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On orders and types of Dirichlet series of slow growth

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Abstract

The present paper has the object of showing some interesting relationship on the maximum modulus, the maximum term, the index of maximum term and the coefficients of entire functions defined by Dirichlet series of slow growth; some properties like Taylor entire functions are obtained.

Key Words: Dirichlet series, generalized order, generalized type.

1. Introduction and main results

The growth and the value distribution of Taylor entire functions

$$f(z) = \sum_{n=0}^{+\infty} b_n z^n$$

were studied for a long time and many important results were obtained in [1],[2] and [3]. For instance, S.K. Bajpai gave some different characterizations on the coefficients and the maximum modulus, the maximum term, and the index of maximum term for the entire functions of fast growth $\rho = \infty$ in [1]. On the other hand, G.P. Kapoor [3] and Ramesh Ganti [2] continued this work and defined a generalized order and a generalized type for the Taylor entire functions of slow growth $\rho = 0$.

Dirichlet series was introduced by L. Dirichlet in 19th century and it has the form:

$$f(s) = \sum_{n=1}^{+\infty} b_n e^{\lambda_n s}, \tag{1}$$

where $\{b_n\} \in \mathbf{C}, 0 < \lambda_n \uparrow +\infty$ and $s = \sigma + it(\sigma,$ t are real variables). It is well known that Dirichlet series are the generalization of Taylor series, because (1) can be Taylor series, provided $e^s = z$ and $\lambda_n = n$. In this paper we do not require λ_n must be integers and we always assume that the series (1) satisfies

$$\overline{\lim_{n\to+\infty}} \frac{\ln n}{\lambda_n} = D, \tag{2}$$

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$$\overline{\lim_{n\to+\infty}}\frac{\ln|b_n|}{\lambda_n}=-\infty. \tag{3}$$

It follows from Valiron formula in [8] that the abscissa of uniform convergence of (1) are $-\infty$ and so the series (1) defines an entire function $f(s)$ in the complex plane. Some relative results on the growth and the convergence of Dirichlet series can be found in [4–7].

Definition 1 *The maximum modulus, the maximum term, and the index of maximum term of* (1) *can be defined as*

$$M(\sigma)=\sup\{|f(\sigma+it)|;t\in\boldsymbol{R}\},$$

$$m(\sigma)=\max\{|b_n|e^{\lambda_n\sigma},n\in\mathbf{N}^+\}.$$

$$N(\sigma)=\max\{\lambda_n;|b_n|e^{\lambda_n\sigma}=m(\sigma),n\in\mathbf{N}^+\}.$$

Note: $N(\sigma)$ is a non-decreasing step function and it plays a key role for the growth of (1). We proceed to replace $N(\sigma)$ by the order of (1).

Normally, the growth of (1) is measured in terms of its order μ and the type ν, the order can be defined by

$$\mu=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\ln M(\sigma)}{\sigma};$$

if $\mu\in(0,\infty)$, we can define the type of (1) by

$$\nu=\overline{\lim_{\sigma\to+\infty}}\frac{\ln M(\sigma)}{e^{\sigma\mu}}.$$

Next, we try to define a new order and a new type of Dirichlet series of slow growth deals with some more general conditions. Let Λ denote the class of functions $h(x)$ satisfying the following conditions:

(i) h(x) is defined on $[a;\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$;

(ii)

$$\lim_{x\to\infty}\frac{d(h(x))}{d(\ln^{[p]}x)}=k\in(0,\infty),\quad p\geq 1,p\in N^+$$

where $\ln^{[0]}x=x,\ln^{[1]}x=\ln x,\ln^{[p]}x=\ln^{[p-1]}\ln x$. we can easy testify that

$$\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1,\ \lim_{x\to\infty}\frac{h(c+x)}{h(x)}=1, \tag{4}$$

for every $c>0$, that is, $h(x)$ is slowly increasing.

Definition 2 *Let $\alpha(x)\in\Lambda$, the generalized order of the entire function $f(s)$ defined by (1) can be defined as*

$$\rho=\rho(\alpha;f)=\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)},$$

if the order is of slow growth i.e. $\rho \in (0,\infty)$, *and then the type* $\tau(\alpha;f)$ *of* (1) *is defined by*

$$\tau = \tau(\alpha;f) = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(M(\sigma))}{[\alpha(e^{\sigma})]^{\rho}} = \overline{\lim_{\sigma\to+\infty}} \frac{\beta(\ln M(\sigma))}{[\beta(\sigma)]^{\rho}},$$

where $\beta(\ln x) = \alpha(x)$.

Theorem 1 *Suppose that Dirichlet series* (1) *satisfies* (2) *and* (3), *then*

$$1^{o} \quad \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)} - 1 = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\lambda_n)}{\alpha(\ln |b_n|^{-\frac{1}{\lambda_n}})}, \quad for \quad p=1,$$

$$2^{o} \quad \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\lambda_n)}{\alpha(\ln |b_n|^{-\frac{1}{\lambda_n}})} \le \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)} \le \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\lambda_n)}{\alpha(\ln |b_n|^{-\frac{1}{\lambda_n}})} + 1, \quad for \quad p=2,3,\cdots$$

Theorem 2 *Suppose that Dirichlet series* (1) *satisfies* (2) *and* (3), *then*

$$\lim_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)} - 1 \le \lim_{\sigma\to+\infty} \frac{\alpha(N(\sigma))}{\alpha(\sigma)} \le \lim_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)}.$$

Theorem 3 *Suppose that Dirichlet series* (1) *satisfies* (2) *and* (3), *and has the generalized order* $\rho \in (1,\infty)$ *then*

$$\tau = \overline{\lim_{\sigma\to+\infty}} \frac{\beta(\ln M(\sigma))}{[\beta(\sigma)]^{\rho}} = \overline{\lim_{n\to+\infty}} \frac{\beta(\lambda_n)}{[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho}}.$$

Note. *Let* $p=1, e^{s}=z$ *and* $\lambda_n = n$, *we can use the same method to prove Theorem 2.1 in [2,P.64].*

2. Preliminary Lemmas

Lemma 1 *Suppose that Dirichlet series* (1) *satisfies* (2) *and* (3), *then*

$$\lim_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma))}{[\alpha(\sigma)]^{E}} = \lim_{\sigma\to+\infty} \frac{\alpha(\ln m(\sigma))}{[\alpha(\sigma)]^{E}}, \quad E \in (0,+\infty)$$

Proof.

Case I: From (2), $\forall \varepsilon > 0$, there exist an integer N, when $n > N$, it follows that

$$\frac{\ln n}{\lambda_n} < D + \frac{\varepsilon}{2},$$

then

$$e^{-\lambda_n} < \left(\frac{1}{n}\right)^{\frac{1}{D+\frac{\varepsilon}{2}}}.$$

By Definition 1,

$$M(\sigma) \le \sum_{n=1}^{\infty} |b_n| e^{\lambda_n \sigma} \le \sum_{n=1}^{N} |b_n| e^{\lambda_n \sigma} + \sum_{n=N+1}^{\infty} |b_n| e^{\lambda_n(\sigma+D+\varepsilon)} e^{-\lambda_n(D+\varepsilon)}$$

$$< (N+1)m(\sigma) + m(\sigma+D+\varepsilon)\sum_{n=N+1}^{\infty} e^{-\lambda_n(D+\varepsilon)}$$

$$= (N+1)m(\sigma) + m(\sigma+D+\varepsilon)\sum_{n=N+1}^{\infty} (\frac{1}{n})^{\frac{D+\varepsilon}{D+\frac{\varepsilon}{2}}}$$

$< K(\varepsilon)m(\sigma+D+\varepsilon)$,

where $K(\varepsilon)$ is a positive constant on ε, then

$$\lim_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma))}{[\alpha(\sigma)]^E} \le \lim_{\sigma\to+\infty}\frac{\alpha(\ln K(\varepsilon)+\ln m(\sigma+D+\varepsilon))}{[\alpha(\sigma+D+\varepsilon)]^E}\cdot\frac{[\alpha(\sigma+D+\varepsilon)]^E}{[\alpha(\sigma)]^E}.$$

By (4), it follows that

$$\lim_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma))}{[\alpha(\sigma)]^E} \le \lim_{\sigma\to+\infty}\frac{\alpha(\ln m(\sigma))}{[\alpha(\sigma)]^E}.$$

Case II: From Lemma 2 in [6, P.559], it follows that $M(\sigma)\ge m(\sigma)$. This proves Lemma 1. □

In the following proof we use C for a real constant. It will not be the same at each occurrence but it is always independent of all variables.

Lemma 2 *Suppose that $\alpha(\sigma)\in\Lambda$ and its inverse function is $\alpha^{-1}(\sigma)$, then*

$$\lim_{\sigma\to\infty}\frac{\alpha\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\}}{\alpha(\sigma)} \le A+1, \qquad A>0, B>0. \tag{5}$$

Proof. From the condition (ii), $\alpha(\sigma) = (k+o(1))\ln^{[p]}\sigma + C$ (let $C\ge 0$ without loss of generality) and $\alpha^{-1}(\sigma) = \exp^{[p]}(\frac{\sigma-C}{k+o(1)})$, where $\exp^{[0]}\sigma=\sigma, \exp^{[1]}\sigma = e^\sigma, \exp^{[p]}\sigma = \exp^{[p-1]}\exp(\sigma)$.

Case I: When $p=1$, it follows that

$$\begin{aligned}\alpha^{-1}[A\alpha(\sigma+B)] &= \exp[\tfrac{A\alpha(\sigma+B)-C}{k+o(1)}]\\ &= \exp[\tfrac{A(k+o(1))\ln(\sigma+B)+AC-C}{k+o(1)}]\\ &= C(\sigma+B)^A + o(1).\end{aligned}$$

Then

$$\begin{aligned}\alpha\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\} &= k\ln\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\} + C\\ &= k\ln\sigma + k\ln[C(\sigma+B)^A+o(1)] + C\\ &= k\ln\sigma + kA\ln(\sigma+B) + C + o(1),\end{aligned}$$

thus

$$\lim_{\sigma\to+\infty}\frac{\alpha\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\}}{\alpha(\sigma)} = \lim_{\sigma\to+\infty}\frac{k\ln\sigma+kA\ln(\sigma+B)+C+o(1)}{(k+o(1))\ln\sigma+C} = A+1.$$

Case II: When $p=2,3,\cdots$, it follows that

$$\alpha^{-1}[A\alpha(\sigma+B)] = \exp^{[p]}[\frac{A(k+o(1))\ln^{[p]}(\sigma+B)+AC-C}{k+o(1)}] = \exp^{[p]}[A\ln^{[p]}(\sigma+B)+C+o(1)].$$

Then

$$\begin{aligned}\alpha\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\} &= k\ln^{[p]}\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\}+C\\ &= k\ln^{[p]}\{\sigma\exp^{[p]}[A\ln^{[p]}(\sigma+B)+C+o(1)]\}+C\\ &= k\ln^{[p-1]}\{\ln\sigma+\exp^{[p-1]}[A\ln^{[p]}(\sigma+B)+C]\}+C+o(1)\\ &\le k\ln^{[p-2]}\{\ln\ln\sigma+\exp^{[p-2]}[A\ln^{[p]}(\sigma+B)+C]\}+C+o(1)\\ &\le \cdots\\ &\le k\ln^{[p]}\sigma+k[A\ln^{[p]}(\sigma+B)]+C+o(1),\end{aligned}$$

thus

$$\lim_{\sigma\to+\infty}\frac{\alpha\{\sigma\alpha^{-1}[A\alpha(\sigma+B)]\}}{\alpha(\sigma)}\le\lim_{\sigma\to+\infty}\frac{k\ln^{[p]}\sigma+k[A\ln^{[p]}(\sigma+B)]+C+o(1)}{((k+o(1))\ln^{[p]}\sigma+C}=A+1.$$

□

Corollary 1 *Suppose that* $\alpha(\sigma)\in\Lambda$ *and its inverse function is* $\alpha^{-1}(\sigma)$, *then*

$$\lim_{\sigma\to+\infty}\frac{\alpha\{(\sigma-B)\alpha^{-1}[A\alpha(\sigma)]\}}{\alpha(\sigma)}\le A+1,\qquad A>0,B>0.$$

Corollary 2 *Suppose that* $\beta(\ln\sigma)=\alpha(\sigma)\in\Lambda$ *and* $\beta^{-1}(\sigma)$ *is the inverse function of* $\beta(\sigma)$, *then*

$$\lim_{\sigma\to+\infty}\frac{\beta\{\sigma\beta^{-1}\{A[\beta(\sigma+B)]^{\rho}\}\}}{[\beta(\sigma)]^{\rho}}\le A,\qquad A>0,B>0,\rho>1.$$

For the convenience of readers, we prove the follow lemma again.

Lemma 3 *Suppose that Dirichlet series* (1) *satisfies* (2) *and* (3), *then*

$$\ln m(\sigma)=\ln m(\sigma_1)+\int_{\sigma_1}^{\sigma}N(\sigma)d\sigma,\quad \sigma_1>0.$$

Proof. Let $m(\sigma)=|b_v|e^{N(\sigma)\sigma}$, where $\lambda_v=N(\sigma)$ is a constant, which satisfies the definition the index of maximum term. We give differential coefficient in each open domain $I_k:=(\sigma_k,\sigma_{k+1})$, it follows that

$m'(\sigma)=N(\sigma)|b_v|e^{N(\sigma)\sigma}=N(\sigma)m(\sigma)$, then

$$\ln m(\sigma_{k+1})-\ln m(\sigma_k)=\int_{\sigma_k}^{\sigma_{k+1}}\frac{m'(\sigma)}{m(\sigma)}d\sigma=\int_{\sigma_k}^{\sigma_{k+1}}N(\sigma)d\sigma.$$

For $\forall\sigma\in(\sigma_1,+\infty),\exists k$, such that $\sigma\in(\sigma_k,\sigma_{k+1})$, combining with the above equality, then

$$\ln m(\sigma)-\ln m(\sigma_1)=\ln m(\sigma)-\ln m(\sigma_k)+\sum_{j=1}^{k-1}(\ln m(\sigma_{j+1})-\ln m(\sigma_j))$$

$$=\int_{\sigma_k}^{\sigma}N(\sigma)d\sigma+\sum_{j=1}^{k-1}\int_{\sigma_j}^{\sigma_{j+1}}N(\sigma)d\sigma=\int_{\sigma_1}^{\sigma}N(\sigma)d\sigma.$$

So Lemma 3 is proved. It can be found in [6]. □

3. Proof of Theorems

Proof of Theorem 1 We prove this theorem in two steps.

Case I: When $p=1, \alpha(\sigma)=k\ln\sigma+C$, then $\alpha^{-1}(\sigma)=\exp(\frac{\sigma-C}{k})=Ce^{\frac{\sigma}{k}}$. By Lemma 1,

$$\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)}=\overline{\lim_{\sigma\to+\infty}}\frac{\alpha(\ln m(\sigma))}{\alpha(\sigma)}=A.$$

We suppose $1\le A<\infty$. Then for every $\varepsilon>0, \exists\sigma_0(\varepsilon)>0$, when $\sigma\ge\sigma_0$, it follows that

$$\frac{\alpha(\ln m(\sigma))}{\alpha(\sigma)}<A+\varepsilon=A^*,$$

it follows that

$$\ln m(\sigma)=\alpha^{-1}[A^*\alpha(\sigma)]=\exp(\frac{A^*\alpha(\sigma)-C}{k})=C\sigma^{A^*},$$

then

$$\ln|b_n|e^{\lambda_n\sigma}\le\ln m(\sigma)=C\sigma^{A^*} \text{ or } \ln|b_n|^{-1}\ge\lambda_n\sigma-C\sigma^{A^*}.$$

When n is large enough, set $\sigma=(\frac{\lambda_n}{A^*})^{\frac{1}{A^*-1}}$, we have

$$\ln|b_n|^{-1}\ge\lambda_n(\frac{\lambda_n}{A^*})^{\frac{1}{A^*-1}}-C(\frac{\lambda_n}{A^*})^{\frac{A^*}{A^*-1}}=(A^*-C)\frac{\lambda_n}{A^*}^{\frac{A^*}{A^*-1}},$$

then

$$\ln[\ln|b_n|^{-\frac{1}{\lambda_n}}]\ge C+\frac{1}{A^*-1}\ln\lambda_n,$$

$$\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})\ge\frac{\alpha(\lambda_n)}{A^*-1}+C,$$

so we obtain

$$A-1\ge\overline{\lim_{n\to+\infty}}\frac{\alpha(\lambda_n)}{\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})}.$$

When $p=2,3\cdots$, we suppose that $A<\infty$. From the above proof, it follows that

$$\ln m(\sigma)<\alpha^{-1}[A^*\alpha(\sigma)] \quad or \quad \ln|b_n|<\alpha^{-1}[A^*\alpha(\sigma)]-\lambda_n\sigma.$$

Choose $\sigma=\sigma(\lambda_n)$ to be the unique root of equation

$$\sigma=\alpha^{-1}[\frac{1}{A^*}\alpha(\lambda_n)], \qquad (\sigma\to\infty\Leftrightarrow n\to\infty).$$

then $\ln|b_n|^{-\frac{1}{\lambda_n}}>\sigma-1$ or $\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})>\alpha(\sigma-1)$.

By (4), when σ is sufficiently large, we have $\alpha(\sigma-1)=(1+o(1))\alpha(\sigma)$, thus

$\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}}) \geq (1+o(1))\alpha(\sigma) = (1+o(1))[\frac{1}{A^*}\alpha(\lambda_n)]$, or

$$A^* = A+\varepsilon \geq \frac{\alpha(\lambda_n)}{\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})}(1+o(1)).$$

Now proceeding to limits, we obtain

$$A \geq \overline{\lim_{n\to+\infty}} \frac{\alpha(\lambda_n)}{\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})}.$$

The above inequality obviously holds when $A=\infty$.

Case II: Conversely, let

$$\overline{\lim_{n\to+\infty}} \frac{\alpha(\lambda_n)}{\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})} = B.$$

We suppose $B<\infty$. Then for $\forall\varepsilon>0$ and for all $n\geq n_0(\varepsilon)$, we have

$$\frac{\alpha(\lambda_n)}{\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})} < B+\varepsilon = B^*$$

or $\alpha(\lambda_n) < B^*\alpha(\ln|b_n|^{-\frac{1}{\lambda_n}})$, then $\alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)] < -\frac{1}{\lambda_n}\ln|b_n|$,

That is to say $\forall\varepsilon>0, \exists n_0>0$, when $n>n_0$

$$|b_n| < \exp\{-\lambda_n\alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)]\}. \tag{6}$$

From (2), there exists $r>0$, such that

$$\lambda_n > r\ln n \quad or \quad e^{-\lambda_n} < \frac{1}{n^r}.$$

In addition, when σ is sufficiently large, there exist $S>n_0$, so that

$$\lambda_S \leq \alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})] \leq \lambda_{S+1}. \tag{7}$$

we have

$$M(\sigma) \leq \sum_{n=1}^{n_0}|b_n|e^{\lambda_n\sigma} + \sum_{n=n_0+1}^{S}|b_n|e^{\lambda_n\sigma} + \sum_{n=S+1}^{\infty}|b_n|e^{\lambda_n\sigma} = A_0+A_1+A_2.$$

$$A_1 \leq e^{\lambda_S\sigma}\sum_{n=n_0+1}^{S}|b_n| \overset{(6)(7)}{\leq} \exp\{\sigma\alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})]\}\cdot\sum_{n=n_0+1}^{S}\exp\{-\lambda_n\alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)]\}$$

$$=\exp\{\sigma\alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})]\}\cdot\sum_{n=n_0+1}^{S}\frac{1}{n^{r\cdot\alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)]}} \leq C\exp\{\sigma\alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})]\}.$$

But in A_2, we can see that $\lambda_n > \alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})]$ then

$$\sigma < \alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)] - \frac{2}{r}.$$

From (6) and the above inequality, it follows that

$$A_2 \leq \sum_{n=S+1}^{\infty} \exp\{-\lambda_n \alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)]\} \cdot \exp\{\lambda_n \alpha^{-1}[\frac{1}{B^*}\alpha(\lambda_n)]\} \cdot e^{-\frac{2\lambda_n}{r}}$$

$$= \sum_{n=S+1}^{\infty} e^{-\frac{2\lambda_n}{r}} \leq \sum_{n=S+1}^{\infty} \frac{1}{n^2} < C.$$

Accordingly,

$$M(\sigma) \leq (1+o(1))C \exp\{\sigma\alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})],$$

by Lemma 2, we have

$$\alpha(\ln M(\sigma)) \leq (1+o(1))\alpha\{\sigma\alpha^{-1}[B^*\alpha(\sigma+\frac{2}{r})]\} = (1+o(1))(B^*+1)\alpha(\sigma),$$

thus

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln M(\sigma))}{\alpha(\sigma)} \leq B+1.$$

So from Case I and II, the proof is completed. □

Proof of Theorem 2 Case I: Let

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\ln m(\sigma))}{\alpha(\sigma)} = A.$$

we suppose $A < \infty$. Then for every $\varepsilon > 0, \exists\sigma_o(\varepsilon) > 0$, when $\sigma \geq \sigma_o$, it follows that

$$\alpha(\ln m(\sigma)) < (A+\varepsilon)\alpha(\sigma).$$

By Lemma 3, we have

$$\alpha[N(\sigma)] = (1+o(1))\alpha[2N(\sigma)] < \int_{\sigma}^{\sigma+2} N(x)dx \leq \alpha[\ln m(\sigma+2)] \leq (A+\varepsilon+o(1))\alpha(\sigma),$$

then

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(N(\sigma))}{\alpha(\sigma)} \leq A.$$

Case II: Let

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(N(\sigma))}{\alpha(\sigma)} = B.$$

we suppose $B < \infty$. Then for every $\varepsilon > 0, \exists\sigma_1(\varepsilon) > 0$, when $\sigma \geq \sigma_1$, it follows that

$$N(\sigma) < \alpha^{-1}[(B+\varepsilon)\alpha(\sigma)].$$

By Lemma 3, there exists $\sigma_1 > 0$ such that

$$\ln m(\sigma) - \ln m(\sigma_1) = \int_{\sigma_1}^{\sigma} N(\sigma)d\sigma \leq (\sigma - \sigma_1)\alpha^{-1}[(B+\varepsilon)\alpha(\sigma)].$$

By Lemma 1 and Corollary 1, Theorem 2 is obtained.

Proof of Theorem 3 We prove this theorem in two steps.

Case I: By Lemma 1, let

$$\overline{\lim_{\sigma\to+\infty}} \frac{\beta(\ln M(\sigma))}{[\beta(\sigma)]^{\rho}} = \overline{\lim_{\sigma\to+\infty}} \frac{\beta(\ln m(\sigma))}{[\beta(\sigma)]^{\rho}} = \tau.$$

we suppose $\tau < \infty$. Then for every $\varepsilon > 0, \exists \sigma_0(\varepsilon) > 0$, when $\sigma \geq \sigma_0$, it follows that

$$\frac{\beta(\ln m(\sigma))}{[\beta(\sigma)]^{\rho}} < \tau + \varepsilon = \overline{\tau},$$

or

$\ln m(\sigma) < \beta^{-1}\{\overline{\tau}[\beta(\sigma)]^{\rho}\}$, then $\ln |b_n| < \beta^{-1}\{\overline{\tau}[\beta(\sigma)]^{\rho}\} - \lambda_n\sigma.$

Choose $\sigma = \sigma(\lambda_n)$ to be the unique root of equation

$$\sigma = \beta^{-1}\{[\frac{1}{\overline{\tau}}\beta(\lambda_n)]^{\frac{1}{\rho}}\}, \qquad (\sigma \to \infty \Leftrightarrow n \to \infty).$$

then $\ln |b_n|^{-\frac{1}{\lambda_n}} > \sigma - 1$ or $\beta(\ln |b_n|^{-\frac{1}{\lambda_n}}) > \beta(\sigma - 1)$.

By (4), when σ is sufficiently large, we have $\beta(\sigma-1) = (1+o(1))\beta(\sigma)$, thus $[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho} \geq (1+o(1))[\beta(\sigma)]^{\rho} = (1+o(1))[\frac{1}{\overline{\tau}}\beta(\lambda_n)]$, or

$$\overline{\tau} = \tau + \varepsilon \geq \frac{\beta(\lambda_n)}{[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho}}(1+o(1)).$$

Now proceeding to limits, we obtain

$$\tau \geq \overline{\lim_{n\to+\infty}} \frac{\beta(\lambda_n)}{[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho}}.$$

The above inequality obviously holds when $\tau = \infty$.

Case II: Conversely, let

$$\overline{\lim_{n\to+\infty}} \frac{\beta(\lambda_n)}{[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho}} = B.$$

We suppose $B < \infty$. Then for $\forall \varepsilon > 0$ and for all $n \geq n_0(\varepsilon)$, we have

$$\frac{\beta(\lambda_n)}{[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^{\rho}} < B + \varepsilon = B^*$$

or $\beta(\lambda_n) < B^*[\beta(\ln |b_n|^{-\frac{1}{\lambda_n}})]^\rho$, then $\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\} < -\frac{1}{\lambda_n}\ln |b_n|$,

That is to say $\forall\varepsilon > 0, \exists n_0 > 0,$ when $n > n_0$

$$|b_n| < \exp\{-\lambda_n\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\}\}. \tag{8}$$

From (2), there exists $r > 0$, such that

$$\lambda_n > r\ln n \quad or \quad e^{-\lambda_n} < \frac{1}{n^r}.$$

In addition, when σ is sufficiently large, there exist $S > n_0$, so that

$$\lambda_S \le \beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\} \le \lambda_{S+1}. \tag{9}$$

we have

$$M(\sigma) \le \sum_{n=1}^{n_0} |b_n|e^{\lambda_n\sigma} + \sum_{n=n_0+1}^{S} |b_n|e^{\lambda_n\sigma} + \sum_{n=S+1}^{\infty} |b_n|e^{\lambda_n\sigma} = A_0' + A_1' + A_2'.$$

$$A_1' \le e^{\lambda_S\sigma} \sum_{n=n_0+1}^{S} |b_n| \overset{(8),(9)}{\le} \exp\{\sigma\beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\} \cdot \sum_{n=n_0+1}^{S} \exp\{-\lambda_n\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\}\}$$

$$= \exp\{\sigma\beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\} \cdot \sum_{n=n_0+1}^{S} \frac{1}{n^{r\cdot\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\}}} \le C\exp\{\sigma\beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\}.$$

But in A_2', we can see that $\lambda_n > \beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\}$ then

$$\sigma < \beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\} - \frac{2}{r}.$$

From (8) and above inequality, it follows that

$$A_2' \le \sum_{n=S+1}^{\infty} \exp\{-\lambda_n\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\}\} \cdot \exp\{\lambda_n\beta^{-1}\{[\frac{1}{B^*}\beta(\lambda_n)]^{\frac{1}{\rho}}\}\} \cdot e^{-\frac{2\lambda_n}{r}}$$

$$= \sum_{n=S+1}^{\infty} e^{-\frac{2\lambda_n}{r}} \le \sum_{n=S+1}^{\infty} \frac{1}{n^2} < C.$$

Accordingly,

$$M(\sigma) \le (1 + o(1))C\exp\{\sigma\beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\}$$

then

$$\beta(\ln M(\sigma)) \le (1 + o(1))\beta\{\sigma\beta^{-1}\{B^*[\beta(\sigma + \frac{2}{r})]^\rho\}\}.$$

Hence, by Corollary 2, it follows that

$$\overline{\lim_{\sigma\to+\infty}} \frac{\beta(\ln M(\sigma))}{[\beta(\sigma)]^\rho} \le B.$$

From Case I and Case II, the result follows. □

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