Turk J Math
33 (2009) , 341 – 350.

doi:10.3906/mat-0711-23

# Oscillation of nonlinear neutral delay differential equations of second-order with positive and negative coefficients

*Mustafa Kemal Yıldız, Başak Karpuz, Özkan Öcalan*

**Abstract**

Some oscillation criteria for the following second-order neutral differential equation

$$\big[x(t) \pm r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t)$$

where $t \geq t_0$, $\gamma, \alpha, \beta \in \mathbb{R}^+$ with $\alpha \geq \beta$, $r \in C^2([t_0,\infty),\mathbb{R}^+)$, $p, q \in C([t_0,\infty),\mathbb{R}^+)$ and $f, g \in C(\mathbb{R},\mathbb{R})$, $s \in C([t_0,\infty),\mathbb{R})$ have been obtained. Our results are not restricted with boundedness of solutions.

**Key word and phrases:** Delay differential equations, neutral, nonlinear, oscillation, second-order.

## 1. Introduction

In this paper, we consider the oscillation of the second-order nonlinear neutral delay differential equations of the form

$$\big[x(t) + r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{1}$$

$$\big[x(t) - r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{2}$$

where $t \geq t_0$, $\gamma \geq 0$, $\alpha \geq \beta \geq 0$, $r \in C^2([t_0,\infty),\mathbb{R}^+)$ and $p, q \in C([t_0,\infty),\mathbb{R}^+)$. Furthermore, we suppose that the following are satisfied:

(H1) $\liminf_{t\to\infty} h(t) > 0$, where $h(t) := p(t) - q(t-\alpha+\beta)$ for $t \geq t_0$.

(H2) $f \in C(\mathbb{R},\mathbb{R})$ is nondecreasing with $f(u)/u > 0$ for $u \neq 0$ and there exists positive constant $M$ such that

$$0 < \frac{f(u)}{u} \leq M, \quad u \neq 0$$

holds.

---

2000 *AMS Mathematics Subject Classification:* 34C10, 34C15, 34K40.

(H3) $g \in C(\mathbb{R}, \mathbb{R})$ with $g(u)/u > 0$ for $u \neq 0$ and there exists positive constants $N_1$ and $N_2$ such that

$$N_1 \leq \frac{g(u)}{u} \leq N_2, \quad u \neq 0$$

holds.

(H4) $s \in C([t_0, \infty), \mathbb{R})$ and there exists a function $S \in C^2([t_0, \infty), \mathbb{R})$ such that $S'' = s$ and $S(t) \to 0$ as $t \to \infty$.

Note that if $S^* \in C^2([t_0, \infty), \mathbb{R})$ is a function satisfying $S^{*\prime\prime} = s$ and $L := \lim_{t\to\infty} S^*(t)$ exists and is finite, then $S := S^* - L$ holds (H4).

For the case $f$ and $g$ are identity functions, we obtain better results than those in [3]. Also in this case our results weaken assumptions on the coefficients. For the first-order case, see the results in [4]. Our results improve results in the literature. We refer readers to [1, 2, 5, 6, 7] for further results.

We restrict our attention only to those solutions $x$ that are not eventually trivial. By a *solution*, we mean a function $x$ identically satisfying the equation and $[x(t) - r(t)f(x(t-\gamma))] \in C^2([t_0, \infty), \mathbb{R})$ for all $t \geq t_0$. A solution is called *nonoscillatory* if it is eventually of single sign; otherwise, the solution is called *oscillatory.* Throughout the paper, we let $\kappa := \max\{\gamma, \alpha\}$.

## 2. Oscillatory behavior of solutions of homogenous equations

We start this section by giving the following sufficient condition on (1).

**Theorem 1** *Assume that (H1)–(H3) hold and $r \in C([t_0, \infty), \mathbb{R}^+)$ is bounded. If*

$$\int_{t_0}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu < \infty \tag{3}$$

*holds, then every solution of (1) is oscillatory.*

**Proof.** Suppose that $x$ is an eventually positive solution of (1). The case where $x$ is eventually negative is similar and is omitted. Let $t_1 \geq t_0$ such that $x(t-\kappa) > 0$ for $t \geq t_1$. Then, considering (3) there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu \leq \frac{1}{2N_2} \tag{4}$$

holds. Now, we set

$$w(t) := x(t) + r(t)f(x(t-\gamma)) \geq 0 \tag{5}$$

and

$$z(t) := w(t) - \int_{t_2}^{t} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \tag{6}$$

for $t \geq t_2$. Then, we have

$$\begin{aligned} z''(t) &= w''(t) - q(t)g(x(t-\beta)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -p(t)g(x(t-\alpha)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -h(t)g(x(t-\alpha)) \leq 0 \end{aligned} \tag{7}$$

for all $t \geq t_2$. Hence, $z'(t)$ and $z(t)$ is strictly monotonic and constant of sign for all $t \geq t_3$, where $t_3 \geq t_2$ is sufficiently large. To prove $z'(t) > 0$ holds for all $t \geq t_2$, we assume contrary that $z'(t) < 0$ holds for all $t \geq t_2$. In the present case, we see that

$$\lim_{t\to\infty} z(t) = -\infty. \tag{8}$$

We also claim that $x$ is bounded. For contrary assume $x$ is unbounded. Thus, there is $t_4 \geq t_3$ such that

$$z(t_4) < 0, \quad x(t_4) = \max\left\{x(t) : t \in [t_3, t_4]\right\}. \tag{9}$$

Then, considering (H3), (4) and (9), we obtain

$$\begin{aligned} 0 > z(t_4) = w(t_4) - \int_{t_2}^{t_4}\int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu, \\ \geq x(t_4) - N_2\int_{t_2}^{t_4}\int_{u-\alpha+\beta}^{u} q(v)x(v-\beta)dvdu, \\ \geq x(t_4)(1 - N_2\int_{t_2}^{t_4}\int_{u-\alpha+\beta}^{u} q(v)dvdu) \geq \frac{1}{2}x(t_4) \geq 0. \end{aligned}$$

This contradiction shows that $x$ must be bounded. There is a positive constant $K$ such that $x(t) \leq K$ holds for all $t \geq t_0$. Accordingly, we see that

$$z(t) \geq -KN_2\int_{t_2}^{\infty}\int_{u-\alpha+\beta}^{u} q(v)dvdu \geq -\frac{K}{2} > -\infty$$

holds, which contradicts with (8) and proves that $z'(t) > 0$ holds for all $t \geq t_2$. By (H1), there exists $t_3 \geq t_2$ and $\varepsilon > 0$ such that $h(t) \geq \varepsilon$ holds for all $t \geq t_3$. Integrating (7) from $t_3$ to $\infty$, we get

$$\infty > z'(t_3) \geq \varepsilon\int_{t_3}^{\infty} g(x(u-\alpha))du \geq \varepsilon N_1\int_{t_3}^{\infty} x(u-\alpha)du,$$

which implies $x \in L^1([t_0,\infty))$. Since $r$ is bounded and (H2) holds, we see from (5) that $w \in L^1([t_2,\infty))$. Hence,

$$\liminf_{t\to\infty} w(t) = 0 \tag{10}$$

is true. On the other hand, we see from (6) that

$$w'(t) = z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du > 0 \tag{11}$$

holds for all $t \geq t_3$. Note that $w$ defined in (4) is positive and increasing by (11), hence (10) is impossible. This is a contradiction. Thus, every solution is oscillatory. □

**Theorem 2** *Assume that (H1)–(H2) hold and* $r \in C([t_0,\infty),\mathbb{R}^+)$ *satisfies*

$$\limsup_{t\to\infty} r(t) < \frac{1}{M}. \tag{12}$$

*If* (3) *holds, then every solution of* (2) *is oscillatory or tending to zero as* $t$ *tends to infinity.*

**Proof.** Suppose that $x$ is a nonoscillatory solution of (2), then we have to show that $\lim_{t\to\infty} x(t) = 0$ is true. Without loss of generality, we suppose that $x$ is an eventually positive solution. There exists $t_1 \geq t_0$ such that $x(t-\kappa) > 0$ holds for all $t \geq t_1$. Considering (12), there exists $t_2 \geq t_1$ and $0 < \delta < 1/M$ such that

$$r(t) \leq \frac{1}{M} - \delta \tag{13}$$

for all $t \geq t_2$. And (3) ensures existence of $t_3 \geq t_2$ such that

$$\int_{t_3}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu < \frac{\delta M}{2N_2}, \tag{14}$$

Now, we set

$$w(t) := x(t) - r(t)f(x(t-\gamma)) \tag{15}$$

and

$$z(t) := w(t) - \int_{t_3}^{t} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \tag{16}$$

for $t \geq t_3$. Then, we have

$$\begin{aligned} z''(t) &= w''(t) - q(t)g(x(t-\beta)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -p(t)g(x(t-\alpha)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -h(t)g(x(t-\alpha)) \leq 0 \end{aligned} \tag{17}$$

for all $t \geq t_3$. Hence, $z'(t)$ and $z(t)$ is strictly monotonic and constant of sign for all $t \geq t_4$, where $t_4 \geq t_3$ is sufficiently large. To prove $z'(t) > 0$ for all $t \geq t_4$, we assume on the contrary that $z'(t) < 0$ holds all $t \geq t_4$. In the present case, since $z'$ is negative and nonincreasing, we see that

$$\lim_{t\to\infty} z(t) = -\infty \tag{18}$$

holds. We also claim that $x$ is bounded. Again on the contrary, assume that $x$ is unbounded. Thus, there exists $t_5 \geq t_4$ such that

$$z(t_5) < 0, \quad x(t_5) = \max\{x(t) : t \in [t_4, t_5]\} \tag{19}$$

hold. Then, from (H2), (H3), (13), (14) and (19), we obtain

$$\begin{aligned} 0 > z(t_5) &= w(t_5) - \int_{t_3}^{t_5} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \\ &\geq x(t_5)\Big(1 - Mr(t_5) - N_2 \int_{t_3}^{t_5} \int_{u-\alpha+\beta}^{u} q(v)dvdu\Big) \\ &\geq x(t_5)\Big(1 - M(\frac{1}{M} - \delta) - \frac{\delta M}{2}\Big) = \frac{\delta M}{2} x(t_5) \geq 0. \end{aligned}$$

This contradiction implies that $x$ is bounded. There is a positive $K$ such that $x(t) \leq K$ for all $t \geq t_0$. Accordingly, for all $t \geq t_4$, we obtain

$$z(t) \geq -\Big(KMr(t) + KN_2 \int_{t_3}^{t} \int_{u-\alpha+\beta}^{u} q(v)dvdu\Big) \geq -\frac{\delta KM}{2} > -\infty,$$

which contradicts with (18) and proves that $z'(t) > 0$ holds for $t \geq t_2$. By (H1), there exists $t_5 \geq t_4$ and $\varepsilon > 0$ such that $h(t) \geq \varepsilon$ holds for all $t \geq t_5$. Integrating (17) from $t_5$ to $\infty$, we get

$$\infty > z'(t_5) \geq z'(t_5) - z'(\infty) \geq \varepsilon \int_{t_5}^{\infty} g(x(u-\alpha))du \geq \varepsilon N_1 \int_{t_5}^{\infty} x(u-\alpha)du$$

which implies $\underline{L} = 0$ and $\overline{L} < \infty$ hold, where $\underline{L} := \liminf_{t\to\infty} x(t)$ and $\overline{L} := \limsup_{t\to\infty} x(t)$. On the other hand, we have from (15) and (14) that

$$w'(t) = z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0$$

holds for all $t \geq t_5$, which implies $w$ is nondecreasing. Therefore, from (H2), (15) and $\overline{L} < \infty$, we see that $-\infty < L < \infty$ holds, where $L := \lim_{t\to\infty} w(t)$.

Now we investigate the following three possible ranges of $L$ as follows:

(i) $0 < L < \infty$. Then, there exists a sufficiently large $t_6 \geq t_5$ such that $w(t) \geq L/2$ holds for all $t \geq t_6$. So, for all $t \geq t_6$, we obtain

$$w(t) = x(t) - r(t)f(x(t-\gamma)) \geq \frac{L}{2},$$

which implies $x(t) \geq L/2$ for all $t \geq t_6$. This contradicts with $\underline{L} = 0$.

(ii) $-\infty < L < 0$. Then, there exists a sufficiently large $t_6 \geq t_5$ such that $w(t) \leq L$ holds for all $t \geq t_6$. So, for all $t \geq t_6$, we see that

$$w(t) = x(t) - r(t)f(x(t-\gamma)) \leq L$$

holds, and together with (H2) and (13), we have

$$-L \leq r(t)f(x(t-\gamma)) \leq M(\frac{1}{M} - \delta)x(t-\gamma),$$

which simply implies $x(t-\gamma) > -L/(M(1/M - \delta))$ for all $t \geq t_6$. This contradicts the fact that $\underline{L} = 0$.

(iii) $L = 0$. Now, we claim that $\bar{L} = 0$. On the contrary, assume that $\overline{L} > 0$. Therefore, from (H2) and (13), we see that

$$w(t) \geq x(t) - \delta x(t-\gamma).$$

holds for all $t \geq t_6$, where $t \geq t_5$ is sufficiently large. Then, there is an increasing divergent sequence $\{u_n\}_{n=1}^{\infty}$ on $[t_7, \infty)$, where $t_7 \geq t_6 + \kappa$ such that $\overline{L} = \lim_{n\to\infty} x(u_n)$ and a sequence $\{v_n\}_{n=1}^{\infty}$ satisfying $x(v_n) = \max\{x(t) : u_n - \kappa \leq t \leq u_n\}$ for all $n \in \mathbb{N}$ Since, $x(v_n) \geq x(u_n)$ for all $n \in \mathbb{N}$, we have $\overline{L} = \lim_{n\to\infty} x(v_n)$ Therefore, from (H2) and (13), we obtain

$$w(u_n) \geq x(u_n) - \delta x(u_n - \gamma),$$

for all $n \in \mathbb{N}$, taking limit as $n \to \infty$, we see that

$$\begin{aligned} L = 0 &\geq \lim_{n\to\infty} [x(u_n) - \delta x(u_n - \gamma)] \\ &\geq \lim_{n\to\infty} x(u_n) - \delta \lim_{n\to\infty} x(v_n) \\ &= \overline{L}(1-\delta) \geq 0, \end{aligned}$$

which implies $\overline{L} = 0$. This contradicts to the assumption that $x$ is not tending to zero as $t \to \infty$.

The proof is complete. □

## 3. Oscillatory behavior of solutions of forced equations

In this section, we shall consider (1) and (2) with forcing terms of the forms:

$$[x(t) + r(t)f(x(t-\gamma))]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{20}$$

$$[x(t) - r(t)f(x(t-\gamma))]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t) \tag{21}$$

for $t \geq t_0$.

**Theorem 3** *Assume that (H1)–(H4) hold and* $r \in C([t_0, \infty), \mathbb{R}^+)$ *is bounded. If* (3) *holds, then every solution of* (20) *is oscillatory or tending to zero as* $t \to \infty$*.*

**Proof.** Suppose that $x$ is an eventually positive solution of (20). Let $t_1 \geq t_0$ satisfy $x(t-\kappa) > 0$ for all $t \geq t_1$. There exists $t_2 \geq t_1$ such that (4) holds.

Let $w$ and $z$ as in (5) and (6) respectively. And if we define

$$W(t) := w(t) - S(t) \text{ and } Z(t) := z(t) - S(t), \tag{22}$$

from (20), we obtain

$$Z''(t) \leq -h(t)g(x(t-\alpha)) \leq 0, \quad t \geq t_1. \tag{23}$$

This shows that $Z'$ is an eventually nonincreasing function. We claim that $Z'$ can not be eventually negative function. Suppose the contrary, i.e. $Z(t) < 0$ for all $t \geq t_3$, for some $t_3 \geq t_2$. Then, we have $\lim_{t\to\infty} Z(t) = -\infty$. We can come to the conclusion that $x$ is bounded from above. As a matter of fact, if $x$ is unbounded from above, there exists an increasing divergent sequence $\{s_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} Z(s_n) = -\infty \text{ and } x(s_n) = \max\{x(t) : t_3 \leq t \leq s_n\} \tag{24}$$

for all $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} x(s_n) = \infty$ holds. Then, from (4) and (24), we have

$$\begin{aligned} Z(s_n) &= x(s_n) + r(s_n)f(x(s_n-\gamma)) - \int_{t_2}^{s_n}\int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu - S(s_n) \\ &\geq x(s_n) - N_2\int_{t_2}^{s_n}\int_{u-\alpha+\beta}^{u} q(v)x(v-\beta)dvdu - S(s_n) \\ &\geq \frac{1}{2}x(s_n) - S(s_n), \end{aligned}$$

and taking the limit as $n \to \infty$, leads the way to the contradiction $\lim_{t\to\infty} Z(t) = \infty$. Since $x$ is bounded from above, there exists a constant $K > 0$ such that $x(t) \leq K$ holds for all $t \geq t_0$. Hence, from (22), we have

$$Z(t) \geq -KN_2\int_{t_0}^{t}\int_{u-\alpha+\beta}^{u} q(v)dvdu + S(t)$$

for all $t \geq t_3$, which according to (4) yields the following:

$$\lim_{t\to\infty} Z(t) \geq -KN_2\int_{t_2}^{\infty}\int_{u-\alpha+\beta}^{u} q(v)dvdu \geq \frac{K}{2}.$$

This contradicts to the fact that $\lim_{t\to\infty} Z(t) = -\infty$.

Therefore, we conclude that $Z$ is an eventually nondecreasing function. Integrating (23) from $t_3$ to $\infty$, we have that $x \in L^1([t_0,\infty))$ because of (H1), and accordingly from (5), this implies that $w \in L^1([t_2,\infty))$ holds since (H2) holds and $r$ is bounded. From (22), we obtain that

$$W'(t) = Z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0$$

holds for all $t \geq t_3$, so that $W$ is nondecreasing. Therefore, using the assumption (H4), we have

$$L := \lim_{t\to\infty} W(t) = \lim_{t\to\infty} w(t),$$

where $0 \leq L < \infty$.

(i) If $0 < L < \infty$. Then, there exists a sufficiently large $t_4 \geq t_3$ such that $w(t) > L/2$ for all $t \geq t_4$. Hence, $w \notin L^1([t_2, \infty))$, and this yields to a contradiction.

(ii) If $L = 0$ is true, then since $x(t) \leq w(t)$ holds for all $t \geq t_2$, we have that $\lim_{t\to\infty} x(t) = 0$.

The proof is therefore completed. □

**Theorem 4** *Assume that (H1)–(H4) hold and* $r \in C([t_0, \infty), \mathbb{R}^+)$ *satisfies* (12). *If* (3) *holds, then every solution of* (21) *is oscillatory or tending to zero as* $t \to \infty$.

**Proof.** Suppose that $x$ is a nonoscillatory solution of (21), which is not tending to zero as $t \to \infty$. Without loss of generality, we suppose that $x$ is eventually positive that is $x(t-\kappa) > 0$ holds for all $t \geq t_1$, where $t_1 \geq t_0$. Then, there exists $t_2 \geq t_1$ such that (13) holds. We have $t_3 \geq t_2$ such that (14) holds because of (3). If we now define $W$ and $Z$ by considering (22) with $w$ and $z$ are defined as in (15) and (16) respectively, from (21), we obtain

$$Z''(t) \leq -h(t)g(x(t-\alpha)) \leq 0, \quad t \geq t_1. \tag{25}$$

for all $t \geq t_3$. We claim that

$$Z'(t) \geq 0, \quad t \geq t_4 \tag{26}$$

holds for some $t_4 \geq t_3$. If this is not a case, then $Z'(t) < 0$ holds for all $t \geq t_4$, implies $\lim_{t\to\infty} Z(t) = -\infty$. On the other hand, $x$ must be bounded from above. Otherwise, there exists an increasing divergent sequence $\{s_n\}_{n=1}^{\infty}$ satisfying (24). Clearly, we have $\lim_{n\to\infty} x(s_n) = \infty$ and $\lim_{n\to\infty} S(s_n) = 0$ from (H4). Since, we have that

$$\begin{aligned} Z(s_n) &= x(s_n) - r(s_n)f(x(s_n-\gamma)) - \int_{t_3}^{s_n} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu - S(s_n) \\ &\geq \frac{\delta M}{2} x(s_n) - S(s_n), \end{aligned}$$

by letting $n \to \infty$ and considering (H4), we get

$$\lim_{t\to\infty} Z(t) = \infty.$$

This is a contradiction. Therefore, $x$ is bounded from above, there exists a constant $K > 0$ such that $x(t) \leq K$ holds for all $t \geq t_0$. Then, we have

$$Z(t) \geq -\frac{\delta KM}{2} + S(t), \quad t \geq t_4.$$

Taking the limit of the above inequality as $t \to \infty$, we have $\lim_{t\to\infty} Z(t) \geq -(\delta KM)/2$ as in the proof of Theorem 2. This is a contradiction. Consequently, we have that (26) holds.

From (H1), (H3) and (25), we obtain $x \in L^1([t_1,\infty))$. Hence, (H2), (5) and boundedness of $r$ implies that $w \in L^1([t_1,\infty))$. On the other hand, we see from (22) that

$$W'(t) = Z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0, \quad t \geq t_4,$$

holds, so that $-\infty < L < \infty$ is true, where

$$L := \lim_{t\to\infty} W(t) = \lim_{t\to\infty} w(t).$$

Now, we investigate possible ranges of $L$ as follows:

(i) $L \neq 0$. In this case, we obtain contradiction as obtained in Theorem 2.

(ii) $L = 0$. In this case, we see that $\lim_{t\to\infty} x(t) = 0$ holds as in Theorem 2.

Proof is done. □

**Remark 1** *Letting $f, g$ as the identity functions, we see that our results are still better than those in [3].*

## 4. An Application

**Example 1** *Consider the forced neutral equation*

$$\begin{aligned}&\left[x(t) - \frac{1}{2}x(t-2\pi)\right]''\\&+ (\mathrm{e}^{-t} + \frac{1}{2}(1 - \frac{1}{\sin^2(t)+2}))\frac{x(t-4\pi)([x(t-4\pi)]^2+2)}{[x(t-4\pi)]^2+1}\\&- \mathrm{e}^{-t}\frac{x(t-2\pi)([x(t-2\pi)]^2+2)}{[x(t-2\pi)]^2+1} = 0 \end{aligned} \tag{27}$$

*for $t \geq 1$. For this equation, we have $r(t) = 1/2$, $\gamma = 2\pi$, $f(u) = u$, $p(t) = (\mathrm{e}^{-t} + (1 - 1/(\sin^2(t)+2))/2)$, $\alpha = 4\pi$, $q(t) = \mathrm{e}^{-t}$, $\beta = 2\pi$, $g(u) = u(u^2+2)/(u^2+1)$. In this case, we may let $M = 1$, and since we have $g(u)/u = 1 + 1/(u^2+1)$ for all $u \neq 0$, we may let $N_1 = 1$ and $N_2 = 2$. On the other hand, we have $\liminf_{t\to\infty} h(t) = 1/4 > 0$, where $h(t) = \mathrm{e}^{-t}(1-\mathrm{e}^{2\pi}) + (1 - 1/(\sin^2(t)+2))/2$, and $\int_1^\infty \int_{u-2\pi}^{u} \mathrm{e}^{-v} dv du = (\mathrm{e}^{2\pi}-1)/\mathrm{e} < \infty$. All the conditions of Theorem 4 are satisfied, thus every solution of (27) is oscillatory or convergent to zero as $t$ tends to infinity. One can see by direct substation that $x(t) = \sin(t)$ is an oscillatory solution.*

**Acknowledgement**

The authors are thankful to the referee for his/her valuable suggestions to improve the presentation of the paper.

## References

[1] Erbe, L. H.; Kong, Q. and Zhang, B. G.: Oscillation theory for functional differential equations, Marcel Dekker, (1995).

[2] Győri, I. and Ladas, G.: Oscillation theory of delay differential equations with applications, Clarendon Press, Oxford, (1991).

[3] Manojlovic, J.; Shoukaku, Y.; Tanigawa, T. and Yoshida, N.: Oscillation criteria for second order differential equations with positive and negative coefficients, Appl. Math. Comput. 181, 853–863 (2006).

[4] Öcalan, Ö.; Yildiz, M. K. and Karpuz, B.: On the oscillation of nonlinear neutral differential equation with positive and negative coefficients, Dynamic Systems and Applications 17, 667–676 (2008).

[5] Parhi, N. and Chand, S.: Oscillation of second order neutral differential equations with positive and negative coefficients, J. Ind.Math. Soc. 66, 227–235 (1999).

[6] Li, W. T. and Quan, H. S.: Oscillation of higher order neutral differential equations with positive and negative coefficients, Ann. of Diff. Eqs. 11, 1, 70–76 (1995).

[7] Zhou, Y. and Zhang, B. G.: Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients, Appl. Math. Lett. 15, 867–874 (2002).

Mustafa Kemal YILDIZ, Başak KARPUZ,
Özkan ÖCALAN,
Department of Mathematics,
Faculty of Science and
Arts, ANS Campus,
Afyon Kocatepe University,
Afyonkarahisar 03200, TURKEY
e-mail: mkyildiz@aku.edu.tr, bkarpuz@aku.edu.tr
e-mail: ozkan@aku.edu.tr

Received 26.11.2007