A gap theorem for complete space-like hypersurface with constant scalar curvature in locally symmetric Lorentz spaces

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Abstract

Let M^n be a complete space-like hypersurface with constant scalar curvature in locally symmetric Lorentz space N_1^{n+1}, S be the squared norm of the second fundamental form of M^n in N_1^{n+1}. In this paper, we obtain a gap property of S: if $nP \leq \sup S \leq D(n, P)$ for some constants P and $D(n, P)$, then either $\sup S = nP$ and M^n is totally umbilical, or $\sup S = D(n, P)$ and M^n has two distinct principal curvatures.

Key Words: Constant scalar curvature, space-like hypersurface, second fundamental form, locally symmetric Lorentz space.

1. Introduction and main theorem

Let N_p^{n+p} be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p (≥ 0). It is called a semi-definite space of index p. In particular, N_1^{n+1} is called a Lorentz space, with de Sitter space S_1^{n+1} as its special case. A hypersurface M of a Lorentz space is said to be space-like if the induced metric on M from that of the Lorentz space is positive definite. When the Lorentz space N_1^{n+1} is of constant curvature c, we call it Lorentz space form, denoted by $N_1^{n+1}(c)$.

The motivation to the study of space-like hypersurfaces in space-times comes from its relevance in general relativity. Moreover, the Goddard's Conjecture [4] encouraged the study of compact or complete space-like hypersurfaces with constant mean curvature in de Sitter space S_1^{n+1} [7, 8]. In fact, constant mean curvature hypersurfaces are relevant for studying propagation of gravitational waves. Another natural Goddard-like problem is to study constant scalar curvature hypersurfaces. Partial results were obtained in [2, 13, 14]. Li [5] proved that a compact space-like hypersurface with constant normalized scalar curvature $R > 1$ in S_1^{n+1} must be totally umbilical. Recently, A. Brasil Jr. et al. [1] generalize Li's results to the complete case and obtain a classification theorem.

In this paper, we substitute the ambient space S_1^{n+1} for more general a large class of Lorentz spaces and study its space-like hypersurfaces with constant normalized scalar curvature. First of all, we recall that, for

constants c_1 and c_2, Jin Ok Baek et al. [9] introduced the class of $(n+1)$-dimensional Lorentz spaces N_1^{n+1} of index 1 which satisfy the following two conditions (here and in the sequel, K_N denotes the sectional curvature on N_1^{n+1}):

(1) for any space-like vector u and any time-like vector v

$$K_N(u,v) = -\frac{c_1}{n};$$
(1.1)

(2) for any space-like vectors u and v

$$K_N(u,v) \geq c_2.$$
(1.2)

This class of Lorentz spaces, denoted by $\mathcal{M}$, contains several examples, for instance,

Example 1. The Lorentz space form $N_1^{n+1}(c) \in \mathcal{M}$, where $-(c_1/n) = c_2 = c$.

Example 2. Semi-Riemannian product manifold $H_1^k(-c_1/n) \times N^{n+1-k}(c_2) \in \mathcal{M}$, $c_1 > 0$, and $\mathbf{R}_1^k \times S^{n+1-k}(1) \in \mathcal{M}$. In particular, $\mathbf{R}_1^1 \times S^n(1)$ is so-called *Einstein Static Universe*. Of course, these are all Lorentz spaces, but not Lorentz space forms.

Example 3. *Robertson-Walker spacetime* $N(c,f) = I \times_f N^3(c) \in \mathcal{M}$, where I denote an open interval of $\mathbf{R}_1^1$ and $f > 0$ an appropriate smooth function defined on the interval I, $N^3(c)$ a 3-dimensional Riemannian manifold of constant curvature c.

In [9], the authors investigated complete space-like hypersurfaces with constant mean curvature in a locally symmetric Lorentz space $N_1^{n+1} \in \mathcal{M}$, They give an optimal estimate of the squared norm of the second fundamental form of such hypersurfaces and the characterization of totally umbilical hypersurfaces. One natural problem is, for the hypersurfaces with constant normalized scalar curvature in locally symmetric Lorentz space $N_1^{n+1} \in \mathcal{M}$, what kind rigidity and classification theorems will be have? In this paper, we shall discuss this problem.

In order to present our theorem, we firstly recall that the scalar curvature of locally symmetric Lorentz spaces is constant. On the other hand, if we denote $\bar{R}_{CD}$ as the components of the Ricci tensor of $N_1^{n+1} \in \mathcal{M}$, then the scalar curvature $\bar{R}$ of N_1^{n+1} is

$$\bar{R} = \sum_A \epsilon_A \bar{R}_{AA} = -2\sum_i K_{(n+1)ii(n+1)} + \sum_{i,j} K_{ijji}$$

$$= 2c_1 + \sum_{i,j} K_{ijji};$$

hence, $\sum\limits_{i,j} K_{ijji}$ is constant. This fact together with the formula (2.3) suggests us to define a constant P by

$$n(n-1)P = n^2H^2 - S = \sum_{i,j} K_{ijji} - n(n-1)R.$$
(1.3)

Using (1.3), we can finally establish our main result:

Theorem 1.1 *Let $M^n (n \geq 3)$ be a complete space-like hypersurface with constant normalized scalar curvature R in a locally symmetric Lorentz space $N_1^{n+1} \in \mathcal{M}$, $0 \leq P \leq c$. If the squared norm S of the second fundamental form of M satisfies: $nP \leq \sup S \leq D(n, P)$, then*

i) $\sup S = nP$ and M^n is totally umbilical hypersurface; or

ii) $\sup S = D(n, P)$ and M^n has two distinct principal curvatures, where

$$D(n, P) = \frac{n}{(n-2)(nP - 2c)}[n(n-1)P^2 - 4c(n-1)P + nc^2],$$

$c = \frac{c_1}{n} + 2c_2$ *and P determined by (1.3).*

In particular, let $N_1^{n+1} = S_1^{n+1}$ in Theorem 1.1, then $-\frac{c_1}{n} = c_2 = c = 1$, so $P = 1 - R$ from (1.3). When $\sup S = D(n, P)$, following from Theorem 1.1, we know M^n has two distinct principal curvatures; in fact M^n is hyperbolic cylinder $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$. In this case, Theorem 1.1 generalizes X. Liu's result in [6] to more general situations. We also refer readers to compare Theorem 1.1 with [1, Theorem 1.1], where they assumed that the supremum of squared mean curvature bounded from above by certain positive constant.

2.   Estimates of the laplacian $\triangle h_{ij}$ and $\triangle S$

In this section, we calculate the Laplacian of the second fundamental form and its squared norm of space-like hypersurface in locally symmetric Lorentz space belong to the class $\mathcal{M}$. We shall make use of the following convention on the ranges of indices throughout this paper, unless otherwise stated:

$$1 \leq A, B, C, \cdots \leq n + 1; \quad 1 \leq i, j, k, \cdots \leq n.$$

2.1 General setting for Lorentz space and its space-like hypersurfaces

We assume that (N, h) is an $(n + 1)$-dimensional Lorentz space and (M, g) is a space-like hypersurface in N. Choose a local field of orthonormal frames $e_1, \ldots, e_{n+1}$ in N such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and the other is normal to M. Namely, $e_1, \ldots, e_n$ are space-like vectors and e_{n+1} is a time-like vector. Let $\{\omega_A\}$ and ω_{AB} be the fields of dual frames and the connection 1-forms of N, respectively. Then the indefinite Riemannian metric tensor h of N is given by $h = \sum_A \epsilon_A \omega_A \otimes \omega_A$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$, the induced metric g of M is given by $g = \sum_i \omega_i \otimes \omega_i$. Restricting the frames to M, we have

$$\omega_{(n+1)i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji},$$

where the h_{ij} are the coefficients of the second fundamental form of M. Then $H = \frac{1}{n}\sum_j h_{jj}$ and $S = \sum_{i,j} h_{ij}^2$ are the mean curvature and squared norm of the second fundamental form of hypersurface M, respectively. As usual, we denote h_{ijk} and h_{ijkl} the first and the second covariant derivatives of h_{ij}, K_{ABCD} the curvature tensor of N. The Gauss equation, components R_{ij} of Ricci tensor and normalized scalar curvature R of M are given by

$$R_{ijkl} = K_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \tag{2.1}$$

$$R_{ij} = \sum_k K_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)R = \sum_{j,k} K_{kjjk} - n^2 H^2 + S, \tag{2.3}$$

The components $K_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor K are defined by

$$\sum_E \epsilon_E K_{ABCD;E}\omega_E$$

$$= \mathrm{d}K_{ABCD} - \sum_E \epsilon_E(K_{EBCD}\omega_{EA} + K_{AECD}\omega_{EB} + K_{ABED}\omega_{EC} + K_{ABCE}\omega_{ED}),$$

restricting on M, $K_{(n+1)ijk;l}$ is given by

$$K_{(n+1)ijk;l} = K_{(n+1)ijkl} + K_{(n+1)i(n+1)k}h_{jl} + K_{(n+1)ij(n+1)}h_{kl} + \sum_m K_{mijk}h_{ml},$$

where $K_{(n+1)ijkl}$ denote the covariant derivative of $K_{(n+1)ijk}$ as a tensor on M so that

$$\sum_l K_{(n+1)ijkl}\omega_l = \mathrm{d}K_{(n+1)ijk} - \sum_l K_{(n+1)ljk}\omega_{li} - \sum_l K_{(n+1)ilk}\omega_{lj} - \sum_l K_{(n+1)ijl}\omega_{lk}.$$

Now, we can write down the Laplacian of the second fundamental form (see [9, formula (2.21)] for a proof) as

$$\begin{aligned}
\triangle h_{ij} =& (nH)_{ij} + \sum_k \left(K_{(n+1)ijk;k} + K_{(n+1)kik;j}\right) \\
& - \sum_k (h_{kk}K_{(n+1)ij(n+1)} + h_{ij}K_{(n+1)k(n+1)k}) \\
& - \sum_{k,l} (2h_{kl}K_{lijk} + h_{jl}K_{lkik} + h_{il}K_{lkjk}) \\
& - nH\sum_l h_{il}h_{lj} + Sh_{ij},
\end{aligned} \tag{2.4}$$

thus

$$\begin{aligned}
\frac{1}{2}\triangle S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\triangle h_{ij} \\
=& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\left(K_{(n+1)ijk;k} + K_{(n+1)kik;j}\right)h_{ij} \\
& - \left(nH\sum_{i,j} h_{ij}K_{(n+1)ij(n+1)} + S\sum_k K_{(n+1)k(n+1)k}\right) \\
& - \sum_{i,j,k,l} 2(h_{kl}h_{ij}K_{lijk} + h_{il}h_{ij}K_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij} + S^2.
\end{aligned} \tag{2.5}$$

2.2 Estimates of $\triangle S$ when $N \in \mathcal{M}$ is a locally symmetric Lorentz space

Now, let $N_1^{n+1} \in \mathcal{M}$ is a locally symmetric Lorentz space, i.e. $K_{ABCD;E} = 0$. We shall continue to estimate $\triangle S$, more precisely, to estimate the middle three terms in the right-hand side of (2.5). In order to do this, we will choose $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$, then local symmetry of N_1^{n+1} implies that

$$\sum_{i,j,k} \left(K_{(n+1)ijk;k} + K_{(n+1)kik;j} \right) h_{ij} = 0. \tag{2.6}$$

Because of $N_1^{n+1} \in \mathcal{M}$, using curvature assumptions (1.1) and (1.2), we get

$$
\begin{aligned}
&- \left(nH \sum_{i,j} h_{ij} K_{(n+1)ij(n+1)} + S \sum_k K_{(n+1)k(n+1)k} \right) \\
&= - \left(nH \sum_k \lambda_k K_{(n+1)kk(n+1)} - S \sum_k K_{(n+1)kk(n+1)} \right) \\
&= \sum_k (S - nH\lambda_k) \frac{c_1}{n} \\
&= c_1 (S - nH^2).
\end{aligned}
\tag{2.7}
$$

Also we have

$$
\begin{aligned}
- \sum_{i,j,k,l} 2(h_{kl} h_{ij} K_{lijk} + h_{il} h_{ij} K_{lkjk}) &= -2 \sum_{j,k} (\lambda_j \lambda_k - \lambda_k^2) K_{kjjk} \\
&\geq c_2 \sum_{j,k} (\lambda_j - \lambda_k)^2 \\
&= 2c_2 (nS - n^2 H^2).
\end{aligned}
\tag{2.8}
$$

Substituting (2.6), (2.7) and (2.8) into (2.5), we finally obtain

$$
\begin{aligned}
\frac{1}{2} \triangle S \geq{} & \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} \\
& + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH \sum_i \lambda_i^3),
\end{aligned}
\tag{2.9}
$$

where λ_j are principal curvatures of M.

3. Key Lemmas

In order to prove the main theorem, we need some lemmas. We quote firstly an asymptotic maximum principle at infinity for complete manifolds due to Omori [11] and Yau [12] (notice that in Lemma 3.1 below, we assumed that C^2-function F bounded from below, there is another version of this maximum principle where F is bounded from above).

Lemma 3.1 ([11], [12]) *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M; then, for any $\epsilon \geq 0$, there exists a point*

$p \in M$ *such that*

$$\|\operatorname{grad} F(p)\| \leq \epsilon, \quad \triangle F(p) \geq -\epsilon, \quad \inf F \leq F(p) \leq \inf F + \epsilon.$$

Lemma 3.2 ([10])   *Let* $\{\mu_i\}_{i=1}^n$ *be real numbers satisfying* $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = B$, *then*

$$\big| \sum_i \mu_i^3 \big| \leq \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}}$$

and the equality holds if and only if at least $n-1$ *of the* μ_i *'s are equal, i.e.*

$$\mu_1 = \cdots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}B, \quad \mu_n = \sqrt{\frac{n-1}{n}}B.$$

Lemma 3.3   *Let* M^n *be a complete space-like hypersurface with constant normalized scalar curvature* R *in locally symmetric Lorentz space* $N_1^{n+1} \in \mathcal{M}$. *If* $P \geq 0$, *then* $\sum_{i,j,k} h_{ijk}^2 \geq n^2 \|\operatorname{grad} H\|^2$.

   Proof   Notice that P is constant, differentiating $n^2 H^2 - S = n(n-1)P$ exteriorly yields $n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}$, then Cauchy-Schwarz inequality leads to

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k \Big(\sum_{i,j} h_{ij} h_{ijk} \Big)^2 \leq \Big(\sum_{i,j} h_{ij}^2 \Big) \Big(\sum_{i,j,k} h_{ijk}^2 \Big),$$

namely,

$$n^4 H^2 \|\operatorname{grad} H\|^2 \leq S \sum_{i,j,k} h_{ijk}^2,$$

together with the fact $n^2 H^2 - S \geq 0$ since $P \geq 0$; we conclude lemma 3.3. $\square$

   The following lemma 3.4 will play a key role in the proof of our Theorem 1.1. Before we state it, we choose $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$, then applying self-adjoint operator $\square$, introduced by Q.M. Cheng and S.T. Yau in [3], to the function nH and using (2.3), we obtain

$$\begin{aligned} \square(nH) :&= \sum_{i,j} (nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \frac{1}{2}\triangle(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\triangle\big(\sum_{i,j} K_{ijji} - n(n-1)R \big) + \frac{1}{2}\triangle S - n^2 \|\operatorname{grad} H\|^2 - \sum_i \lambda_i (nH)_{ii}. \end{aligned} \quad (3.1)$$

According to (1.3), the first term on the right-hand side of (3.1) vanishes; and substituting (2.9) into (3.1), we conclude that

$$\begin{aligned} \square(nH) \geq & \sum_{i,j,k} h_{ijk}^2 - n^2 \|\operatorname{grad} H\|^2 \\ & + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH\sum_i \lambda_i^3). \end{aligned} \quad (3.2)$$

Lemma 3.4 *With the same assumptions as Theorem 1.1, we have*

$$\frac{n-1}{n}(S - nP)\phi_P(S) \leq \square(nH) \leq nC|\triangle(nH)|,$$

where $\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{(n(n-1)P + S)(S - nP)}, c = \frac{c_1}{n} + 2c_2$, *C is an upper bound of* $|H| + \sqrt{S}$, *and* $\phi_P(\sup S) \geq 0$.

    Proof   Putting $\mu_i = \lambda_i - H, B = \sum_i \mu_i^2$, then

$$\sum_i \mu_i = 0, \quad B = S - nH^2,$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3HB + nH^3.$$

Applying lemma 3.2, we get

$$-nH\sum_i \lambda_i^3 = -n^2H^4 - 3nH^2B - nH\sum_i \mu_i^3$$

$$\geq 2n^2H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{3}{2}}.$$

Putting into (3.2) and using lemma 3.3 leads to

$$\square(nH) \geq B\left\{nc - nH^2 + B - \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{1}{2}}\right\}. \tag{3.3}$$

It follows from (1.3) that

$$B = S - nH^2 = \frac{n-1}{n}(S - nP). \tag{3.4}$$

Substituting into (3.3), we have

$$\square(nH) \geq \frac{n-1}{n}(S - nP)\phi_H(S), \tag{3.5}$$

where

$$\phi_H(S) = nc - 2nH^2 + S - \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|\sqrt{S - nH^2}. \tag{3.6}$$

Using (3.4), we can rewrite (3.6) as

$$\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S$$

$$- \frac{n-2}{n}\sqrt{(n(n-1)P + S)(S - nP)},$$

so, (3.5) becomes

$$\square(nH) \geq \frac{n-1}{n}(S - nP)\phi_P(S). \tag{3.7}$$

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On the other hand,

$$\Box(nH) \leq |\sum_i (nH - h_{ii})(nH)_{ii}|$$

$$= n(\|H\| + \sqrt{S})|\triangle(nH)|$$

$$\leq nC|\triangle(nH)|.$$

Finally, we shall prove $\phi_P(\sup S) \geq 0$. It is easy to check that the assumption $\sup S \leq D(n, P)$ implies

$$\left(nc - 2(n-1)P + \frac{n-2}{n} \sup S\right)^2 \geq \frac{(n-2)^2}{n^2}(n(n-1)P + \sup S)(\sup S - nP). \tag{3.8}$$

Because of $P \leq c$, $\sup S \geq nP$, (3.8) equivalent to

$$nc - 2(n-1)P + \frac{n-2}{n} \sup S \geq \frac{(n-2)}{n}\sqrt{(n(n-1)P + \sup S)(\sup S - nP)},$$

which completes the proof of Lemma 3.4. $\qquad\qquad\square$

4. Proof of Theorem 1.1

Since $N_1^{n+1} \in \mathcal{M}$ is a locally symmetric Lorentz space, from (2.2), it is easy to check that $\text{Ric}^M \geq nc_2 - \frac{n^2 H^2}{4}$. Since S is bounded by assumption, we know that the Ricci curvature of M is bounded from below. Thus we may apply Omori and Yau's maximum principle to the function F defined by $F = \frac{1}{\sqrt{1+(nH)^2}}$, which is a positive smooth function on M. A straightforward calculation will give

$$\|\text{grad}\, F\|^2 = \frac{1}{4}\frac{\|\text{grad}\,(nH)^2\|^2}{(1+(nH)^2)^3}, \tag{4.1}$$

$$\triangle F = -\frac{1}{2}\frac{\triangle(nH)^2}{(1+(nH)^2)^{\frac{3}{2}}} + \frac{3}{4}\frac{\|\text{grad}\,(nH)^2\|^2}{(1+(nH)^2)^{\frac{5}{2}}}. \tag{4.2}$$

According to Lemma 3.1, there exists a point $\{p_k\}$ such that

$$\lim_{k\to\infty} F(p_k) = \inf F, \quad \triangle F(p_k) > -\frac{1}{k}, \quad \|\text{grad}\, F\|^2(p_k) < \frac{1}{k^2}. \tag{4.3}$$

Applying (4.3) to (4.1) and (4.2), we obtain

$$-\frac{1}{k} \leq -\frac{1}{2}\frac{\triangle(nH)}{(1+(nH)^2)^{\frac{3}{2}}}(p_k) + \frac{3}{k^2}\left(1+(nH)^2(p_k)\right)^{\frac{1}{2}},$$

hence,

$$\frac{\triangle(nH)}{(1+(nH)^2)^2}(p_k) < \frac{2}{k}\left(\frac{1}{\sqrt{1+(nH)^2(p_k)}} + \frac{3}{k}\right). \tag{4.4}$$

Evaluating at the points p_k, then Lemma 3.4 gives

$$\frac{n-1}{n}(S(p_k) - nP)\phi_P(S(p_k)) \leq \Box(nH)(p_k) \leq nC|\triangle(nH)|(p_k). \tag{4.5}$$

Let $k \to \infty$, it follows from (4.4) and the fact $\lim\limits_{k\to\infty}(nH)(p_k) = \sup(nH)$ that the right-hand side of (4.5) goes to zero. Therefore, we have

$$\frac{n-1}{n}(\sup S - nP)\phi_P(\sup S) \leq 0.$$

On the other hand, we have already shown in lemma 3.4 that $\phi_P(\sup S) \geq 0$, together with the assumption $\sup S \geq nP$, it must be

$$\frac{n-1}{n}(\sup S - nP)\phi_P(\sup S) = 0,$$

which implies

$$\sup S = nP,$$

or

$$\phi_P(\sup S) = 0.$$

(i) When $\sup S = nP$, it follows from (3.4) that $\sup B = \frac{n-1}{n}(\sup S - nP) = 0$, that is, $B \equiv 0$. Thus $S = nH^2$ and we infer that M is totally umbilical.

(ii) When $\phi_P(\sup S) = 0$, that is

$$nc - 2(n-1)P + \frac{n-2}{n}\sup S = \frac{n-2}{n}\sqrt{(n(n-1)P + \sup S)(\sup S - nP)},$$

which leads to

$$\sup S = \frac{n}{(n-2)(nP - 2c)}[n(n-1)P^2 - 4c(n-1)P + nc^2],$$

at that time, the equalities in lemma 3.2 and 3.3 hold. Hence, M^n has two distinct principal curvatures. We complete the proof of Theorem 1.1. $\qquad\qquad\square$

If M is a compact space-like hypersurface in locally symmetric Lorentz space $N_1^{n+1} \in \mathcal{M}$, then $\int_M \Box(nH)\mathrm{dv} = 0$ since $\Box$ is self-adjoint operator. Similar arguments as in the proof of theorem 1.1 will show that either $S = nP$, that is $S = nH^2$ and M^n is totally umbilical, or

$$S = \frac{n}{(n-2)(nP - 2c)}[n(n-1)P^2 - 4c(n-1)P + nc^2],$$

which implies that M^n has two distinct principal curvatures. Thus we obtain the following theorem.

Theorem 4.1   *Let M be a compact space-like hypersurface with constant normalized scalar curvature in locally symmetric Lorentz space $N_1^{n+1} \in \mathcal{M}$, $0 \leq P \leq c$. If the squared norm S of the second fundamental form of M satisfies: $nP \leq S \leq D(n, P)$, where $D(n, P)$ the same as in Theorem 1.1, then*

i) $S = nP$ *and* M^n *is totally umbilical; or*

ii) $S = D(n, P)$ *and* M^n *has two distinct principal curvatures.*

Acknowledgememt

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