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# C1 modules with respect to a hereditary torsion theory

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**Abstract**

An R-module M is said to be a C1-module if every closed submodule of M is a direct summand. In this paper we introduce and investigate the concept of the $\tau$-C1 module for a hereditary torsion theory $\tau$ on Mod-R. $\tau$-C1 modules are a generalization of C1-modules.

**Key word and phrases:** Torsion theory, C1-module, closed submodule.

## 1. Introduction

Throughout the paper R will denote an associative ring with identity, Mod-R will be the category of unitary right R-modules, and all modules and module homomorphisms will belong to Mod-R. If $\tau = (\Gamma, \digamma)$ is a torsion theory on Mod-R, then $\tau$ is uniquely determined by its associated torsion class $\Gamma$ of $\tau$-torsion modules. Modules in $\Gamma$ will be called $\tau$-torsion and modules in $\digamma$ will be called $\tau$-torsionfree modules. Also, for any module M, $\tau(M)$ denotes the sum of the $\tau$-torsion submodules of M and so $\tau(M)$ is the unique largest $\tau$-torsion submodule of M. For a torsion theory $\tau = (\Gamma, \digamma)$, $\Gamma \cap \digamma = 0$ and the torsion class $\Gamma$ is closed under homomorphic images, direct sums and extensions. In this paper $\tau$ is assumed to be hereditary, that is, we assume that submodules of $\tau$-torsion modules are $\tau$-torsion. (See [1] and [2] for more details ). An $R$-submodule K is called a $\tau$*-essential submodule* of the $R$-module M if $K \cap A \neq 0$ for all nonzero submodules A of M such that $M/A \in \Gamma$, denoted by $K \subseteq^{\tau-ess} M$. Then every essential submodule of M (see [3] and [4] for more details) is a $\tau$-essential submodule. This is a generalization of essential submodules and it is of interest to know how far the old theories extend to the new situation. The following example shows that there is an example of $\tau$-essential submodule but not essential submodule. And also if every nonzero submodule of $M \in$ Mod-R is $\tau$-essential, then M is called a $\tau$*-uniform module*. Moreover, modules satisfying condition C1 are also called CS modules or extending modules. In this respect, the paper [5] has been also considered C1 modules with respect to a torsion theory (and in particular $\tau-$essential submodules). However this paper's definitions are quite different.

**Example 1.1** *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$*, where* $F = \mathbb{Z}_2$*. The right nonzero R-submodules of R are* $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$,

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$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, $\{\begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} : a \in F\}$ *and R itself. Let* $\Gamma = \{A \in$ *Mod-R* $: AX = 0\}$, *where* $X = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. *Since* $\begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bigoplus \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *is not an essential submodule of R. But since* $R/\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \notin \Gamma$, $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *is a* $\tau$*-essential submodule of R. In fact R is a* $\tau$*-uniform module but not a uniform module.*

**Lemma 1.2** *Let* $K \subseteq S \subseteq M \in$ *Mod-R. The following are satisfied.*
*(1) If* $K \subseteq^{\tau-ess} M$ *and* $M/S \in \Gamma$, *then* $K \subseteq^{\tau-ess} S$.
*(2) If* $K \subseteq^{\tau-ess} M$, *then* $S \subseteq^{\tau-ess} M$.
*(3) If* $K \subseteq^{\tau-ess} S$ *and* $S \subseteq^{\tau-ess} M$, *then* $K \subseteq^{\tau-ess} M$.
*(4)If* $\alpha : M_1 \to M_2$ *is an epic R-linear morphism and* $K \subseteq^{\tau-ess} M_2$, *then* $\alpha^{-1}(K) \subseteq^{\tau-ess} M_1$.
*(5) Let* $M = \oplus_{i\in I} M_i$. *If* $K_i \subseteq^{\tau-ess} M_i$ *for all* $i \in I$, *then* $\oplus_{i\in I} K_i \subseteq^{\tau-ess} M$.
*(6) Let* $M = \oplus_{i\in I} M_i$ *and* $\oplus_{i\in I} K_i \subseteq^{\tau-ess} M$ *where* $K_i \subseteq M_i$ *for all* $i \in I$. *If* $M/M_i \in \Gamma$ *for some* $i \in I$, *then* $K_i \subseteq^{\tau-ess} M_i$.

**Proof.** (1),(2),(3) are routine verifications.
(4) Let $M_1/Y \in \Gamma$ and $\beta : M_1/Y \to M_2/\alpha(Y)$ be a function such that $\beta(a+Y) = \alpha(a)+\alpha(Y)$ for all $a \in M_1$. Then $\beta$ is an R-module epimorphism and so $M_2/\alpha(Y) \in \Gamma$. Then $K \cap \alpha(Y) \neq 0$, and so $\alpha^{-1}(K) \cap Y \neq 0$.
(5) Let $M/Y \in \Gamma$. If $M_i \cap Y = 0$ for every $i \in I$, then $M \in \Gamma$ and by Lemma 1.1(4) in [4] $\oplus_{i\in I} K_i \subseteq^{\tau-ess} M$. If not, there is at least one $i \in I$ such that $K_i \cap Y \neq 0$ and so $\oplus_{i\in I} K_i \cap Y \neq 0$.
(6) Let $M_i/S \in \Gamma$ where $S \neq 0$. Then $M/M_i \cong \frac{M/S}{M_i/S}$ implies $M/S \in \Gamma$. Since $\oplus_{i\in I} K_i \subseteq^{\tau-ess} M$, $\oplus_{i\in I} K_i \cap S \neq 0$ and so $K_i \cap S \neq 0$. □

A module U is called $\tau$*-essentially M injective* if every diagram in R-mod with exact row $0 \to K \to M$ and $g : K \to U$ and $Ker(g) \subseteq^{\tau-ess} K$ can be extended commutatively by some homomorphism $M \to U$.

**Lemma 1.3** *Let M and U be R-modules.*
*(1) Any product* $\Pi_\lambda U_\lambda$ *is* $\tau$*-essentially M injective if and only if every* $U_\lambda$ *is* $\tau$*-essentially M injective.*

*(2) If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of R-modules and U is* $\tau$*-essentially M injective then U is* $\tau$*-essentially* $M'$ *injective and* $\tau$*-essentially* $M''$ *injective.*

**Proof.** (1) Follow the proof of [8] 16.1. (2)Using Lemma 1.2(4) follow the proof of [3] 2.15. □

## 2. $\tau$-closed submodules

Let $M \in$Mod-R. A submodule A of M is called a $\tau$*-closed submodule* of M, if there is no submodule B of M such that $A \subset^{\tau-ess} B \subseteq M$. We denote this by $A \subset^{\tau-closed} M$. Note that if A is a $\tau$-closed submodule of M, then A is a closed submodule of M. But a closed submodule may not be a $\tau$-closed submodule by Example 1.1. And also if A is a $\tau$-closed submodule of M, then $A \subseteq \tau(M)$, i.e. A is $\tau$-torsion. Because there is a submodule H such that $A \cap H = 0$ and $M/H \in \Gamma$ and since $\tau$ is a hereditary torsion theory $A + H/H \in \Gamma$ and so $A \in \Gamma$.

**Lemma 2.1** *Let M be a module in Mod-R. Then the following are satisfied.*
*i) If A is $\tau$-closed submodule in M, then $A \subseteq K \subseteq^{\tau-ess} M$ implies $K/A \subset^{\tau-ess} M/A$. But the converse may not be true.*

*ii) If $L \subset^{\tau-closed} M$, then $L/K \subset^{\tau-closed} M/K$ for all submodules K of L. If $L/K \subset^{\tau-closed} M/K$ and $K \subset^{\tau-closed} M$, then $L \subset^{\tau-closed} M$.*

**Proof.** i) Let $A \subseteq K \subseteq^{\tau-ess} M$. Assume that there exists a nonzero submodule $S/A$ such that $\frac{M/A}{S/A} \in \Gamma$ and $K/A \cap S/A = 0$. Then $A = K \cap S \subseteq^{\tau-ess} S$ since $K \subseteq^{\tau-ess} M$ and $M/S \in \Gamma$. Since $A \subset^{\tau-closed} M$, $S = A$. But this is a contradiction. So $K/A \subset^{\tau-ess} M/A$. For the converse we can give the following counterexample:

Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F = \mathbb{Z}_2$. Let $\Gamma = \{A \in \text{Mod-R}: AX = 0\}$ where $X = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$.

Then $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \subset^{\tau-ess} \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \subset^{\tau-ess} \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and also $\dfrac{\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}} \subset^{\tau-ess} \dfrac{\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}}{\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}}$ but $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \not\subseteq^{\tau-closed} \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$.

ii) Let $L \subset^{\tau-closed} M$. Assume that there exists a submodule $S/K$ of $M/K$ such that $L/K \subset^{\tau-ess} S/K \subseteq M/K$. By Lemma 1.2 (4) $L \subset^{\tau-ess} S$ which is a contradiction.

Now let $L/K \subset^{\tau-closed} M/K$. Assume that there exists a submodule S of M such that $L \subset^{\tau-ess} S \subseteq M$. Since $K \subset^{\tau-closed} M$, $K \subset^{\tau-closed} S$ and by the part (i) we can write $L/K \subset^{\tau-ess} S/K$ which is a contradiction.

□

**Proposition 2.2** *If $A \subset^{\tau-closed} B \subset^{\tau-closed} M$, then $A \subset^{\tau-closed} M$.*

**Proof.** Assume that there is a submodule K such that $A \subset^{\tau-ess} K \subseteq M$. Since $A \subset^{\tau-closed} B$, $K \nsubseteq B$. Then $B + K \neq B$ and $B \subset B + K \subseteq M$. If $S \cap K \neq 0$ for all nonzero S such that $B + K/S \in \Gamma$, then

$A \subset^{\tau-ess} B+K$ since $A \subset^{\tau-ess} K$ and so $B \subset^{\tau-ess} B+K$. This is a contradiction. Then there is a nonzero submodule S of B+K such that $S \cap K = 0$ and $B+K/S \in \Gamma$. Then by $K+S/S \in \Gamma$, $K \in \Gamma$ and so $B+K \in \Gamma$ since $B \subset^{\tau-closed} M$ and so $B \in \Gamma$. Since $A \subset^{closed} B \subset^{closed} B+K$, we can write that $A \subset^{closed} B+K$, that is $A \subset^{\tau-closed} B+K$. But this contradicts $A \subset^{\tau-ess} K \subseteq B+K$. □

Remark: By Zorn's Lemma for every non $\tau$-essential submodule N of a module M, we can find a submodule A of M which is maximal with respect to the property that $N \subseteq^{\tau-ess} A$ and, in this case, A is a $\tau$-closed submodule of M. Therefore if $\tau(M) \nsubseteq^{\tau-ess} M$, then there is a submodule A of M such that $\tau(M) \subseteq^{\tau-ess} A \subset^{\tau-closed} M$ and we obtain $\tau(M) \subset^{\tau-closed} M$ since $A \in \Gamma$. That is, it is either $\tau(M) \subseteq^{\tau-ess} M$ or $\tau(M) \subset^{\tau-closed} M$.

Remark: Let $N \nsubseteq^{\tau-ess} M$. Then by Zorn's Lemma we can find that a maximal submodule A such that $M/A \in \Gamma$ and $A \cap N = 0$. Then $N \oplus A \subseteq^{\tau-ess} M$. If $M \notin \Gamma$, then also $A \subset^{\tau-ess} M$.

**Lemma 2.3** *Let $A \subset^{closed} M$ and $A \nsubseteq^{\tau-ess} M$. Then $A \subset^{\tau-closed} M$.*

**Proof.** Assume that $A \nsubseteq^{\tau-closed} M$. Then there is a submodule B such that $A \subset^{\tau-ess} B \subset^{\tau-closed} M$ since $A \nsubseteq^{\tau-ess} M$. Then $B \in \tau(M)$ and so $A \subset^{ess} B \subset^{\tau-closed} M$ and this is a contradiction. □

We call a module M a $\tau$-*C1 module* if every $\tau$-closed submodule of M is a direct summand of M.

**Lemma 2.4** *Let M be a $\tau$-torsionfree module. Then M is a $\tau$-C1 module.*

**Proof.** Since M is a $\tau$-torsionfree module, any $\tau$-closed submodule of M is zero. □

**Lemma 2.5** *Let M be a $\tau$-C1 module. Then every $\tau$-torsion direct summand of M is a $\tau$-C1 module.*

**Proof.** Let B be a $\tau$-torsion direct summand of M. First we prove $B \subset^{\tau-closed} M$. Assume that $B \subset^{\tau-ess} X \subseteq M$. Since there is a submodule $B' \subseteq M$ such that $M = B \oplus B'$, $X = B \oplus (X \cap B')$ and so $B \subset^{\tau-ess} B \oplus (X \cap B') \subseteq M$ where $X \cap B' \neq 0$. Then $\frac{B \oplus (X \cap B')}{X \cap B'} \in \Gamma$ and $X \cap B' \cap B = 0$, but this is a contradiction. If $A \subset^{\tau-closed} B$, then since $B \subset^{\tau-closed} M$, $A \subset^{\tau-closed} M$ by Proposition 2.2 . Since M is a $\tau$-C1 module, A is a direct summand of M and so A is a direct summand of B. □

**Lemma 2.6** *Let M be a $\tau$-C1 module and $\tau(M) \nsubseteq^{\tau-ess} M$. Then every direct summand of $M$ is also a $\tau$-C1 module.*

**Proof.** Since $\tau(M) \subset^{\tau-closed} M$ and M is a $\tau$-C1 module, $\tau(M)$ is a direct summand of M, it is denoted by $\tau(M) \subseteq^{\oplus} M$. Let $A \subset^{\tau-closed} M_1 \subseteq^{\oplus} M$ and $M = M_1 \oplus M_2$. Then $A \in \Gamma$.

i) If $\tau(M_2) = 0$, then $\tau(M_1) \subseteq^{\oplus} M$ since $\tau(M) = \tau(M_1) \oplus \tau(M_2)$. Then $A \subset^{\tau-closed} \tau(M_1) \subset^{\oplus} M$ and by Lemma 2.5 A is a direct summand of M and so a direct summand of $M_1$.

ii) If $\tau(M_2) \neq 0$, then $A \subset^{\tau-closed} \tau(M_1) \subset^{\tau-closed} \tau(M) \subseteq^{\oplus} M$ and so $A \subset^{\tau-closed} \tau(M) \subseteq^{\oplus} M$. By Lemma 2.5 A is a direct summand of M and so a direct summand of $M_1$. $\square$

**Example 2.7** *Let $\tau_G$ be the Goldie torsion theory. Let M be a $\tau_G$-C1 module. Then $\tau_G(M) = \mathbb{Z}_2(M)$ and if there is a torsionfree submodule N of M such that $\mathbb{Z}_2(M/N) = M/N$, then $\mathbb{Z}_2(M)$ is a $\tau$-closed submodule of M and every direct summand of M is a $\tau_G$-C1 module.*

**Lemma 2.8** *Let M be a $\tau$-C1 module and $M_1$ be a closed submodule of M such that $M/M_1 \in \Gamma$. Then $M_1$ is also a $\tau$-C1 module.*

**Proof.** Let $A \subset^{\tau-closed} M_1$. Therefore $A \subset^{closed} M_1 \subset^{closed} M$ and so $A \subset^{closed} M$. If $A \not\subset^{\tau-ess} M$, then $A \subset^{\tau-closed} M$ and so A is a direct summand of $M_1$.
If $A \subset^{\tau-ess} M$, then by Lemma 1.2(1) $A \subset^{\tau-ess} M_1$ since $M/M_1 \in \Gamma$. This is a contradiction. $\square$

By Lemma 2.5 and 2.8 every direct summand $M_1$ of a $\tau$-C1 module M such that $M_1 \in \Gamma$ or $M/M_1 \in \Gamma$ is also a $\tau$-C1 module. But we don't know whether or not any direct summand $M_1$ of a $\tau$-C1 module M with $\tau(M) \subset^{\tau-ess} M$ is also a $\tau$-C1 module.

**Example 2.9** *Every C1- module is a $\tau$-C1 module since every $\tau$-closed submodule is a closed submodule. But the converse may not hold.*

**Proof.** Let $R = \begin{pmatrix} F & V \\ 0 & F \end{pmatrix}$ where F is a field and $V = F \oplus F$. If we take $\Gamma = \{A \in$Mod-R: $A \begin{pmatrix} 0 & V \\ 0 & F \end{pmatrix} = 0\}$, then R is $\tau$-uniform and so R is a $\tau$-C1 module. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $eR = \begin{pmatrix} F & V \\ 0 & 0 \end{pmatrix}$ is indecomposable (in fact $eRe \cong F$). Assume that R is a C1-module. Then $eR$ is also a C1-module. Since $eR$ is indecomposable, it is a uniform module. But since $\begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix} = 0$, it cannot be a uniform module. (See [4] for more details). Thus R is a $\tau$-C1 module, but not a C1-module. $\square$

**Lemma 2.10** *If N is a $\tau$-closed submodule of a $\tau$-C1 module M, then $M/N$ is also a $\tau$-C1 module.*

**Proof.** Let $A/N \subseteq^{\tau-closed} M/N$. Then $A \subseteq^{\tau-closed} M$. Otherwise there is a submodule B such that $A \subset^{\tau-ess} B \subseteq M$ and by Lemma 2.1 $A/N \subseteq^{\tau-ess} B/N \subseteq M/N$. But this is a contradiction. Since $A \subseteq^{\tau-closed} M$ and M is a $\tau$-C1 module, A is a direct summand of M and so $A/N$ is a direct summand of $M/N$. $\square$

**Lemma 2.11** *If $\tau(M)$ is a C1-module and a direct summand of M, then M is a $\tau$-C1 module.*

**Proof.** Let $A \subset^{\tau-closed} M$. Then $A \in \tau(M)$ and $A \subset^{closed} \tau(M)$. Therefore A is a direct summand of $\tau(M)$ and A is a direct summand of M. □

Let $\tau soc(M) = \bigcap\{A : A \subseteq^{\tau-ess} M\}$. Then $\tau soc(M) \subseteq soc(M)$ and $\tau soc(M)$ is a direct summand of $soc(M)$ and so a semisimple submodule of M.

**Example 2.12** *We give examples such that*
*i)* $\tau soc(M) \subset soc(M)$
*ii) if* $A \subset B$*, then* $\tau soc(B) \subset \tau soc(A)$
*iii) if* $A = B \oplus C$*, then* $\tau soc(A) \neq \tau soc(B) \oplus \tau soc(C)$ *and* $B \bigcap \tau soc(A) \neq \tau soc(B)$.

**Proof.** i) Let R and $\Gamma$ be as in Example 1.1. Then
$\tau soc\left(\left(\begin{array}{cc} 0 & F \\ 0 & F \end{array}\right)\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)$, but $soc\left(\left(\begin{array}{cc} 0 & F \\ 0 & F \end{array}\right)\right) = \left(\begin{array}{cc} 0 & F \\ 0 & F \end{array}\right)$.

ii) Let $A = \left(\begin{array}{cc} 0 & 0 \\ 0 & F \end{array}\right) \subset B = \left(\begin{array}{cc} 0 & F \\ 0 & F \end{array}\right)$. Then

$$\tau soc(A) = A \supset \tau soc(B) = 0$$

iii) Let $A = \left(\begin{array}{cc} 0 & F \\ 0 & F \end{array}\right)$, $B = \left(\begin{array}{cc} 0 & F \\ 0 & 0 \end{array}\right)$ and $C = \left(\begin{array}{cc} 0 & 0 \\ 0 & F \end{array}\right)$. Then $A = B \oplus C$ but $\tau soc(A) \neq \tau soc(B) \oplus \tau soc(C)$ and $B \bigcap \tau soc(A) \neq \tau soc(B)$. □

Note that if there is a simple submodule $A$ of M such that $M/A \in \Gamma$, then $A \subseteq \tau soc(M)$. Also if there is a simple submodule A of M such that A is not in $\Gamma$, then $A \subseteq^{\tau-ess} M$ and therefore $\tau soc(M)$ is either A or $0$. Otherwise $soc(M) = soc(\tau(M))$ and $\tau soc(M) = \bigcap\{\tau(A) : A \subseteq^{\tau-ess} M\}$.

Note that if $mR \nsubseteq^{\tau-ess} M$ for all $m \in M$, then $mR \in \Gamma$ for all $m \in M$, and so $M \in \Gamma$. If $M \notin \Gamma$, then there is at least one $m \in M$ such that $mR \subseteq^{\tau-ess} M$ and hence $\tau soc(M) \subseteq mR$.

(C3-condition): A module M is said to *satisfy condition C3* if, whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M.

**Proposition 2.13** *If* $M = K \oplus N$ *is a* $\tau$*-C1 module satisfying condition C3 and* $N \in \Gamma$*, then K is an N-injective module.*

**Proof.** If $X \subseteq N$ and $\alpha : X \to K$ is R-linear, we must extend $\alpha$ to $N \to K$. Put $Y = \{x - \alpha(x) : x \in X\}$. Then $Y \cap K = 0$, so let $C \supseteq Y$ be a complement of K in M. Then C is a $\tau$-closed submodule of M. Otherwise there is a submodule A such that $C \subset^{\tau-ess} A \subseteq M$.Since $M/K \in \Gamma$ and so $(A+K)/K \cong A/(A \cap K) \in \Gamma$ and also C is a maximal submodule satisfying $C \cap K = 0$, $A \cap K \neq 0$ and so $A \cap K \cap C \neq 0$ since $C \subset^{\tau-ess} A$.

But this is a contradiction.
Thus C is a $\tau$-closed submodule in M. Since M is a $\tau$-C1 module, $C \subseteq^{\oplus} M$. Since M satisfies condition-C3, $C \oplus K$ is a direct summand of M, that is there is a submodule D such that $M = C \oplus K \oplus D$. Let $\pi : M \to K$ be a projection with $\mathrm{Ker}(\pi) = C \oplus D$. Then $Y \subseteq \mathrm{Ker}(\pi)$ and so $\pi(x) = \pi(\alpha(x)) = \alpha(x)$ for any $x \in X$. Thus the restriction of $\pi$ to N extends $\alpha$. □

**Proposition 2.14** *If $M = \tau(M) \oplus N$ is a $\tau$-C1 module , then N is a $\tau(M)$-injective module.*

**Proof.** Let $\varphi : X \to N$ be a module homomorphism such that $0 \neq X \subseteq \tau(M)$. Take $X' = \{x - \varphi(x) : x \in X\}$. Since $M/N \in \Gamma$ and $N \cap X' = 0$, $X' \nsubseteq^{\tau-ess} M$. Therefore there is a submodule K of M such that $X' \subseteq^{\tau-ess} K \subset^{\tau-closed} M$, and then $M = K \oplus K'$ for some submodule $K'$. Let $\pi : K \oplus \tau(K') \oplus N \to N$ be the projection. Then the restriction of $\pi$ to X extends $\varphi$ since $\tau(M) = \tau(K) \oplus \tau(K')$ and $K \subseteq \tau(M)$, $\tau(M) = K \oplus \tau(K')$ . □

We know that it is not necessary that the direct sum of two C1 modules is a C1 module by the $\mathbb{Z}-$module example $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. So under arbitrary hereditary torsion theory we can say that it is not necessary that the direct sum of two $\tau$-C1 modules is a $\tau$-C1 module. Now we investigate when this case is possible.

**Lemma 2.15** *Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are both $\tau$-C1 modules and $M_2 \in \Gamma$. Then M is a $\tau$-C1 module if and only if every $\tau$-closed submodule K of M with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.*

**Proof.** The necessity is clear. Conversely, let $K \subset^{\tau-closed} M$ and $K \cap M_2 \neq 0$. Then there is a submodule H such that $K \cap M_2 \subseteq H \subset^{\tau-closed} K$ since $K \cap M_2 \nsubseteq^{\tau-ess} M$. By Proposition 2.2 $H \subset^{\tau-closed} M$. Clearly $H \cap M_1 = 0$ since $H + M_1/M_1 \in \Gamma$. By hypothesis $M = H \oplus H'$ for some submodule $H'$ of M, and so $K = H \oplus (K \cap H')$. Since $K \in \Gamma$, $K \cap H' \subset^{\tau-closed} K \subset^{\tau-closed} M$ and hence $K \cap H' \subset^{\tau-closed} M$. Also $K \cap H' \cap M_2 = 0$ and by hypothesis $K \cap H'$ is a direct summand of M and hence also of $H'$. It follows that K is a direct summand of M. Thus M is a $\tau$-C1 module. □

**Proposition 2.16** *Let $M = M_1 \oplus M_2$ and $M_1$ and $M_2$ be relatively injective modules and $M_2 \in \Gamma$. Then M is a $\tau$-C1 module if and only if $M_1$ and $M_2$ are $\tau$-C1 modules.*

**Proof.** The necessity is clear by Lemma 2.5 and Lemma 2.8. Let $K \subset^{\tau-closed} M$ and $K \cap M_1 = 0$. By [6] there exists a submodule $M_1'$ such that $M = M_1 \oplus M_1'$ and $K \subseteq M_1'$. Then $M_1' \cong M_2$ and so $M_1'$ is also a $\tau$-C1 module. Therefore K is a direct summand of $M_1'$ and so of M since $K \subset^{\tau-closed} M_1'$. Similarly if $K \cap M_2 = 0$, then K is a direct summand of M. By Lemma 2.15 M is a $\tau$-C1 module. □

**Proposition 2.17** *Let M be a module containing a $\tau$-essential submodule of the form $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ where each $U_i$ is a $\tau$-uniform submodule of M. If N is a submodule of M with $N \cap U_i \neq 0$ for every $i = 1, \cdots, n$, then*

$N \subseteq^{\tau-ess} M$. *Also any direct sum of non-zero submodules of M which is not* $\tau$*-essential in M has at most n summands.*

**Proof.** Let $\frac{U_1\oplus U_2\oplus\cdots\oplus U_n}{K} \in \Gamma$. If $U_i \cap K = 0$ for every $i = 1, \cdots, n$, then $U_1 \oplus U_2 \oplus \cdots \oplus U_n \in \Gamma$. By the same proof of 5.6 in [3] we can obtain that $N \subseteq^{\tau-ess} M$. Otherwise there is at least one $U_i$ with $U_i \cap K \neq 0$. Thus $N \cap U_i \cap K \neq 0$ since $U_i$ is $\tau$-uniform and hence $N \cap (U_1 \oplus U_2 \oplus \cdots \oplus U_n) \subseteq^{\tau-ess} U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and the proof is completed by Lemma 1.2(3) . The last part of the lemma can be seen using 5.7 in [3]. □

We say that a module M a *$\tau\pi$-injective module* if there exist submodules $A_1 \supseteq A$ and $A_2 \supseteq B$ such that $M = A_1 \oplus A_2$ whenever there exist submodules A and B such that $A \cap B = 0$ and $M/A \in \Gamma$.

**Proposition 2.18** *Let M be a* $\tau$*-C1 module such that whenever* $M = M_1 \oplus M_2$ *then* $M_1$ *and* $M_2$ *are relatively injective. Then M is* $\tau\pi$*-injective .*

**Proof.** Let $L_1$ and $L_2$ be submodules of M such that $L_1 \cap L_2 = 0$ and $M/L_2 \in \Gamma$. There exists a submodule A such that $L_1 \subseteq^{\tau-ess} A \subset^{\tau-closed} M$ since $M/L_2 \in \Gamma$, so there exists a submodule B such that $M = A \oplus B$. And also $A \cap L_2 = 0$. Otherwise $A \cap L_2 \neq 0$ and so $L_1 \cap A \cap L_2 \neq 0$ since $L_1 \subseteq^{\tau-ess} A$. But this is a contradiction. By [3] 7.5 there exists a submodule $B'$ such that $M = A \oplus B'$ such that $L_2 \subseteq B'$. □

We say that a R-module M is a *$\tau$-nonsingular module* if $\alpha = 0$ whenever $r_M(\alpha) = \{m \in M : \alpha(m) = 0\} \subseteq^{\tau-ess} M$ where $\alpha \in End_R(M)$ .

**Lemma 2.19** *Let M be a* $\tau$*-nonsingular module. If* $X \subseteq^{\tau-ess} N \subset^{\oplus} M$*, then N is unique.*

**Proof.** Assume $X \subseteq^{\tau-ess} N_1 \subset^{\oplus} M$ and $X \subseteq^{\tau-ess} N_2 \subset^{\oplus} M$ such that $M = N_1 \oplus M_1 = N_2 \oplus M_2$ and $N_1 \neq N_2$. Then there exists either $x \in N_1 - N_2$ or $x \in N_2 - N_1$. Assume that $x \in N_1 - N_2$. Take $\alpha = \pi_{M_2}(\pi_{N_1})$. Then $\alpha \neq 0$. Now we show that $\mathrm{Ker}(\alpha) \subseteq^{\tau-ess} M$. Let $M/K \in \Gamma$. Then $(N_1 + K)/K \in \Gamma$. If $N_1 \cap K = 0$, then $N_1 \in \Gamma$ and the proof is the same as [7] Proposition 2.27. Otherwise $N_1 \cap K \neq 0$ and there exists $k \in N_1 \cap K \cap X$. Therefore $\alpha(k) = 0$. Therefore $k \in K \cap \mathrm{Ker}(\alpha)$. Then $\mathrm{Ker}(\alpha) \subseteq^{\tau-ess} M$ and $\alpha \neq 0$. This is a contradiction. □

An R-module M is called *a Baer module* if for all submodules N of M we have $l_S(N) = \{f \in S : f(n) = 0$ for all n$\in N\} = Se$ where $End_R(M) = S$ and $e^2 = e$. (See [7] for more details). If also $l_S(N) = Se$ and $eM \in \Gamma$ for all submodules N of M , then a Baer module M is called *a* $\tau$*-Baer module* .

**Lemma 2.20** *A* $\tau$*-Baer module M is a* $\tau$*-nonsingular module.*

**Proof.** This is trivial. □

**Lemma 2.21** *Every* $\tau$*-nonsingular* $\tau$*-C1 module is a Baer module.*

**Proof.** This is proved in the same way as [7] Lemma 2.14 using the property of a $\tau$-C1 module and particularly Lemma 1.2(6). □

An R-module M is called *a $\tau$-cononsingular module* if $N \subseteq^{\tau-ess} M$ whenever $l_S(N) = 0$ for all submodule N of M where $S = End_R(M)$. (See [7] for more details).

**Lemma 2.22** *Let M be a $\tau$-cononsingular and $\tau$-Baer module. Then M is a $\tau$-C1 module.*

**Proof.** Let $0 \neq N \subset^{\tau-closed} M$. Since M is a $\tau$-Baer module there exists an idempotent $e \in S$ such that $l_S(N) = Se$ and $eM \in \Gamma$. Since $l_S(N) = Se$ we can write that $N \subseteq (1-e)M$. Assume that $N \neq (1-e)M$. Since $N \subset^{\tau-closed} (1-e)M$ there exists a submodule K such that $(1-e)M/K \in \Gamma$ and $N \cap K = 0$. Also we can find a submodule $N_1 \supseteq N$ maximal with respect to the property of having zero intersection with K. By $eM \in \Gamma$, $M/K \in \Gamma$ and hence $N_1 \nsubseteq^{\tau-ess} M$ and by the $\tau$-cononsingularity of M there exists an $0 \neq \alpha \in S$ such that $\alpha(N_1) = 0$ and hence $\alpha(N_1 \oplus K) = 0$ and $N_1 \oplus K \subseteq^{ess} M$. Since $M$ is also a Baer module and so a nonsingular module by [7] and hence $\alpha = 0$. But this is a contradiction. □

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