$  is nilpotent of class at most  $n$ . Since  $D$  is nilpotent of class at most  $k$ , Theorem 7 of [3] implies that  $\langle g \rangle^L$  is nilpotent of class at most  $n \binom{k+1}{2} - \binom{k}{2} = t(n, k)$ . Since  $[h, g] \in \langle g \rangle^L$ ,  $[h, {}_m g] = 1$ , where  $m = m(n, k) = t(n, k) + 1$ . It follows that  $h \in \mathbf{E}_{G,m}(g)$ .  $\square$

**Proposition 1.4** *Let  $G$  be a group and  $A$  be a normal abelian subgroup of  $G$ ,  $B$  a  $G$ -invariant subgroup of  $A$  and let  $g$  be an element of  $G$  such that  $C_G(B) \neq gC_G(B) \in \zeta(G/C_G(B))$ . Suppose that  $G$  satisfies the following conditions:*

- (1)  $G/A$  is nilpotent;
- (2)  $[G, g]$  is nilpotent;
- (3) there is a positive integer  $k$  such that  $[A, {}_k g] \leq B$ ;
- (4) there is a positive integer  $r$  such that  $D = [B, {}_r g] = [B, {}_{r+1} g] \neq \langle 1 \rangle$ .

*Then there exists a subgroup  $L$  of  $G$  such that  $G = LD$  and  $L \cap B \subseteq E_{G,m}(g)$  for some positive integer  $m$ , and  $L \cap D$  is a proper  $G$ -invariant subgroup of  $D$ .*

**Proof.** We observe that condition (4) implies  $D = [D, g]$ . Since this is not possible for a nilpotent group, we can conclude that  $G$  is not nilpotent. Let  $x$  be an arbitrary element of  $G$ . Then the choice of  $g$  implies  $[x, g] \in C_G(B)$ . By condition (1)  $G/A$  is nilpotent, so there is a positive integer  $n$  such that  $[x, {}_n g] \in A$ . Condition (3) shows that  $[x, {}_{n+k} g] \in B$ . Applying condition (4), we obtain that  $[x, {}_{n+k+r} g] \in D$ . Put  $t = n + k + r$ , then  $[x, {}_t g] = a$  for a suitable element  $a \in D$ . The equation  $[D, g] = D$  implies that there is an element  $a_1 \in D$  such that  $a = [d_1, g]$ . By the same reasons,  $d_1 = [d_2, g]$  for some element  $d_2 \in D$ . Applying ordinary induction we come to an element  $d_t \in D$  such that  $a = [d_t, {}_t g]$ . Thus

$$[x, {}_t g] = a = [d_t, {}_t g].$$

By Lemma 1.1 we obtain that

$$[xd_t^{-1}, {}_t g] = [x, {}_t g][d_t^{-1}, {}_t g].$$

Lemma 1.2 implies that  $[d_t^{-1}, t g] = [d_{t,t} g]^{-1}$ , and therefore

$$[x d_t^{-1}, t g] = [x, t g][d_{t,t} g]^{-1};$$

so we come to the equation

$$[x d_t^{-1}, t g] = 1.$$

This equation justifies that the element  $x d_t^{-1}$  belongs to the left  $t$ -Engelizer  $\mathbf{E}_{G,t}(g)$  of the element  $g$ . Put  $L = \langle \mathbf{E}_{G,t}(g) \rangle$ . Then  $x d_t^{-1} \in L$  and  $x \in DL$ . Since  $x$  is an arbitrary element of  $G$ , the equation  $G = DL$  is valid.

By condition (2)  $[G, g]$  is nilpotent. Then by Lemma 1.3 there exists a positive integer  $m$  such that  $L = \langle \mathbf{E}_{G,t}(g) \rangle \subseteq \mathbf{E}_{G,m}(g)$ . In particular,  $D \cap L \subseteq \mathbf{E}_{G,m}(g)$ . Suppose that  $D \cap L = D$ . Then  $D$  has the finite series

$$D = D_0 \geq D_1 = [D, g] \geq [[D, g], g] \geq \dots \geq [D, m g] = \langle 1 \rangle.$$

In particular, this shows that  $D \neq [D, g]$ . This contradiction proves that  $D \cap L$  is a proper subgroup of  $D$ . Obviously,  $D \cap L$  is a  $G$ -invariant subgroup.  $\square$

**Corollary 1.5** *Let  $G$  be a group and  $A$  be a normal abelian subgroup of  $G$ ,  $B$  be a  $G$ -invariant subgroup of  $G$ . Suppose that  $G$  satisfies the following conditions:*

- (1)  $G/A$  is nilpotent;
- (2)  $A$  has the series of  $G$ -invariant subgroups

$$\langle 1 \rangle = B_0 \leq B_1 \leq \dots \leq B_n = A$$

*such that the centralizers  $C_G(B_j/B_{j-1}) \geq C_G(B_1)$ ,  $2 \leq j \leq n$ , and  $C_G(B_1)$  is nilpotent;*

- (3)  $A$  satisfies  $\text{Min}-G$ .

*Suppose that  $g$  is an element of  $G$  such that  $C_G(B_1) \neq g C_G(B_1) \in \zeta(G/C_G(B_1))$ . If  $G$  has no proper contranormal subgroups, then there is a positive integer  $t$  such that  $[A, t g] = \langle 1 \rangle$ . In particular, the subgroup  $\langle A, g \rangle$  is nilpotent.*

**Proof.** Put  $U = B_{n-1}$ . Since  $g C_G(B_1) \in \zeta(G/C_G(B_1))$  and  $C_G(A/U) \geq C_G(B_1)$ ,  $g C_G(A/U) \in \zeta(G/C_G(A/U))$ . By the choice of element  $g$  the subgroups  $[A/U, n g U]$  are  $G$ -invariant for all  $n \in \mathbb{N}$ . Since  $A$  satisfies  $\text{Min}-G$ , there is a positive integer  $k$  such that  $D/U = [A/U, k g U] = [A/U, k+1 g U]$ . Suppose that  $D/U \neq \langle 1 \rangle$ . By the choice of element  $g$ ,  $[G, g]$  is nilpotent, thus we may apply Proposition 1.4. By this Proposition  $G/U$  includes a subgroup  $L/U$  such that  $G/U = (D/U)(L/U)$  and  $D/U \cap L/U \neq D/U$ . Hence  $G = DL$ . In particular,  $L$  is a proper subgroup. Furthermore, as we can see from the proof of Proposition 1.4,  $g U \in L/U$ . Since  $D = [D, g]$ , for each element  $a \in D$  there is an element  $b \in D$  such that  $a = b^{-1} g^{-1} b g$ . It follows that  $b^{-1} g^{-1} b = a g^{-1}$ . This equation shows that  $D \leq L^G$ . Together with  $G = DL$  it justifies that  $G = L^G$ . In other words,  $L$  is a proper contranormal subgroup of  $G$ . This is impossible. The contradiction proves that  $D = U$ . In other words,  $[A, k g] \leq B_{n-1}$ .

Put  $V = B_{n-2}$ . Since  $gC_G(B_1) \in \zeta(G/C_G(B_1))$  and  $C_G(B_{n-1}/B_{n-2}) \geq C_G(B_1)$ ,  $gC_G(U/V) \in \zeta(G/C_G(U/V))$ . By the choice of element  $g$ , the subgroups  $[U/V, {}_n gV]$  are  $G$ -invariant for all  $n \in \mathbb{N}$ . Since  $A$  satisfies  $\text{Min-}G$ , there is a positive integer  $s$  such that  $W/V = [U/V, {}_s gV] = [U/V, {}_{s+1} gV]$ . Suppose that  $W/V \neq \langle 1 \rangle$ . By the choice of element  $g$ ,  $[G, g]$  is nilpotent and we may apply Proposition 1.4. Contradiction. Therefore  $W = V$ . In other words,  $[A, {}_{k+s} g] \leq B_{n-2}$ . Repeating similar arguments, after finitely many steps we come to a positive integer  $t$  such that  $[A, {}_t g] = \langle 1 \rangle$ .  $\square$

**Theorem 1.6** *Let  $G$  be a group and  $A$  be a normal abelian subgroup of  $G$ . Suppose that  $A$  satisfies  $\text{Min-}G$  and  $G/A$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof.** Put  $C = C_G(A)$ . Since  $G/A$  is nilpotent,  $C$  is nilpotent too. If  $G = C$ , then all is proved. Therefore we may suppose that  $G \neq C$ . In the center  $\zeta(G/C)$  we choose a non-identity element  $g_1 C$ . Corollary 1.5 shows that the subgroup  $\langle A, g_1 \rangle$  is nilpotent. Let

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \dots \leq Z_n = A$$

be the upper  $\langle g_1 \rangle$ -central series of  $A$ , that is,  $Z_1 = C_A(g)$ ,  $Z_{j+1}/Z_j = C_{A/Z_j}(g)$ ,  $1 \leq j \leq n-1$ .

The choice of element  $g_1$  and the fact that  $A$  is abelian imply that the mapping  $\phi : a \rightarrow [a, g_1]$ ,  $a \in A$ , is an endomorphism of  $A$ . Moreover, it is a  $G$ -endomorphism of  $A$ . Indeed, for each  $x \in G$ ,  $a \in A$  we have  $\phi(a)^x = [a, g_1]^x = [a^x, g_1^x]$ . The inclusion  $g_1 C \in \zeta(G/C)$  implies that  $g_1^x = g_1 c$  for some element  $c \in C$ . Since the choice of  $c$  implies  $[a^x, c] = 1$  and  $[a^x, g_1]^c = [a^x, g_1]$  we obtain

$$[a^x, g_1^x] = [a^x, g_1 c] = [a^x, c][a^x, g_1]^c = [a^x, g_1] = \phi(a^x).$$

It follows that  $Z_2/Z_1$  is  $G$ -isomorphic to some  $G$ -invariant subgroup of  $Z_1$ . In particular, if  $x \in C_G(Z_1)$ , then  $x \in C_G(Z_2/Z_1)$ . In other words,  $C_G(Z_1) \leq C_G(Z_2/Z_1)$ . By the same reason,  $C_G(Z_2/Z_1) \leq C_G(Z_3/Z_2)$ , and hence  $C_G(Z_1) \leq C_G(Z_3/Z_2)$ . Using the same arguments we obtain that  $C_G(Z_1) \leq C_G(Z_{j+1}/Z_j)$ ,  $1 \leq j \leq n-1$ . If  $x \in C_G(Z_1)$ , then  $x$  acts trivially in all factors of series

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \dots \leq Z_n = A.$$

In other words, the last series is a central series for the subgroup  $C_G(Z_1)$ . Since  $C_G(Z_1)/A$  is nilpotent, it follows that  $C_G(Z_1)$  is a nilpotent subgroup.

If  $G = C_G(Z_1)$ , then all is proved. Therefore assume that  $G \neq C_G(Z_1)$ . In the center  $\zeta(G/C_G(Z_1))$  we choose a non-identity element  $g_2 C_G(Z_1)$ . Since the subgroup  $C_G(Z_1)$  is nilpotent, we may apply Corollary 1.5. This Corollary shows that the subgroup  $\langle A, g_2 \rangle$  is nilpotent, in particular,  $\langle Z_1, g_2 \rangle$  is nilpotent too. Let

$$\langle 1 \rangle = Z_{10} \leq Z_{11} \leq \dots \leq Z_{1m} = Z_1$$

be the upper  $\langle g_2 \rangle$ -central series of  $Z_1$ . We observe that the selection of  $g_2$  implies that  $Z_1 \neq Z_{11}$ . Since  $Z_{j+1}/Z_j$  is  $G$ -isomorphic to some  $G$ -invariant subgroup of  $Z_1$ , we obtain that  $Z_{j+1}/Z_j$  has finite  $\langle g_2 \rangle$ -central series

$$Z_j = Z_{j0} \leq Z_{j1} \leq \dots \leq Z_{jm} = Z_{j+1}, 1 \leq j \leq n-1.$$

Using the above arguments we obtain the inclusion  $C_G(Z_{11}) \leq C_G(Z_{1j+1}/Z_{1j}), 1 \leq j \leq m-1$ . In turn, it follows that  $C_G(Z_{11}) \leq C_G(Z_{kj+1}/Z_{kj}), 1 \leq j \leq m-1, 2 \leq k \leq n-1$ . In other words, if  $x \in C_G(Z_1)$ , then  $x$  acts trivially in all factors  $Z_{kj+1}/Z_{kj}, 1 \leq j \leq m-1, 1 \leq k \leq n-1$ . This means that  $A$  has a finite  $C_G(Z_{11})$ -central series. Since  $C_G(Z_{11})/A$  is nilpotent,  $C_G(Z_{11})$  is nilpotent.

If  $G = C_G(Z_{11})$ , then all is proved. Therefore assume that  $G \neq C_G(Z_{11})$ . In the center  $\zeta(G/C_G(Z_{11}))$  we choose a non-identity element  $g_3 \in C_G(Z_{11})$ . Repeating the above arguments we obtain that  $Z_{11}$  has an upper  $\langle g_2 \rangle$ -central series, so that  $Z_{11} \neq Z_{11} \cap C_A(g_3) = Z_{111}$ . Thus we obtain the strictly descending series  $Z_1 > Z_{11} > Z_{111}$ . Observe that the subgroups of this series is  $G$ -invariant. If  $G = C_G(Z_{111})$ , then all is proved. If  $G \neq C_G(Z_{111})$ , we can continue the above process and obtain the strictly descending series  $Z_1 > Z_{11} > Z_{111} > Z_{1111}$  of  $G$ -invariant subgroups. Since  $A$  satisfies Min- $G$ , this process is finite. Hence after finitely many steps we obtain that  $G$  is nilpotent.  $\square$

**Corollary 1.7** *Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Suppose that  $H$  has a finite series of  $G$ -invariant subgroups*

$$\langle 1 \rangle = A_0 \leq A_1 \leq \dots \leq A_n = H$$

*such that  $A_{j+1}/A_j$  is abelian and satisfies Min- $G$  for every  $j, 0 \leq j \leq n-1$ . Assume that  $G/H$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary 1.8** *Let  $G$  be a polynilpotent group satisfying minimal condition for normal subgroups. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary 1.9** *Let  $G$  be a group and  $H$  be a normal subgroup of  $G$  such that  $G/H$  is nilpotent. Suppose that  $H$  has a finite series of  $G$ -invariant subgroups*

$$\langle 1 \rangle = A_0 \leq A_1 \leq \dots \leq A_n = H$$

*such that  $A_{j+1}/A_j$  is abelian and  $G$ -simple for every  $j, 0 \leq j \leq n-1$ . If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Corollary 1.10** *Let  $G$  be a Chernikov group and let  $D$  be a divisible part of  $G$ . If  $G$  has no proper contranormal subgroups, then  $D \leq \zeta(G)$  and  $G/D$  is a finite nilpotent group.*

**Proof.** A characteristic subgroup  $D$  is abelian and satisfies Min. Since  $G/D$  is a finite group without proper contranormal subgroups,  $G/D$  is nilpotent. Theorem 1.6 implies that  $G$  is nilpotent. Now we recall that in a nilpotent Chernikov group the center includes a divisible part (see, for example, [1, Corollary 1.7]).  $\square$

**Proposition 1.11** *Let  $G$  be a group and  $K$  be a finite normal subgroup of  $G$ . Suppose that  $G/K$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof.** Put  $C = C_G(K)$ . Then  $G/C$  is finite. Since the finite group  $G/C$  does not include proper contranormal subgroups,  $G/C$  is nilpotent. By the isomorphism  $K/(K \cap C) \cong KC/C \leq G/C$  and the inclusion



$K \cap C \leq \zeta(K)$ ,  $K$  is nilpotent. Being finite  $K$  has a finite series of  $G$ -invariant subgroups, whose factors are abelian and  $G$ -simple. Applying Corollary 1.9 we obtain that  $G$  is nilpotent.  $\square$

**Corollary 1.12** *Let  $G$  be a group and  $H$  be a normal Chernikov subgroup of  $G$ . Suppose that  $G/H$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof of Theorem A** This Theorem is an immediate consequence of Corollary 1.7 and Proposition 1.11.  $\square$

We observe, that an analogy of Theorem 1.6 for the condition Max- $G$  is not valid. The following example justifies it.

Let  $R = \mathbb{Z}_{2^\infty}$  be the ring of 2-adic numbers and  $A$  be its additive group,  $U = U(\mathbb{Z}_{2^\infty})$ . Then  $U$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times H$ , where  $H \cong A$  (see, for example, [2, § 128]). Every element of  $R$  has a form  $2^n u$  where  $u \in U$ . It follows that the  $\mathbb{Z}H$ -submodule of  $R$ , generated by 1, coincides with  $A$ . Moreover, every  $\mathbb{Z}H$ -submodule of  $A$  coincides with some ideal of a ring  $R$ , that is, has a form  $2^n A$  for some positive integer  $n$ . Let  $G$  be a natural semidirect product of  $A$  and  $H$ . Clearly,  $G$  is a non-nilpotent group and  $C_G(A) = A$ . Choose in  $A$  a non-identity  $G$ -invariant subgroup  $K$ . By the above arguments,  $A/K$  is a cyclic 2-group. Since the automorphisms group of cyclic 2-group is a 2-group, it is worth nothing to see that  $G/K$  is nilpotent. Suppose that  $G$  has a proper contranormal subgroup  $L$ . Since  $G/A$  is abelian,  $(LA/A)^{G/A} = L^G A/A = LA/A$ . The equation  $L^G = G$  shows that  $LA = G$ . If  $D = L \cap A \neq \langle 1 \rangle$ , then clearly  $D$  is a  $G$ -invariant subgroup. We proved above that  $A/D$  is nilpotent. On the other hand, the equation  $L^G = G$  implies  $L^G/D = G/D$ . For a nilpotent group this means that  $L/D = G/D$  or  $L = G$ . Suppose now that  $L \cap A = \langle 1 \rangle$ . In particular,  $L$  is abelian. Consider the factor-group  $G/B$  where  $B = 2A$ . It includes a normal subgroup  $A/B$  of order 2. Then  $A/B \leq \zeta(G/B)$ . It follows that  $G/B = A/B \rtimes (LB/B) = A/B \times (LB/B)$ , so  $G/B$  is abelian. It follows that  $L^G B/B = LB/B$  is a proper subgroup of  $G/B$ . Therefore  $LG$  is also a proper subgroup of  $G$ . Hence  $G$  has no proper contranormal subgroups. As we see above, every a  $G$ -invariant subgroup of  $A$  has finite index in  $A$ , in particular,  $A$  satisfies Max- $G$ .

## 2. Groups having a normal subgroup similar to minimax

A group  $G$  is said to be minimax if  $G$  has a finite subnormal series whose factors satisfy the condition Min (the minimal condition for all subgroups) or the condition Max (the maximal condition for all subgroups). If  $A$  is an abelian minimax group, then  $A$  includes a finitely generated subgroup  $B$  such that  $A/B$  is a Chernikov group. Therefore a soluble minimax group has a finite subnormal series whose factors are abelian and either are finitely generated or Chernikov groups.

*If  $G$  is a group then denote by  $\Pi(G)$  the set of all prime divisors of orders of all elements of  $G$*

**Lemma 2.1** *Let  $G$  be a group and  $A$  be a normal abelian minimax torsion-free subgroup of  $G$ . Suppose that  $G/A$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof.** Being minimax  $A$  includes a free abelian subgroup  $B$  such that  $A/B$  is Chernikov. In particular, the set  $\Pi(A/B)$  is finite. Let  $p$  be a prime such that  $p \in \Pi(A/B)$ . Then  $B/B^p$  is the Sylow  $p$ -subgroup of  $A/B^p$ ,

therefore  $A/B^p = B/B^p \times C/B^p$  where  $C/B^p$  is the Sylow  $p$ -subgroup of  $A/B^p$ . In particular,  $A/C$  is a non-identity elementary abelian  $p$ -group. It follows that  $A \neq A^p$ , moreover  $A^p \leq C$  thus  $A^p \cap B \leq C \cap B = B^p$ . In turn, the following inclusion follows:

$$\left( \bigcap_{p \notin \Pi(A/B)} A^p \right) \cap B = \bigcap_{p \notin \Pi(A/B)} (A^p \cap B) \leq \bigcap_{p \notin \Pi(A/B)} B^p = \langle 1 \rangle.$$

Since  $A/B$  is periodic and  $A$  is torsion-free, this means that  $\bigcap_{p \notin \Pi(A/B)} A^p = \langle 1 \rangle$ . Consider the factor-group  $G/A^p$ . Since  $A$  is minimax, then  $A$  has finite 0-rank. Let  $r_0(A) = k$ . The factor group  $A/A^p$  is finite, moreover  $|A/A^p| \leq p^k$ . By Proposition 1.11,  $G/A^p$  is nilpotent. Since  $A/A^p$  is a normal finite  $p$ -subgroup of a nilpotent group  $G/A^p$  and  $|A/A^p| \leq p^k$ ,  $[A, {}_k G] \leq A^p$ . Recall that this is valid for each  $p \notin \Pi(A/B)$ , therefore

$$[A, {}_k G] \leq \bigcap_{p \notin \Pi(A/B)} A^p = \langle 1 \rangle.$$

This shows that the hypercenter of  $G$  having a natural number  $k$  includes  $A$ . Since  $G/A$  is nilpotent, it follows that  $G$  is nilpotent. □

**Corollary 2.2** *Let  $G$  be a group and  $A$  be a normal soluble minimax subgroup of  $G$ . Suppose that  $G/A$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof.** Being soluble  $A$  has a series of  $G$ -invariant subgroup whose factors are abelian. Since every periodic minimax abelian group is Chernikov,  $A$  has a series of  $G$ -invariant subgroup whose factors either are abelian minimax torsion-free or abelian Chernikov group. Now we can apply Corollary 1.12 and Lemma 2.1. □

**Corollary 2.3** *Let  $G$  be a group and  $A$  be a normal hyperabelian minimax subgroup of  $G$ . Suppose that  $G/A$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

Since a hyperabelian group with  $Max$  clearly polycyclic, and hyperabelian group with  $Min$  is Chernikov [1, Theorem 1.3], we can conclude that a hyperabelian minimax group is soluble.

**Corollary 2.4** *Let  $G$  be a group and  $K$  be a normal finitely generated hyperabelian subgroup of  $G$ . Suppose that  $G/K$  is nilpotent. If  $G$  has no proper contranormal subgroups, then  $G$  is nilpotent.*

**Proof.** Let  $H$  be a normal subgroup of  $K$  having finite index. Let  $|K/H| = t$ . Then  $K^t \leq H$ . Clearly the subgroup  $K^t$  is  $G$ -invariant. Since  $K$  is soluble and finitely generated and  $K/K^t$  is bounded,  $K/K^t$  is finite. By Proposition 1.11 the factor-group  $G/K^t$  is nilpotent, in particular,  $K/K^t$  is nilpotent. Thus every finite factor-group of  $K$  is nilpotent. By a result due to D. Robinson (see, for example, [6, Theorem 13.8]),  $K$  is nilpotent. Being finitely generated  $K$  is minimax, so that we may apply Corollary 2.3. □

**Proof of Theorem B** This result follows directly from Corollary 1.7, Proposition 1.11 and Corollaries 2.3 and 2.4. □

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