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Uniqueness for meromorphic functions and differential polynomials

Chao Meng

Abstract

In this article, we deal with the uniqueness problems on meromorphic functions concerning differential polynomials and prove the following result: Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha) = o(T(r,f) + T(r,g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that m, n are positive integers such that $n > m+10$. If Ψ'_f and Ψ'_g share "$(0,2)$", then (i) if $m \geq 2$, then $f(z) \equiv g(z)$; (ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$. The results in this paper improve the results of Xiong-Lin-Mori [14] and the author [12].

Key word and phrases: Uniqueness; meromorphic function; differential polynomials.

1. Introduction, definitions and results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in [5]. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM. Let m be a positive integer or infinity and $a \in C \cup \{\infty\}$. We denote by $E_{m)}(a,f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than m. Let $N_E(r,a;f,g)(\overline{N}_E(r,a;f,g))$ be the counting function(reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)(\overline{N}_0(r,a;f,g))$ be the counting function(reduced counting function) of all common zeros of $f-a$ and $g-a$, ignoring multiplicities. If

$$\overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,\frac{1}{g-a}) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

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then we say that f and g share a " CM ". On the other hand, if

$$\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g)\,,$$

then we say that f and g share a "IM". We now explain in the following definition the notion of weighted sharing which was introduced by I. Lahiri [8] or [9].

Definition 1 [8]. For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in C \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f-a$ with multiplicity $m(> k)$ if and only if it is a zero of $g-a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 2 [8]. Let p be a positive integer and $a \in C \cup \{\infty\}$. We denote by $N_{p)}(r, \frac{1}{f-a})$ the counting function of the zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not greater than p, $N_{(p+1}(r, \frac{1}{f-a})$ to denote the counting function of zeros of $f-a$ whose multiplicities are not less than $p+1$. And $\overline{N}_{p)}(r, \frac{1}{f-a})$, $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ denote their corresponding reduced counting functions (ignoring multiplicities), respectively.

W.K. Hayman proposed the following well-known conjecture in [6].

Hayman's Conjecture. If an entire function f satisfies $f^n f' \neq 1$ for all positive integers $n \in N$, then f is a constant.

It has been verified by Hayman himself in [7] for the case $n > 1$ and Clunie in [3] for the case $n \geq 1$, respectively.

It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. In 1997, corresponding to the famous conjecture of Hayman, Yang and Hua studied the unicity of differential monomials and obtained the following theorem.

Theorem A [16]. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be an integer and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some (n+1)-th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.*

In 2001, Fang and Hong studied the unicity of differential polynomials of the form $f^n(f-1)f'$ and proved the following uniqueness theorem.

Theorem B [4]. *Let f and g be two transcendental entire functions, $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi extended the above theorem in view of the fixed-point. They proved the following results.

Theorem C [11]. *Let f and g be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then either $f(z) \equiv g(z)$ or*

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$$

where h is a nonconstant meromorphic function.

In 2005, Xiong, Lin and Mori considered the function $\Psi_f = f^{n+1}(f^m + a) + \alpha$, where a is a constant and f, g, α are meromorphic functions. They improved Theorem C and obtained the following result.

Theorem D [14]. *Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha) = o(T(r,f) + T(r,g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that m, n, k are positive integers such that $(k-1)n > 14 + 3m + k(10+m)$. If $E_{k)}(\Psi'_f) = E_{k)}(\Psi'_g)$, then (i) if $m \geq 2$, then $f(z) \equiv g(z)$; (ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.*

In 2007, Shen and Lin improved Theorem D by deriving the following.

Theorem E [13]. *Suppose the condition $(k-1)n > 14 + 3m + k(10+m)$ is placed by $n > 6 + (4+m)k$ in Theorem D, then the conclusion remains valid.*

Recently, the author improved Theorem D by the idea of weighted shared value and obtained the following theorem.

Theorem F [12]. *Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha) = o(T(r,f) + T(r,g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that m, n are positive integers such that $n > m + 10$. If Ψ'_f and Ψ'_g share $(0,2)$, then*

(i) if $m \geq 2$, then $f(z) \equiv g(z)$;

(ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.

Now one may ask the following question which is the motivation for this paper: In Theorem F, can the nature of sharing value be further relaxed other than the concept of weighted sharing?

To answer the above question, we need the following notion of weakly weighted sharing introduced by Lin and Lin [10].

Definition 3 [10]. *Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}^{E}_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k. $\overline{N}^{O}_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.*

Definition 4 [10]. *For $a \in C \cup \{\infty\}$, if k is a positive integer or ∞ and*

$$\overline{N}_{k)}(r, \frac{1}{f-a}) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{k)}(r, \frac{1}{g-a}) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, g),$$

$$\overline{N}_{(k+1}(r, \frac{1}{f-a}) - \overline{N}^{O}_{(k+1}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{(k+1}(r, \frac{1}{g-a}) - \overline{N}^{O}_{(k+1}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}(r, \frac{1}{f-a}) - \overline{N}_0(r, a; f, g) = S(r, f), \overline{N}(r, \frac{1}{g-a}) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f and g weakly share a with weight k. Here, we write f, g share "(a, k)" to mean that f, g weakly share a with weight k.

Now it is clear from Definition 1 and Definition 4 that weighted sharing and weakly weighted sharing are respectively scalings between IM, CM and "IM", "CM". Also weakly weighted sharing includes the definition of weighted sharing.

In this paper, we prove the following theorem which improves Theorem F by the notion of weakly weighted sharing.

Theorem 1. *Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r, \alpha) = o(T(r, f) + T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that m, n are positive integers such that $n > m + 10$. If Ψ'_f and Ψ'_g share "$(0, 2)$", then*

(i) if $m \geq 2$, then $f(z) \equiv g(z)$;

(ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.

Recently, A. Banerjee [2] introduced another sharing notion which is also a scaling between "IM" and "CM" but weaker than weakly weighted sharing.

Definition 5 [2]. *We denote by $\overline{N}(r, a; f| = p; g| = q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.*

Definition 6 [2]. *Let f, g share a "IM". Also let k be a positive integer or ∞ and $a \in C \cup \{\infty\}$. If*

$$\sum_{p, q \leq k (p \neq q)} \overline{N}(r, a; f| = p; g| = q) = S(r),$$

then we say f and g share a with weight k in a relaxed manner. Here we write f and g share $(a, k)^*$ to mean that f and g share a with weight k in a relaxed manner.

With the notion of weight in a relaxed manner, we prove the following theorem which also improves Theorem F.

Theorem 2. *Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha)=o(T(r,f)+T(r,g))$ and $\alpha\not\equiv 0,\infty$. Let a be a nonzero constant. Suppose that m,n are positive integers such that $n>2m+14$. If Ψ'_f and Ψ'_g share $(0,2)^*$, then*

(i) if $m\geq 2$, then $f(z)\equiv g(z)$;

(ii) if $m=1$, either $f(z)\equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\varpi_1,\varpi_2)=(n+1)(\varpi_1^{n+2}-\varpi_2^{n+2})-(n+2)(\varpi_1^{n+1}-\varpi_2^{n+1})$.

Without the notions of weakly weighted sharing and weight in a relaxed manner, we prove the following theorem which also improves Theorem F.

Theorem 3. *Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha)=o(T(r,f)+T(r,g))$ and $\alpha\not\equiv 0,\infty$. Let a be a nonzero constant. Suppose that m,n are positive integers such that $n>m+10$. If $\overline{E}_{4)}(0,\Psi'_f)=\overline{E}_{4)}(0,\Psi'_g)$ and $E_{2)}(0,\Psi'_f)=E_{2)}(0,\Psi'_g)$ then*

(i) if $m\geq 2$, then $f(z)\equiv g(z)$;

(ii) if $m=1$, either $f(z)\equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\varpi_1,\varpi_2)=(n+1)(\varpi_1^{n+2}-\varpi_2^{n+2})-(n+2)(\varpi_1^{n+1}-\varpi_2^{n+1})$.

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H=\left(\frac{F''}{F'}-2\frac{F'}{F-1}\right)-\left(\frac{G''}{G'}-2\frac{G'}{G-1}\right).$$

Lemma 1 [15]. *Let f be a nonconstant meromorphic function, and let $a_1,a_2,...,a_n$ be finite complex numbers, $a_n\neq 0$. Then*

$$T(r,a_nf^n+\cdots+a_2f^2+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2 [2]. *Let H be defined as above. If F and G share "$(1,2)$" and $H\not\equiv 0$, then*

$$T(r,F)\leq N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)-\sum_{p=3}^{\infty}\overline{N}_{(p}(r,\frac{G}{G'})+S(r,F)+S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 3 [2]. *Let H be defined as above. If F and G share $(1,2)^*$ and $H \not\equiv 0$, then*

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F)$$
$$-m(r,\frac{1}{G-1}) + S(r,F) + S(r,G)\,,$$

the same inequality holds for $T(r,G)$.

Lemma 4 [1]. *Let H be defined as above. If $\overline{E}_{4)}(1,F) = \overline{E}_{4)}(1,G)$ and $E_{2)}(1,F) = E_{2)}(1,G)$, and $H \not\equiv 0$, then*

$$T(r,F) + T(r,G) \le 2\{N_2(r,\frac{1}{F}) + N_2(r,F) + N_2(r,\frac{1}{G}) + N_2(r,G)\} + S(r,F) + S(r,G)\,.$$

Lemma 5 [13]. *Let f, g, Ψ_f, Ψ_g and α be defined as in Theorem 1. If Ψ'_f and Ψ'_g share 0 CM and $n > 10+m$, then (i) if $m \ge 2$, then $f(z) \equiv g(z)$; (ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\varpi_1,\varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.*

Lemma 6 [17]. *Let f be a nonconstant meromorphic function. Then*

$$N(r,\frac{1}{f^{(k)}}) \le N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f)\,.$$

3. Proof of Theorem 1

Let

$$F = \frac{f^n(f^m+a_1)f'}{\alpha_1}\,, \qquad G = \frac{g^n(g^m+a_1)g'}{\alpha_1}\,, \tag{1}$$

$$F_1 = \frac{1}{n+m+1}f^{n+m+1} + \frac{a_1}{n+1}f^{n+1}\,, \qquad G_1 = \frac{1}{n+m+1}g^{n+m+1} + \frac{a_1}{n+1}g^{n+1}\,, \tag{2}$$

where

$$a_1 = \frac{(n+1)a}{n+m+1}\,, \qquad \alpha_1 = \frac{-\alpha'}{n+m+1}\,. \tag{3}$$

Then F and G share "$(1,2)$". By Lemma 1, we have

$$T(r,F_1) = (n+m+1)T(r,f) + S(r,f)\,, \qquad T(r,G_1) = (n+m+1)T(r,g) + S(r,g)\,. \tag{4}$$

Since $F_1' = \alpha_1 F$, $G_1' = \alpha_1 G$, we deduce that

$$T(r,F_1) + T(r,G_1) \le T(r,F) + T(r,G) + N(r,\frac{1}{F_1}) + N(r,\frac{1}{G_1})$$
$$-N(r,\frac{1}{F}) - N(r,\frac{1}{G}) + S(r,f) + S(r,g)\,. \tag{5}$$

From (1) we get

$$N_2(r,F)+N_2(r,G)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})\le 2\overline{N}(r,f)+2\overline{N}(r,g)+2N(r,\frac{1}{f})$$
$$+2N(r,\frac{1}{g})+N(r,\frac{1}{f^m+a_1})+N(r,\frac{1}{g^m+a_1})+N(r,\frac{1}{f'})$$
$$+N(r,\frac{1}{g'})+S(r,f)+S(r,g)\,. \tag{6}$$

Let $H\not\equiv 0$, thus by Lemma 2, we have

$$T(r,F)+T(r,G)\le 2\{N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)\}+S(r,f)+S(r,g)\,. \tag{7}$$

By (5), (6), (7), we obtain

$$T(r,F_1)+T(r,G_1)\le 4\{\overline{N}(r,f)+\overline{N}(r,g)\}+5\{N(r,\frac{1}{f})+N(r,\frac{1}{g})\}+\{N(r,\frac{1}{f^m+a_1})$$
$$+N(r,\frac{1}{g^m+a_1})\}+\{N(r,\frac{1}{f'})+N(r,\frac{1}{g'})\}+\{N(r,\frac{1}{f^m+a})+N(r,\frac{1}{g^m+a})\}$$
$$+S(r,f)+S(r,g)\,. \tag{8}$$

Thus we have from (8) and Lemma 6

$$(n+m+1)\{T(r,f)+T(r,g)\}\le(2m+11)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\,. \tag{9}$$

Thus we can deduce that $n\le m+10$, which contradicts with $n>m+10$. Therefore $H\equiv 0$. That is,

$$\frac{F''}{F'}-2\frac{F'}{F-1}\equiv\frac{G''}{G'}-2\frac{G'}{G-1}\,. \tag{10}$$

By integration, we have from (10)

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B\,. \tag{11}$$

where $A\ne 0$ and B are constants. Thus F and G share 1 CM, and hence, we obtain from (1) that Ψ_f' and Ψ_g' share 0 CM. Therefore by Lemma 5, we get the conclusion. □

4. Proof of Theorem 2

Let

$$F=\frac{f^n(f^m+a_1)f'}{\alpha_1}\,,\qquad G=\frac{g^n(g^m+a_1)g'}{\alpha_1}\,, \tag{12}$$

$$F_1=\frac{1}{n+m+1}f^{n+m+1}+\frac{a_1}{n+1}f^{n+1}\,,\qquad G_1=\frac{1}{n+m+1}g^{n+m+1}+\frac{a_1}{n+1}g^{n+1}\,, \tag{13}$$

where

$$a_1 = \frac{(n+1)a}{n+m+1}, \qquad \alpha_1 = \frac{-\alpha'}{n+m+1}. \tag{14}$$

Then F and G share $(1,2)^*$. Similar to that which proceeded proof of Theorem 1, we also have (4), (5) and (6). Let $H \not\equiv 0$. Thus by Lemma 3, we have

$$T(r,F)+T(r,G) \le 2\{N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)\}+\overline{N}(r,\frac{1}{F}) \\ +\overline{N}(r,F)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,G)+S(r,f)+S(r,g). \tag{15}$$

By (15) and (1), (5), (6), we have that

$$T(r,F_1)+T(r,G_1) \le 5\{\overline{N}(r,f)+\overline{N}(r,g)\}+6\{N(r,\frac{1}{f})+N(r,\frac{1}{g})\}+2\{N(r,\frac{1}{f^m+a_1}) \\ +N(r,\frac{1}{g^m+a_1})\}+2\{N(r,\frac{1}{f'})+N(r,\frac{1}{g'})\}+\{N(r,\frac{1}{f^m+a})+N(r,\frac{1}{g^m+a})\} \\ +S(r,f)+S(r,g). \tag{16}$$

Thus we have from (16) and Lemma 6

$$(n+m+1)\{T(r,f)+T(r,g)\} \le (3m+15)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \tag{17}$$

Thus we can deduce that $n \le 2m+14$, which contradicts with $n > 2m+14$. Therefore $H \equiv 0$. That is

$$\frac{F''}{F'} - 2\frac{F'}{F-1} \equiv \frac{G''}{G'} - 2\frac{G'}{G-1}. \tag{18}$$

By integration, we have from (18)

$$\frac{1}{G-1} \equiv \frac{A}{F-1} + B. \tag{19}$$

where $A \ne 0$ and B are constants. Thus F and G share 1 CM, and hence, we obtain from (1) that Ψ'_f and Ψ'_g share 0 CM. Therefore by Lemma 5, we get the conclusion.

5. Proof of Theorem 3

Let

$$F = \frac{f^n(f^m+a_1)f'}{\alpha_1}, \qquad G = \frac{g^n(g^m+a_1)g'}{\alpha_1}, \tag{20}$$

$$F_1 = \frac{1}{n+m+1}f^{n+m+1} + \frac{a_1}{n+1}f^{n+1}, \qquad G_1 = \frac{1}{n+m+1}g^{n+m+1} + \frac{a_1}{n+1}g^{n+1}, \tag{21}$$

where

$$a_1 = \frac{(n+1)a}{n+m+1}, \qquad \alpha_1 = \frac{-\alpha'}{n+m+1}. \tag{22}$$

Then $\overline{E}_{4)}(1,F) = \overline{E}_{4)}(1,G)$ and $E_{2)}(1,F) = E_{2)}(1,G)$. Similar with the proceeding of the proof of Theorem 1, we also have (4), (5) and (6). Let $H \not\equiv 0$. Thus by Lemma 4, we have

$$T(r,F) + T(r,G) \le 2\{N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G)\} + S(r,f) + S(r,g). \tag{23}$$

By (23) and (1), (5), (6), we have that

$$\begin{aligned} T(r,F_1) + T(r,G_1) &\le 4\{\overline{N}(r,f) + \overline{N}(r,g)\} + 5\{N(r,\frac{1}{f}) + N(r,\frac{1}{g})\} + \{N(r,\frac{1}{f^m+a_1}) \\ &\quad + N(r,\frac{1}{g^m+a_1})\} + \{N(r,\frac{1}{f'}) + N(r,\frac{1}{g'})\} + \{N(r,\frac{1}{f^m+a}) + N(r,\frac{1}{g^m+a})\} \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{24}$$

Thus we have from (24) and Lemma 6

$$(n+m+1)\{T(r,f) + T(r,g)\} \le (2m+11)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{25}$$

Thus we can deduce that $n \le m+10$, which contradicts with $n > m+10$. Therefore $H \equiv 0$. That is

$$\frac{F''}{F'} - 2\frac{F'}{F-1} \equiv \frac{G''}{G'} - 2\frac{G'}{G-1}. \tag{26}$$

By integration, we have from (26)

$$\frac{1}{G-1} \equiv \frac{A}{F-1} + B. \tag{27}$$

where $A \neq 0$ and B are constants. Thus F and G share 1 CM, and hence, we obtain from (1) that Ψ'_f and Ψ'_g share 0 CM. Therefore by Lemma 5, we get the conclusion. □

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Chao MENG
Department of Mathematics, Shandong University,
Jinan 250100, People's Republic of CHINA
e-mail: mengchao-syiae@sohu.com

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