Strong differential subordination

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Abstract

The concept of differential subordination was introduced in [4] by S. S. Miller and P. T. Mocanu and the concept of strong differential subordination was introduced in [1] by J. A. Antonino and S. Romaguera. This last concept was applied in the special case of Briot-Bouquet strong differential subordination. In this paper we study the strong differential subordinations in the general case, following the general theory of differential subordinations presented in [4].

Key Words: Analytic function, differential subordination, subordination, strong subordination, univalent function.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H};\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in U,$$

which are analytic in the unit disk.

Definition 1 *[1], [2], [3] Let $H(z, \xi)$ be analytic in $U \times \overline{U}$ and let $f(z)$ analytic and univalent in U. The function $H(z, \xi)$ is strongly subordinate to $f(z)$, written $H(z, \xi) \prec\prec f(z)$ if for $\xi \in \overline{U}$, the function of z, $H(z, \xi)$ is subordinate to $f(z)$.*

Remark 1 Since $f(z)$ is analytic and univalent, Definition 1 is equivalent to

$$H(0, \xi) = f(0) \text{ and } H(U \times \overline{U}) \subset f(U). \tag{1}$$

2. Main results

Let Ω and Δ be any sets in $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(r, s, t; z, \xi) : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. As in [4], in this article we consider conditions on Ω, Δ and ψ for which the following implication holds:

$$\{\psi(p(z), zp'(z), z^2 p''(z); z, \xi \mid z \in U, \ \xi \in \overline{U}\} \subset \Omega \ \Rightarrow \ p(U) \subset \Delta. \tag{2}$$

There are three distinct cases to consider in analyzing this implication, which we list as the following problems.

Problem 1. Given Ω and Δ, find conditions on the function ψ so that (2) holds.

We call such a ψ an **admissible function**.

Problem 2. Given ψ and Ω, find a set Δ such that (2) holds. Furthermore, find the "smallest" such Δ.

Problem 3. Given ψ and Δ, find a set Ω such that (2) holds. Furthermore, find the "largest" such Ω.

If either Ω or Δ in (2) is a simply connected domain, then it may be possible to rephrase (2) in terms of strong subordination.

If p is analytic in U, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = p(0)$. In this case (2) can be rewritten as

$$\{\psi(p(z), zp'(z), z^2 p''(z); z, \xi \mid z \in U, \ \xi \in \overline{U}\} \subset \Omega \ \Rightarrow \ p(z) \prec q(z). \tag{2$'$}$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(p(0), 0, 0; 0, 0)$. If in addition, the function $\psi(p(z), zp'(z), z^2 p''(z); z, \xi)$ is analytic in $\overline{U}$, then (2$'$) can be rewritten as

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \prec\prec h(z) \ \Rightarrow \ p(z) \prec q(z). \tag{2$''$}$$

This implication also has meaning if h and q are analytic and not necessarily univalent. This last result leads us to some of the important definitions that will be used in this article.

Definition 2 *Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p is analytic in U and satisfies the (second-order) strong differential subordination*

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \prec\prec h(z), \quad z \in U, \tag{3}$$

*then p is called a **solution** of the strong differential subordination.*

*The univalent function q is called a **dominant of the solutions of the strong differential subordination**, or more simply a **dominant**, if $p \prec q$ for all p satisfying (3).*

*A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (3$'$) is said to be the best dominant of (3). Note that the best dominant is unique up to a rotation of U. If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)-**solution**, q an (a, n)-**dominant**, and $\widetilde{q}$ the **best (a, n)-dominant**.*

Let Ω be a set in $\mathbb{C}$ and suppose (3) is replaced by

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \subset \Omega, \ for \ z \in U, \ \xi \in \overline{U}. \tag{3'}$$

Although this is a differential inclusion and $\psi(p(z), zp'(z), z^2 p''(z); z, \xi)$ may not be analytic in U, we shall refer to (3') as a (**second-order**) strong differential subordination, and use the same definitions of solution, dominant and best dominant as given above.

In the case when Ω and Δ in (2) are simply connected domains, we have seen that (2) can be rewritten in terms of strong subordinations such as given in (2').

In the special case when the set inclusion (2) can be replaced by the strong subordination in (3) we can reinterpret the three problems referred to above as follows:

Problem 1'. Given univalent functions h and q find a class of admissible functions $\psi[h, q]$ such that (2") holds.

Problem 2'. Given the strong differential subordination in (2") find a dominant q. Moreover, find the best dominant.

Problem 3'. Given ψ and dominant q, find the largest class of univalent functions h such that (2") holds.

Before obtaining some of our main results we need to introduce a class of univalent functions defined on the unit disk that have some nice boundary properties.

Definition 3 *[4], [5, Definition 2.2b, p. 21] We denote by Q the set of functions q that are analytic and injective on $\overline{U} - E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Remark 2. If $q \in Q$ then the domain $\Delta = q(U)$ is simply connected and its boundary consists of either a simple closed regular curve or the union (possibly infinite) of pairwise disjoint simple regular curves, each of which converges to ∞ in both directions. The functions $q_1(z) = z$ and $q_2(z) = \dfrac{1+z}{1-z}$ are examples of these two cases.

We will use the following Lemma [5, 2.2d, p. 24] from the theory of differential subordinations to determine subordinants of strong differential subordinations.

Lemma A. *[4], [5, 2.2.d, p. 24]. Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + \ldots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ *and*

(iii) Re $\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m$Re $\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right].$

We next define the class of admissible functions referred to in the introduction.

Definition 4 *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The* **class of admissible functions** $\Psi_n[\Omega, q]$, *consists of those functions $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r, s, t; z, \xi) \notin \Omega \tag{A}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$

$$\text{Re} \left[\frac{t}{s} + 1 \right] \geq m\text{Re} \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right],$$

$z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is a conformal mapping of U onto Ω we denote this class by $\Psi_n[h, q]$.

If $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\psi(q(\zeta), m\zeta q'(\zeta); z, \xi) \notin \Omega,$$

when $z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

If $\psi : \mathbb{C} \times U \times \overline{U} \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\psi(q(\zeta); z, \xi) \notin \Omega$$

when $z \in U$, $\xi \in \overline{U}$, and $\zeta \in \partial U \setminus E(q)$.

A careful check of the definition shows that $\psi_n[\overline{\Omega}, q] \subset \psi_n[\Omega, q]$ when $\Omega \subset \overline{\Omega}$, so enlarging Ω decreases the class of admissible functions. Also note that $\psi_n[\Omega, q] \subset \psi_{n+1}[\Omega, q]$.

The next theorem is a foundation result in the theory of first and second order strong differential subordinations.

Theorem 1 *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \in \Omega \tag{4}$$

then

$$p(z) \prec q(z), \quad z \in U.$$

Proof. Assume $p \not\prec q$. By Lemma A there exist points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ that satisfy (i)-(iii) of Lemma A. Using these conditions with $r = p(z_0)$, $s = z_0 p'(z_0)$, $t = z_0^2 p''(z_0)$ and $z = z_0$ in Definition 4 we obtain

$$\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0, \xi) \notin \Omega.$$

Since this contradicts (4) we must have $p(z) \prec q(z)$. □

From this theorem we see that we can obtain dominants of a strong differential subordination of the form (4) by merely checking that the function ψ is an admissible function. This requires that ψ satisfies (A). This simple algebraic check yields various strong differential subordinations and strong differential inequalities that would be difficult to derive directly.

Upon examining the proof of Theorem 1, it is easy to see that the theorem also holds if condition (4) is replaced by

$$\psi(p(z), zp'(z), z^2p''(z); w(z), \xi) \subset \Omega, \quad z \in U \tag{5}$$

where $w(z)$ is any function mapping U into U,.

On checking the definitions of Q and $\Psi_n[\Omega, q]$ we see that the hypothesis of Theorem 1 requires that q behave very nicely on the boundary of U. If this is not true or if the behavior of q is not known, it may still be possible to prove that $p \prec q$ by the following limiting procedure.

Corollary 1 *Let $\Omega \subset \mathbb{C}$ and let q be univalent in U, with $q(0) = a$. Let $\psi \in \Psi_n[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $p \in \mathcal{H}[a, n]$ and*

$$\psi(p(z), zp'(z), z^2p''(z), z, \xi) \subset \Omega, \quad z \in U,$$

then

$$p(z) \prec q(z), \quad z \in U.$$

Proof. The function q_ρ is univalent on $\overline{U}$, and therefore $E(q_\rho)$ is empty and $q_\rho \in Q$. The class $\Psi_n[\Omega, q]$ is an admissible class and from Theorem 1 we obtain $q_\rho \prec p$. Since $q_\rho \prec q$., we deduce $p \prec q$. □

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω and the class $\psi_n[h(U), q]$ is written as $\psi_n[h, q]$.

The following result is an immediate consequence of Theorem 1.

Theorem 2 *Let $\psi \in \Psi_n[h, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z, \xi)$ is analytic in U, and*

$$\psi(p(z), zp'(z), z^2p''(z); z, \xi) \prec\prec h(z) \tag{6}$$

then

$$p(z) \prec q(z), \quad z \in U.$$

This result can be extended to those cases in which the behavior of q on the boundary of U is unknown by the following theorem.

Theorem 3 *Let h and q be univalent in U, with $q(0) = a$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h, q_\rho]$, for some $\rho \in (0, 1)$; or

(ii) there exists $\rho_0 \in (0,1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a,b]$, $\psi(p(z), zp'(z), z^2p''(z); z, \xi)$ is analytic in U, and

$$\psi(p(z), zp'(z), z^2p''(z); z, \xi) \prec\prec h(z),$$

then

$$p(z) \prec q(z), \quad z \in U.$$

Proof.   Case (i). By applying Theorem 1 we obtain $p \prec q_\rho$. Since $q_\rho \prec q$ we deduce $p \prec q$.

Case (ii). If we let $p_\rho(z) = p(\rho z)$, then

$$\psi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z, \xi) = \psi(p(\rho z), \rho zp'(\rho z), \rho^2 z^2 p''(z); \rho z, \xi) \in h_\rho(U).$$

By using Theorem 1 and the comment associated with (5), with $w(z) = \rho z$, we obtain $p_\rho(z) \prec q_\rho(z)$, for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1$ we obtain $p \prec q$. □

We next apply Theorem 1 to two important particular cases corresponding to $q(U)$ being a disk and $q(U)$ being a half-plane.

Case 1. The disk $\Delta = \{w : |w| < M\}$.

The function

$$q(z) = M\frac{Mz + a}{M + \overline{a}z}$$

with $M > 0$ and $|a| < M$, satisfies $\Delta = q(U) = U_M$, $q(0) = a$, $E(q) = \emptyset$ and $q \in Q$.

We first describe the class of admissible functions for this particular q, as given by Definition 4. We set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\Psi_n[M, a]$. Since $q(\zeta) = Me^{i\theta}$, with $\theta \in \mathbb{R}$, when $|\zeta| = 1$, the condition of admissibility (A) becomes

$$\psi(r, s, t; z, \xi) \notin \Omega, \text{ when } r = Me^{i\theta} \tag{A'_0}$$

$$s = m\frac{M|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2}e^{i\theta}$$

$$\text{Re } \frac{t}{s} + 1 \geq m\frac{|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2}$$

$z \in U$, $\xi \in \overline{U}$, $\theta \in \mathbb{R}$ and $m \geq n$.

If $a = 0$, then (A'_0) simplifies to

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z, \xi) \notin \Omega, \text{ when } K \geq nM, \tag{A''_0}$$

$$\text{Re } [Le^{-i\theta}] \geq (n-1)K,$$

$z \in U$, $\xi \in \overline{U}$ and $\theta \in \mathbb{R}$, a condition much easier to check.

In this particular case Theorem 1 becomes the following theorem.

Theorem 4 *Let $p \in \mathcal{H}[a, n]$.*

(i) If $\Psi \in \Psi_n[\Omega, M, a]$, then

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \in \Omega \;\Rightarrow\; |p(z)| < M$$

(ii) If $\psi \in \Psi_n[M, a]$, then

$$|\psi(p(z), zp'(z), z^2 p''(z); z, \xi)| < M \;\Rightarrow\; |p(z)| < M.$$

Case 2. The half-plane $\Delta = \{w : \operatorname{Re} w > 0\}$.

The function

$$q(z) = \frac{a + \overline{a}z}{1 - z}$$

with $\operatorname{Re} a > 0$, satisfies $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$ and $q \in Q$. We first describe the class of admissible functions, as defined in Definition 4 for this q. We set $\psi_n\{\Omega, a\} \equiv \psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$ we denote the class by $\psi_n\{a\}$. Since $\operatorname{Re} q(\zeta) = 0$ when $\zeta \in \partial U \setminus \{1\}$, the condition of admissibility (A) becomes

$$\psi(\rho i, \mu + \nu i; z, \xi) \notin \Omega, \text{ when } \rho, \sigma, \mu, \nu \in \mathbb{R} \qquad (A')_1$$

$$\sigma \leq -\frac{n}{2} \frac{|a - i\rho|^2}{\operatorname{Re} a}, \quad \sigma + \mu \leq 0$$

$z \in U$, $\xi \in \overline{U}$ and $n \geq 1$.

If $a = 1$ then (A'_1) implies

$$\psi(\rho i, \sigma, \mu + \nu i; z, \xi) \notin \Omega, \text{ when } \rho, \sigma, \mu, \nu \in \mathbb{R} \qquad (A''_1)$$

$$\sigma \leq -\frac{n}{2}[1 + \rho^2], \quad \sigma + \mu \leq 0,$$

$z \in U$, and $n \geq 1$.

In this particular case Theorem 1 becomes this:

Theorem 5 *Let $p \in \mathcal{H}[a, n]$.*

(i) If $\psi \in \Psi_n[\Omega, a]$, then

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \in \Omega \;\Rightarrow\; \operatorname{Re} p(z) > 0.$$

(ii) If $\psi \in \psi_n[a]$, then

$$\operatorname{Re}\,[\psi(p(z), zp'(z), z^2 p''(z); z, \xi)] > 0 \;\Rightarrow\; \operatorname{Re} p(z) > 0.$$

Example 1 *Let $\Omega = \Delta = \{w; \operatorname{Re} w > 0\}$.*

If $p \in \mathcal{H}[1, 1]$ and $0 < \operatorname{Re} B(\xi) \leq \dfrac{1}{2}$, then

$$zp'(z) + p(z) + B(\xi) \prec\prec \frac{1 + z}{1 - z}, \quad z \in U, \ \xi \in \overline{U}$$

implies

$$p(z) \prec \frac{1 + z}{1 - z}, \quad z \in U.$$

Proof.   Let

$$\psi(p(z), zp'(z); z, \xi) = zp'(z) + p(z) + B(\xi)$$

$$\operatorname{Re} \psi(p(z), zp'(z); z, \xi) > 0$$

$$\operatorname{Re} \psi(\rho i, \sigma; z, \xi) = \operatorname{Re} [\sigma + \rho i + B(\xi)] = \sigma + \operatorname{Re} B(\xi)$$

$$\leq -\frac{1}{2}(1 + \rho^2) + \operatorname{Re} B(\xi) \leq -\frac{1}{2}\rho^2 + \operatorname{Re} B(\xi) - \frac{1}{2} \leq 0 \ \Rightarrow \ \psi \in \psi_n\{1\}.$$

By using (ii) from Theorem 5 we have

$$p(z) \prec \frac{1 + z}{1 - z}.$$

$\square$

Example 2 $\Omega = \Delta = U_M$, $q(z) = Mz$, $q(U) = U_M$.

If $p \in \mathcal{H}[0, 1]$, and

$$|zp'(z) + p(z) + B(\xi)| \prec\prec 2M \ \Rightarrow \ |p(z)| < M.$$

Proof.   Let $\psi(p(z), zp'(z), z; \xi) = zp'(z) + p(z) + B(\xi)$

$$|\psi(Me^{i\theta}, Ke^{i\theta}, z; \xi)| = |Me^{i\theta} + Ke^{i\theta} + A(\xi)|$$

$$\geq |M + K| + |A(\xi)| \geq M + K + |A(\xi)| \geq M + M + |A(\xi)| \geq 2M \ \Rightarrow \ \psi \in \psi_n[M, 0].$$

By using (ii) from Theorem 4 we have

$$|p(z)| < M, \quad z \in U.$$

$\square$

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