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Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Minkowski 3-space E_1^3

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Abstract

H. K. Nickerson and Gerald S. Manning [8] derived the intrinsic equations for a relaxed elastic line on an oriented surface in the Euclidean 3-dimensional space E^3. In this paper, we define a generalized relaxed elastic line and derive the intrinsic equations for a generalized of relaxed elastic line on an oriented surface in the Minkowski 3-dimensional space E_1^3 and give some applications of the result.

Key Words: Minkowski space, A generalized relaxed elastic line, Intrinsic equation, Variational problem.

1. Introduction

In this section, we will give some fundamental definitions and theorems.

Definition 1.1 *E^n with the metric*

$$\langle v, w\rangle = -\sum_{i=1}^{n-\nu} v_i w_i + \sum_{j=n-\nu+1}^{n} v_j w_j, \qquad v, w \in E^n,\ 0 \leq \nu \leq n,$$

is called semi-Euclidean space and is denoted by E_ν^n, where ν is called the index of the metric. For $n = 3$, E_1^3 is called Minkowski 3-space ([3],[5],[9]).

Among the natural variational integrals in geometry are the inevitable integrals on space curves $\alpha(s)$. These include length $L(\alpha) = \int ds$, total torsion $T(\alpha) = \int \tau ds$, total squared curvature $K(\alpha) = \int \kappa^2 ds$ used in ([7],[8]) and the integral $H(\alpha) = \int \kappa^2 \tau ds$.

Let $\alpha(s)$ denote an arc on a connected oriented surface S in E_1^3, parameterized by arc lengths, $0 \leq s \leq l$, with curvature $\kappa(s)$ and torsion $\tau(s)$. Let the energy density be given as some function of the curvature and torsion, $f(\kappa, \tau)$. Then

$$H = \int f(\kappa, \tau) ds \tag{1}$$

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is an Hamiltonian for curves [2]. Thus the following integral can be taken as a special case of Hamiltonians for curves:

$$H = \int_0^l \kappa^2 \tau ds. \tag{2}$$

Definition 1.2 *The arc α is called a generalized relaxed elastic line if it is an extremal for variational problem of minimizing the value of H within the family of all arcs of length l on S having the same initial point and initial direction as α in the Minkowski 3-space E_1^3.*

In this study, we would like to calculate the intrinsic equations for the curve α which is an extremal for (2).

We shall require that the coordinate functions of S are sufficiently smooth and that the equations of α, as functions of s, are sufficiently smooth in these coordinates.

Definition 1.3 *Let E_ν^n be a semi-Euclidean space furnished with a metric tensor $\langle\ ,\ \rangle$. A vector v to E_ν^n is called*

$$\begin{aligned} \text{spacelike if } \langle v, v\rangle &> 0 \text{ or } v = 0, \\ \text{null if } \langle v, v\rangle &= 0 \text{ and } v \neq 0, \\ \text{timelike if } \langle v, v\rangle &< 0 \text{ [9].} \end{aligned}$$

Definition 1.4 *Apart from the Frenet frame $\{T, n, b\}$, there also exists a second frame at every point of curve α. At a point $\alpha(s)$ of α, let T denote the unit tangent vector to α, N the unit normal to M, and*

$$N \times T = \varepsilon Q(s), \qquad \varepsilon = \pm 1; \tag{3}$$

respectively, then with the respect to which inner product $\{T, Q, N\}$, gives an orthonormal basis in E_1^3. If M is spacelike surface, $T \times Q = N, Q \times N = -T, N \times T = -Q$. Similarly if M is timelike surface $T \times Q = N, Q \times N = \pm T, N \times T = -Q$ [10].

Theorem 1.5 *Let M be a surface in E_1^3 and α a curve on M. The analogue of the Frenet–Serret formulas is given by*

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix}, \tag{4}$$

where $\langle T, T\rangle = \varepsilon_1$, $\langle Q, Q\rangle = \varepsilon_2$, $\langle N, N\rangle = \varepsilon_3$. Here $k_g(s) = \langle T'(s), Q(s)\rangle$, $\tau_g(s) = \langle Q'(s), N(s)\rangle$ and $k_n(p) = \langle II(T(s), T(s)), N(s)\rangle$ are respectively the geodesic curvature, the geodesic torsion and the normal curvature [1].

Theorem 1.6 *Let α be any regular curve on a surface in E_1^3. Then, the following formulas hold:*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \text{ and } \tau = \varepsilon_1 \varepsilon_2 \frac{\langle \alpha' \times \alpha'', \alpha'''\rangle}{\|\alpha' \times \alpha''\|^2}.$$

2. Obtaining the equations

Now, assume that α lies in a coordinate patch $(u,v)\rightarrow x(u,v)$ of S, and let $x_u=\dfrac{\partial x}{\partial u}$, $x_v=\dfrac{\partial x}{\partial v}$. Then α is expressed as $\alpha(s)=x(u(s),v(s))$,
$0\leq s\leq l$ with

$$T(s)=\alpha'(s)=\frac{du}{ds}x_u+\frac{dv}{ds}x_v$$

and

$$Q(s)=p(s)x_u+q(s)x_v$$

for suitable scalar functions $p(s)$ and $q(s)$.

Next, we must define variational fields for our problem. In order to obtain variational arcs of length l, it is generally necessary to extend α to an arc $\alpha^*(s)$ defined for $0\leq s\leq l^*$, with $l^*>l$ but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0\leq s\leq l^*$, be a scalar function sufficiently smooth, not vanishing identically. Define

$$\eta(s)=\mu(s)p^*(s),\quad \xi(s)=\mu(s)q^*(s).$$

Then

$$\eta(s)x_u+\xi(s)x_v=\mu(s)Q(s) \tag{5}$$

along α. Also, assume that

$$\mu(0)=0,\ \mu'(0)=0 \text{ and } \mu''(0)=0. \tag{6}$$

No further restrictions may be placed on μ. Now define

$$\beta(\sigma;t)=x\left(u(\sigma)+t\eta(\sigma),v(\sigma)+t\xi(\sigma)\right) \tag{7}$$

for $0\leq\sigma\leq l^*$. For $|t|<\varepsilon$ (where $\varepsilon>0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma;t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma;t)$ gives an arc with the same initial point and initial direction as α, because of (6). For $t=0$, $\beta(\sigma;0)$ is the same as α^* and σ is arc length. For $t\neq0$, the parameter σ is not arc length in general.

For fixed t, $|t|<\varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma;t)$,
$0\leq\sigma\leq l^*$. Then

$$L^*(t)=\int_0^{l^*}\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\sigma}(\sigma;t),\frac{\partial\beta}{\partial\sigma}(\sigma;t)\right\rangle\right|}d\sigma \tag{8}$$

with

$$L^*(0)=l^*>l. \tag{9}$$

It is clear from (7) and (8) that $L^*(t)$ is continuous in t. In particular, it follows from (9) that

$$L^*(t)>\frac{l+l^*}{2}>l,\qquad |t|<\varepsilon_* \tag{10}$$

for a suitable ε_* satisfying $0 < \varepsilon_* \leq \varepsilon_.$ Because of (10) we can restrict $\beta(\sigma; t),\ 0 \leq |t| < \varepsilon_*,$ to an arc of length l by restricting the parameter σ to an interval $0 \leq \sigma \leq \lambda(t) \leq l^*$ by requiring

$$\int_0^{\lambda(t)} \sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|} d\sigma = l. \tag{11}$$

Note that $\lambda(0) = l.$ The function $\lambda(t)$ needs not be determined explicitly, but we shall need

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \varepsilon_1 \int_0^l \mu k_g ds. \tag{12}$$

The proof of (12) and other results below will be depended on calculations from (7); such as

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T, \qquad 0 \leq \sigma \leq l \tag{13}$$

Further differentiations of (13) give

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = T' = \varepsilon_2 k_g Q + \varepsilon_3 k_n N. \tag{14}$$

and

$$\begin{aligned} \left.\frac{\partial^3\beta}{\partial\sigma^3}\right|_{t=0} &= \left(-\varepsilon_1\varepsilon_2 k_g^2 - \varepsilon_1\varepsilon_3 k_n^2\right) T + \left(\varepsilon_2 k_g' - \varepsilon_2\varepsilon_3 k_n \tau_g\right) Q \\ &+ \left(\varepsilon_3 k_n' + \varepsilon_2\varepsilon_3 k_g \tau_g\right) N. \end{aligned} \tag{15}$$

Also,

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q, \tag{16}$$

because of (5). Further differentiation of (16) gives

$$\left.\frac{\partial^2\beta}{\partial t\partial\sigma}\right|_{t=0} = \left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0} = -\varepsilon_1\mu k_g T + \mu' Q + \varepsilon_3\mu\tau_g N \tag{17}$$

using (4), and

$$\begin{aligned} \left.\frac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0} &= \left(-2\varepsilon_1\mu' k_g - \varepsilon_1\mu k_g' - \varepsilon_1\varepsilon_3\mu\tau_g k_n\right) T \\ &+ \left(\mu'' - \varepsilon_1\varepsilon_2\mu k_g^2 - \varepsilon_2\varepsilon_3\mu\tau_g^2\right) Q \\ &+ \left(2\varepsilon_3\mu'\tau_g - \varepsilon_1\varepsilon_3\mu k_g k_n + \varepsilon_3\mu\tau_g'\right) N \end{aligned} \tag{18}$$

and

$$\left.\frac{\partial^4\beta}{\partial t\partial\sigma^3}\right|_{t=0} = [\mu\varepsilon_2 k_g^3 - 3\varepsilon_1\mu' k_g' - 3\varepsilon_1\mu'' k_g - 3\varepsilon_1\varepsilon_3\mu'\tau_g k_n - 2\varepsilon_1\varepsilon_3\mu\tau_g' k_n$$
$$-\varepsilon_1\varepsilon_3\mu\tau_g k_n' + \varepsilon_1\varepsilon_2\varepsilon_3\mu\tau_g^2 k_g + \varepsilon_3\mu k_g k_n^2 - \varepsilon_1\mu k_g'']T$$
$$+[\mu''' - 3\varepsilon_1\varepsilon_2\mu' k_g^2 - 3\varepsilon_2\varepsilon_3\mu'\tau_g^2 - 3\varepsilon_1\varepsilon_2\mu k_g k_g' - 3\varepsilon_2\varepsilon_3\mu\tau_g\tau_g']Q \tag{19}$$
$$+[-2\varepsilon_1\varepsilon_3\mu k_g' k_n - \varepsilon_1\varepsilon_3\mu k_g k_n' - \varepsilon_2\mu\tau_g^3 + 3\varepsilon_3\mu'\tau_g' + 3\varepsilon_3\mu''\tau_g$$
$$-\varepsilon_1\varepsilon_2\varepsilon_3\mu k_g^2\tau_g + \varepsilon_3\mu\tau_g'' - 3\varepsilon_1\varepsilon_3\mu' k_g k_n - \mu\tau_g\varepsilon_1 k_n^2]N.$$

To prove (12), differentiate (11) with respect to t, remembering that l is constant, and evaluate at $t=0$ using (13) and (17), with $\lambda(0)=l$.

$$\left.\frac{d\lambda}{dt}\right|_{t=0}\sqrt{\left|\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle\right|} + \int_0^l \left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0}\right\rangle \frac{\sqrt{\left|\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle\right|}}{\left\langle \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}\right\rangle} ds = 0.$$

Thus, we can now calculate $H(t)$ for the arc $\beta(\sigma;t),\ 0\le\sigma\le\lambda(t),\ |t|<\varepsilon_*$. Since σ is not generally arc length for $t\neq 0,\ H(t)$ is

$$H(t) = \varepsilon_1\varepsilon_2 \int_0^{\lambda(t)} < \frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} > \left|< \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} >\right|^{-5/2} d\sigma.$$

A necessary condition that α be an extremal is that $H'(0)=0$ for arbitrary μ satisfying (6). We now calculate the $H'(t)$:

$$H'(t) = \varepsilon_1\varepsilon_2\frac{d\lambda}{dt}\left\{< \frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} > \left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-5/2}\right\}_{\sigma=\lambda(t)}$$
$$+\varepsilon_1\varepsilon_2\int_0^{\lambda(t)}\left\{< \frac{\partial^2\beta}{\partial t\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} > + < \frac{\partial\beta}{\partial\sigma}\times\frac{\partial^3\beta}{\partial t\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} >\right.$$
$$\left. + < \frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^4\beta}{\partial t\partial\sigma^3} >\right\}\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-5/2} d\sigma$$
$$-5\varepsilon_1\varepsilon_2\int_0^{\lambda(t)} < \frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} > \left|< \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} >\right|^{-7/2}\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\frac{\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|}{\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle} d\sigma.$$

Using (11), (12), (13), (14), (15) (17), (18) and (19), we find

$$\begin{aligned} H'(0) &= \varepsilon_1\varepsilon_2(\int_0^l \mu k_g ds)[\varepsilon_2 k_g(l)k_n'(l) - \varepsilon_3 k_n(l)k_g'(l) + [k_g^2(l) + k_n^2(l)]\tau_g(l)] \\ &+\int_0^l \{[5\tau_g\varepsilon_2 k_g k_n^2 - 2\varepsilon_1 k_g k_g' k_n - 3\varepsilon_1\varepsilon_2 k_g^3\tau_g - \varepsilon_2\varepsilon_3\tau_g' k_g' \\ &+\varepsilon_1\varepsilon_2\varepsilon_3\tau_g k_g^3 - 3\varepsilon_1 k_g^2 k_n' + 5\varepsilon_1\varepsilon_2\varepsilon_3 k_n k_g k_g' - \varepsilon_3\tau_g^2 k_n' \\ &-\varepsilon_1\varepsilon_3 k_g\tau_g k_n^2 + 5\varepsilon_2 k_g^3\tau_g + 4\varepsilon_2 k_n\tau_g\tau_g' + k_g\tau_g'' + 5k_g^2 k_n' \\ &-5\varepsilon_2 k_g\varepsilon_3 k_n k_g' - 2\varepsilon_2\varepsilon_3 k_g\tau_g^3 - 2\varepsilon_1\varepsilon_2 k_g k_n^2\tau_g + k_g\varepsilon_1 k_n^2\tau_g]\mu \\ &+[3k_g\tau_g' + 3\varepsilon_1\varepsilon_2\varepsilon_3 k_n k_g^2 - 2\varepsilon_2\varepsilon_3\tau_g k_g' - \varepsilon_1\varepsilon_3 k_n k_g^2 - \varepsilon_1\varepsilon_2 k_n^3 \\ &-3\varepsilon_1 k_g^2 k_n + 5\varepsilon_2 k_n\tau_g^2]\mu' + [4k_g\tau_g + \varepsilon_2 k_n']\mu'' - \varepsilon_3 k_n\mu'''\}ds. \end{aligned}$$

However, with integration by parts and (6),

$$-\varepsilon_3\int_0^l \mu''' k_n ds = -\varepsilon_3\mu''(l)k_n(l) + \varepsilon_3\mu'(l)k_n'(l) - \varepsilon_3\mu(l)k_n''(l) + \varepsilon_3\int_0^l \mu k_n''' ds$$

and

$$\begin{aligned} \int_0^l \mu''[4k_g\tau_g + \varepsilon_2 k_n']ds &= \mu'(l)[4k_g(l)\tau_g(l) + \varepsilon_2 k_n'(l)] \\ &-\mu(l)[4\tau_g(l)k_g'(l) + 4k_g(l)\tau_g'(l) + \varepsilon_2 k_n''(l)] \\ &+\int_0^l \mu[8\tau_g' k_g' + 4\tau_g k_g'' + 4k_g\tau_g'' + \varepsilon_2 k_n''']ds \end{aligned}$$

and

$$\begin{aligned} &\int_0^l \mu'[3k_g\tau_g' + \varepsilon_1(3\varepsilon_2\varepsilon_3 - \varepsilon_3 - 3)k_n k_g^2 + 5\varepsilon_2 k_n\tau_g^2 - 2\varepsilon_2\varepsilon_3\tau_g k_g' \\ &-\varepsilon_1\varepsilon_2 k_n^3]ds = [3k_g(l)\tau_g'(l) + \varepsilon_1(3\varepsilon_2\varepsilon_3 - \varepsilon_3 - 3)k_n(l)k_g^2(l) \\ &\qquad -2\varepsilon_2\varepsilon_3\tau_g(l)k_g'(l) + 5\varepsilon_2 k_n(l)\tau_g^2(l) - \varepsilon_1\varepsilon_2 k_n^3(l)]\mu(l) \\ &\qquad -\int_0^l \mu[(3 - 2\varepsilon_2\varepsilon_3)\tau_g' k_g' + \varepsilon_1(3\varepsilon_2\varepsilon_3 - \varepsilon_3 - 3)k_n' k_g^2 \\ &\qquad +3k_g\tau_g'' - 2\varepsilon_2\varepsilon_3\tau_g k_g'' - \varepsilon_1(2\varepsilon_3 + 6 - 6\varepsilon_2\varepsilon_3)k_n k_g k_g' \\ &\qquad +5\varepsilon_2 k_n'\tau_g^2 - 3\varepsilon_1\varepsilon_2 k_n^2 k_n' + 10\varepsilon_2 k_n\tau_g\tau_g']ds. \end{aligned}$$

Thus $H'(0)$ can be written as

$$\begin{aligned} H'(0) = & \int_0^l \big\{(\varepsilon_2+\varepsilon_3)k_n''' - (\varepsilon_3+5\varepsilon_2)\tau_g^2 k_n' + \varepsilon_2(5+\varepsilon_1\varepsilon_3-3\varepsilon_1)k_g^3\tau_g \\ & -\varepsilon_2(\varepsilon_1\varepsilon_3+5\varepsilon_3-4\varepsilon_1\varepsilon_2-2\varepsilon_1\varepsilon_2\varepsilon_3)k_nk_gk_g' - 6\varepsilon_2k_n\tau_g\tau_g' + 2k_g\tau_g'' \\ & +(5+\varepsilon_2\varepsilon_3)k_g'\tau_g' - 2\varepsilon_2\varepsilon_3k_g\tau_g^3 + \varepsilon_1(1-\varepsilon_3-2\varepsilon_2+5\varepsilon_2)k_g\tau_gk_n^2 \\ & +3\varepsilon_1\varepsilon_2k_n^2k_n' + (2\varepsilon_2\varepsilon_3+4)\tau_gk_g'' + (5-3\varepsilon_1\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_3)k_g^2k_n' \\ & +\varepsilon_1k_g\left(\varepsilon_2k_g(l)k_n'(l) + [k_g^2(l)+k_n^2(l)]\tau_g(l) - \varepsilon_3k_n(l)k_g'(l)\right)\big\}\mu ds \\ & + \big[-(2\varepsilon_2\varepsilon_3+4)\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) + k_n(l)\left(5\varepsilon_2\tau_g^2(l)\right. \\ & \left.+\varepsilon_1(3\varepsilon_2\varepsilon_3-3-\varepsilon_3)k_g^2(l) - \varepsilon_1\varepsilon_2k_n^2(l)\right) - (\varepsilon_2+\varepsilon_3)k_n''(l)\big]\mu(l) \\ & +[4k_g(l)\tau_g(l) + (\varepsilon_2+\varepsilon_3)k_n'(l)]\mu'(l) - \varepsilon_3k_n(l)\mu''(l). \end{aligned} \tag{20}$$

2.1. Intrinsic equations for a generalized elastic line on a timelike surface for timelike arc α

If T is timelike, Q and N are spacelike, then $\langle T,T\rangle = \varepsilon_1 = -1$, $\langle Q,Q\rangle = \varepsilon_2 = 1$, $\langle N,N\rangle = \varepsilon_3 = 1$. Thus $H'(0)$ can be written as

$$\begin{aligned} H'(0) = & \int_0^l \big\{2k_n''' - [6\tau_g^2 + 3k_n^2 - 7k_g^2]k_n' - 3k_g\tau_gk_n^2 + 6\tau_gk_g'' \\ & -10k_nk_gk_g' + 7k_g^3\tau_g - 2k_g\tau_g^3 - 6k_n\tau_g\tau_g' + 6k_g'\tau_g' + 2k_g\tau_g'' \\ & -k_g\left(k_g(l)k_n'(l) + [k_g^2(l)+k_n^2(l)]\tau_g(l) - k_n(l)k_g'(l)\right)\big\}\mu ds \\ & + \big[-6\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) + k_n(l)\left(5\tau_g^2(l) + k_g^2(l) + k_n^2(l)\right) \\ & -2k_n''(l)\big]\mu(l) + [4k_g(l)\tau_g(l) + 2k_n'(l)]\mu'(l) - k_n(l)\mu''(l). \end{aligned} \tag{21}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l), \mu'(l)$ and $\mu''(l)$, the given timelike arc α must satisfy three boundary conditions

(BC I) $\quad k_n(l) = 0$

(BC II) $\quad 2k_n'(l) = -4k_g(l)\tau_g(l)$

(BC III) $\quad 2k_n''(l) = -6\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) + k_n(l)\left[5\tau_g^2(l) + k_g^2(l) + k_n^2(l)\right]$

and the differential equation

$$\text{(DE)} \quad \begin{aligned} &2k_n''' - [6\tau_g^2 + 3k_n^2 - 7k_g^2]k_n' - 3k_g\tau_g k_n^2 - 10k_nk_gk_g' \\ &+6\tau_g k_g'' + 7k_g^3\tau_g - 2k_g\tau_g^3 - 6k_n\tau_g\tau_g' + 6k_g'\tau_g' + 2k_g\tau_g'' \\ &-k_g\left(k_g(l)k_n'(l) + [k_g^2(l) + k_n^2(l)]\tau_g(l) - k_n(l)k_g'(l)\right) = 0. \end{aligned} \tag{22}$$

2.2. Intrinsic equations for a generalized elastic line on a timelike surface for spacelike arc α

If Q is timelike, T and N are spacelike, then $\langle T,T\rangle = \varepsilon_1 = 1$, $\langle Q,Q\rangle = \varepsilon_2 = -1$, $\langle N,N\rangle = \varepsilon_3 = 1$. Thus $H'(0)$ can be written as

$$\begin{aligned} H'(0) = &\int_0^l \left\{4\tau_g^2k_n' - 3k_g^3\tau_g + 12k_nk_gk_g' + 6k_n\tau_g\tau_g' + 2k_g\tau_g'' \right. \\ &+4k_g'\tau_g' + 2k_g\tau_g^3 - 3k_g\tau_g k_n^2 - 3k_n^2k_n' + 2\tau_g k_g'' + 9k_g^2k_n' \\ &\left. +k_g\left(-k_g(l)k_n'(l) + [k_g^2(l) + k_n^2(l)]\tau_g(l) - k_n(l)k_g'(l)\right)\right\}\mu ds \\ &+\left[-2\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) + k_n(l)\left(-5\tau_g^2(l) + k_n^2(l) \right.\right. \\ &\left.\left. -7k_g^2(l)\right)\right]\mu(l) + [4k_g(l)\tau_g(l)]\mu'(l) - k_n(l)\mu''(l). \end{aligned} \tag{23}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l), \mu'(l)$ and $\mu''(l)$, the given spacelike arc α must satisfy three boundary conditions

(BC I) $\quad k_n(l) = 0$

(BC II) $\quad 4k_g(l)\tau_g(l) = 0$

(BC III) $\quad 2\tau_g(l)k_g'(l) + k_g(l)\tau_g'(l) + k_n(l)\left[5\tau_g^2(l) + 7k_g^2(l) - k_n^2(l)\right] = 0$

and the differential equation

$$\text{(DE)} \quad \begin{aligned} &4\tau_g^2k_n' - 3k_g^3\tau_g + 12k_nk_gk_g' + 6k_n\tau_g\tau_g' + 2k_g\tau_g'' \\ &+4k_g'\tau_g' + 2k_g\tau_g^3 - 3k_g\tau_g k_n^2 - 3k_n^2k_n' + 2\tau_g k_g'' + 9k_g^2k_n' \\ &+k_g\left(-k_g(l)k_n'(l) + \left[k_g^2(l) + k_n^2(l)\right]\tau_g(l) - k_n(l)k_g'(l)\right) = 0. \end{aligned} \tag{24}$$

2.3. Intrinsic equations for a generalized elastic line on a spacelike surface

If T and Q are spacelike and N is timelike, then $\langle T,T\rangle = \varepsilon_1 = 1$, $\langle Q,Q\rangle = \varepsilon_2 = 1$, $\langle N,N\rangle = \varepsilon_3 = -1$. Thus $H'(0)$ can be written as

$$\begin{aligned} H'(0) = \; & \int_0^l \{[-4\tau_g^2 + 3k_n^2 + 7k_g^2]k_n' + 5k_g\tau_g k_n^2 + k_g^3\tau_g + 2k_g\tau_g^3 \\ & +8k_nk_gk_g' + 2\tau_gk_g'' - 6k_n\tau_g\tau_g' + 4k_g'\tau_g' + 2k_g\tau_g'' \\ & +k_g\left(k_g(l)k_n'(l) + [k_g^2(l)+k_n^2(l)]\tau_g(l) + k_n(l)k_g'(l)\right)\}\,\mu ds \\ & +\left[-2\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) + k_n(l)\left(5\tau_g^2(l) - 5k_g^2(l)\right.\right. \\ & \left.\left.-k_n^2(l)\right)\right]\mu(l) + [4k_g(l)\tau_g(l)]\mu'(l) + k_n(l)\mu''(l). \end{aligned} \tag{25}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (6), with arbitrary values of $\mu(l), \mu'(l)$ and $\mu''(l)$, the given spacelike arc α must satisfy three boundary conditions

(BC I) $\quad k_n(l) = 0$

(BC II) $\quad 4k_g(l)\tau_g(l) = 0$

(BC III) $\quad 2\tau_g(l)k_g'(l) + k_g(l)\tau_g'(l) + k_n(l)\left[-5\tau_g^2(l) + 5k_g^2(l) + k_n^2(l)\right] = 0$

and the differential equation

$$\begin{aligned} \text{(DE)} \quad & \left[-4\tau_g^2 + 3k_n^2 + 7k_g^2\right]k_n' + 5k_g\tau_gk_n^2 + k_g^3\tau_g + 2k_g\tau_g^3 \\ & +8k_nk_gk_g' + 2\tau_gk_g'' - 6k_n\tau_g\tau_g' + 4k_g'\tau_g' + 2k_g\tau_g'' \\ & +k_g\left(k_g(l)k_n'(l) + [k_g^2(l)+k_n^2(l)]\tau_g(l) + k_n(l)k_g'(l)\right) = 0. \end{aligned} \tag{26}$$

If $\tau = 1,\ \nu = 0$ and $\tau = 1,\ \nu = 1$ and $\tau = 1,\ n = 3$, then we obtain intrinsic equations in [8] and in [4], [6] and in [5], respectively.

3. Applications

Theorem 3.1 *In the timelike plane in E_1^3, there is not generalized relaxed elastic line.*

Proof. In the timelike plane in E_1^3, the geodesic torsion τ_g vanishes for all curves and $k_n^2 = \dfrac{1}{r^2} \neq 0$ [6]. Therefore BCI can not be satisfied. □

Theorem 3.2 *In the spacelike plane in E_1^3, any arc is a generalized relaxed elastic line.*

Proof. In the spacelike plane in E_1^3, the geodesic torsion τ_g vanishes for all curves and $k_n = 0$ [6]. Therefore any arc in the spacelike plane satisfies (26), trivially. □

Theorem 3.3 *On a pseudo-sphere $S_1^2(r)$, there is no generalized relaxed elastic line.*

Proof. Since $k_n = \dfrac{1}{r}$ and $\tau_g = 0$, the proof is clear. □

Theorem 3.4 *If α is both an asymtotic and a geodesic line on any surface S in E_1^3, then α is a generalized relaxed elastic line.*

Proof. Since $k_g = k_n = 0$, then proof is straightforward. □

Theorem 3.5 *If α is any ruling of the ruled surface in E_1^3, then α is a generalized relaxed elastic line.*

Proof. Since any ruling of the ruled surface is both asymtotic and geodesic, it follows that $k_g = k_n = 0$. Hence the proof is clear. □

4. Conclusion

In [8], "Intrinsic equations for a relaxed elastic line on an oriented surface" was studied by H. K. Nickerson and G. S. Manning. In their study, the authors calculated the internal equations of elastic lines with the aid of k_g just by using curvature of an elastic curve. In [4], [5] and [6], Gürbüz, N. and Görgülü A. showed similar equations in Minkowski space. In this study, since the energy density is given as some function of the curvature and torsion, $f(\kappa, \tau)$, the equations were given in Minkowski space with the aid of k_n by using both the curvature and the torsion of an elastic curve.

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