

B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms

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Abstract

In this paper some B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms are established.

Key Words: Chen's invariant, squared mean curvature, quaternionic space form, slant submanifold.

1. Introduction

One of the most powerful tools to find relationships between intrinsic invariants and extrinsic invariants of a submanifold is provided by Chen's invariants. This theory was initiated in [9] where B.-Y. Chen established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature (both being intrinsic invariants) and squared mean curvature (the main extrinsic invariant).

On the other hand, the slant submanifolds of complex manifolds were defined in [8] and Chen-like inequalities for slant submanifolds in complex space forms and in generalized complex space forms were obtained in [27] and [23]. The slant submanifolds of contact manifolds were introduced in [24], and Chen-like inequalities for slant submanifolds of Sasakian space forms were obtained in [14]. The study of slant submanifolds in S -manifolds and B.-Y. Chen inequalities in S -space forms has been realized in [6] and [7]. Other Chen-like inequalities in different settings and submanifolds satisfying Chen's equality can be found in [1], [2], [3], [5], [12], [13], [15], [17], [18], [19], [20], [28], [29], [30], [32], [34].

Some B.-Y. Chen inequalities for totally real submanifolds in quaternionic space forms are established in [33]. Recently, Şahin [31] introduced the slant submanifolds of quaternionic Kähler manifolds, as a natural generalization of both quaternionic and totally real submanifolds. Motivated by the above considerations, we'll be studying Chen-like inequalities in the context of slant submanifolds in quaternionic space forms.

The paper is organized as follows: in Section 2, following [31], we collect basic definitions, some formulas and results concerning the slant submanifolds of quaternionic Kähler manifolds for later use. In Section 3, following [11], we recall a string of Riemannian invariants on a manifold. In Section 4 we establish a Chen-like inequality between Chen's δ invariant and squared mean curvature for θ -slant submanifolds in a quaternionic

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space form. In Section 5 we give another Chen-like inequality between Chen's $\delta(n_1, \dots, n_k)$ -invariant and squared mean curvature for θ -slant submanifolds in a quaternionic space form. In Section 6 we establish a sharp inequality between Ricci curvature and the squared mean curvature for slant submanifolds in quaternionic space forms.

2. Slant submanifolds of quaternionic kähler manifolds

Let \overline{M} be a differentiable manifold and assume that there is a rank 3-subbundle σ of $End(T\overline{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J_\alpha^2 = -Id, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \quad (1)$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ . Moreover, (\overline{M}, σ) is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $4m$.

A Riemannian metric \overline{g} on \overline{M} is said to be adapted to the almost quaternionic structure σ if it satisfies the relation

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y), \forall \alpha \in \{1, 2, 3\} \quad (2)$$

for all vector fields X, Y on \overline{M} and any canonical local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, $(\overline{M}, \sigma, \overline{g})$ is said to be an almost quaternionic Hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of \overline{g} , then $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that we have for all $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \quad (3)$$

for any vector field X on \overline{M} , where the indices are taken from $\{1, 2, 3\}$ modulo 3 (see [22]).

Remark 2.1 For a submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$, we denote by g the metric tensor induced on M . If ∇ is the covariant differentiation induced on M , the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM) \quad (4)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp), \quad (5)$$

where h is the second fundamental form of M , ∇^\perp is the connection on the normal bundle and A_N is the shape operator of M with respect to N . The shape operator A_N is related to h by

$$g(A_N X, Y) = \overline{g}(h(X, Y), N) \quad (6)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

If we denote by \overline{R} and R the curvature tensor fields of $\overline{\nabla}$ and ∇ we have the Gauss equation:

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \overline{g}(h(X, W), h(Y, Z)) \\ &\quad - \overline{g}(h(X, Z), h(Y, W)), \end{aligned} \tag{7}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

If $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \dots, e_{4m}\}$ is an orthonormal basis of $T_p^\perp M$, where $p \in M$, we denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 4m\}$$

and

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

The submanifold M is called totally geodesic if the second fundamental form vanishes identically and totally umbilical if there is a real number λ such that $h(X, Y) = \lambda g(X, Y)H$ for any tangent vectors X, Y on M .

If $H = 0$, then the submanifold M is said to be minimal.

A submanifold M of a quaternionic Kähler manifold \overline{M} is called a quaternion submanifold (resp. totally real submanifold) if each tangent space of M is carried into itself (resp. into the normal space) by each section in σ .

Definition 2.2 [31] *A submanifold M of a quaternionic Kähler manifold \overline{M} is said to be a slant submanifold if for each non-zero vector X tangent to M at p , the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM , $\alpha \in \{1, 2, 3\}$ is constant, i.e. it does not depend on choice of $p \in M$ and $X \in T_pM$.*

We can easily see that quaternionic submanifolds are slant submanifolds with $\theta = 0$ and totally-real submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a quaternionic Kähler manifold is said to be proper (or θ -slant proper) if it is neither quaternionic nor totally real.

B. Şahin obtained the next characterization for slant submanifolds of quaternionic Kähler manifolds [31]:

Theorem 2.3 *Let M be a submanifold of a quaternionic Kähler manifold \overline{M} . Then M is slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$P_\beta P_\alpha X = \lambda X, \quad \forall X \in \Gamma(TM), \quad \alpha, \beta \in \{1, 2, 3\}, \tag{8}$$

where $P_\alpha X$ denote the tangential component of $J_\alpha X$. Furthermore, in such case, if θ is the slant angle of M , then it satisfies $\lambda = -\cos^2 \theta$.

From the above theorem we can deduce that if M is a θ -slant submanifold of a quaternionic Kähler manifold \overline{M} , then we have for any $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \{1, 2, 3\}$ (see [31])

$$g(P_\alpha X, P_\beta Y) = \cos^2 \theta g(X, Y). \quad (9)$$

Moreover, if M is a θ -slant proper submanifold, we can choose a canonical orthonormal local frame $\{e_1, e_2, \dots, e_{2s}\}$, called an adapted slant frame, as follows: let e_1 be a local unit vector field tangent to M and we settle $\alpha \in \{1, 2, 3\}$. We define now the unit tangent vector $e_2 = \sec \theta P_\alpha e_1$. If $\dim M > 2$, then, by induction, for each $i \in \{1, \dots, s-1\}$, we may choose a unit vector e_{2i+1} of M orthogonal to e_1, \dots, e_{2i} and we can define $e_{2i+2} = \sec \theta P_\alpha e_{2i+1}$. So, we can conclude that every proper slant submanifold of a quaternionic Kähler manifold is of even dimension.

3. Riemannian invariants

Let M be an n -dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, $p \in M$. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space $T_p M$, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i < j} K(e_i \wedge e_j) \quad (10)$$

One denotes

$$(\inf K)(p) = \inf \{K(\pi) | \pi \subset T_p M, \dim \pi = 2\} \quad (11)$$

and Chen first invariant is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p). \quad (12)$$

Suppose L is an r -dimensional subspace of $T_p M$, $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L . We define the scalar curvature $\tau(L)$ of the r -plane section L by

$$\tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta). \quad (13)$$

For an integer $k \geq 0$ we denote by $S(n, k)$ the set of k -tuples (n_1, \dots, n_k) of integers ≥ 2 satisfying $n_1 < n$, $n_1 + \dots + n_k \leq n$. We denote by $S(n)$ the set of unordered k -tuples with $k \geq 0$ for a fixed n .

For each k -tuples $(n_1, \dots, n_k) \in S(n)$, Chen introduced a Riemannian invariant $\delta(n_1, \dots, n_k)$ defined by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - S(n_1, \dots, n_k)(p), \quad (14)$$

where

$$S(n_1, \dots, n_k)(p) = \inf \{\tau(L_1) + \dots + \tau(L_k)\}; \quad (15)$$

L_1, \dots, L_k run over all k mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j \in \{1, \dots, k\}$. Also, we denote by $d(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ the real constants given by

$$d(n_1, \dots, n_k) = \frac{n^2(n+k-1 - \sum_{j=1}^k n_j)}{2(n+k - \sum_{j=1}^k n_j)}$$

and

$$b(n_1, \dots, n_k) = \frac{n(n-1) - \sum_{j=1}^k n_j(n_j-1)}{2}.$$

4. First Chen-like inequality for slant submanifolds in quaternionic space forms

Let $(\overline{M}, \overline{g}, \sigma)$ be a quaternionic Kähler manifold and let X be a non-null vector on \overline{M} . Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c . It is well-known that a quaternionic Kähler manifold $(\overline{M}, \overline{g}, \sigma)$ is a quaternionic space form (denoted $\overline{M}(c)$) if and only if its curvature tensor is given by (see [22])

$$\begin{aligned} \overline{R}(X, Y)Z &= \frac{c}{4} \{ \overline{g}(Z, Y)X - \overline{g}(X, Z)Y + \sum_{\alpha=1}^3 [\overline{g}(Z, J_\alpha Y)J_\alpha X - \\ &\quad - \overline{g}(Z, J_\alpha X)J_\alpha Y + 2\overline{g}(X, J_\alpha Y)J_\alpha Z] \} \end{aligned} \quad (16)$$

for all vector fields X, Y, Z on \overline{M} and any local basis $\{J_1, J_2, J_3\}$ of σ .

We recall now the following lemma of Chen [9] for later uses.

Lemma 4.1 *If a_1, \dots, a_n, b are $n+1$ real numbers, with $n \geq 2$, such that:*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right),$$

then

$$2a_1a_2 \geq b,$$

with equality holding if and only if

$$a_1 + a_2 = a_3 = \dots = a_n.$$

We prove next a Chen-like inequality for proper slant submanifolds in quaternionic space forms.

Theorem 4.2 *Let M^n be a θ -slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then, for each point $p \in M$, we have:*

$$\delta_M(p) \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + \frac{c}{4}(n+1+9\cos^2\theta) \right\}. \quad (17)$$

Equality in (17) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{4m}\}$ of $T_p^\perp M$ such that the shape operators $A_r \equiv A_{e_r}$, $r \in \{n+1, \dots, 4m\}$, take

the following forms:

$$A_{n+1} = \begin{pmatrix} c & 0 & 0 & \dots & 0 \\ 0 & c & 0 & \dots & 0 \\ 0 & 0 & c+d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c+d \end{pmatrix} \quad (18)$$

and

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r \in \{n+2, \dots, 4m\}. \quad (19)$$

Proof. Let $p \in M$ and $\pi \subset T_p M$ a plane section. We choose an adapted slant basis $\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\}$ of $T_p M$, where $2s = n$, and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, such that $\pi = \text{Span}\{e_1, e_2\}$ and the normal vector e_{n+1} is in the direction of the mean curvature vector H .

Since $\overline{M}^{4m}(c)$ is a quaternionic space form, from (16) and Gauss equation we can easily obtain the relation

$$n^2 \|H\|^2 = 2\tau(p) + \|h\|^2 - \frac{n(n-1)c}{4} - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{i,j=1}^n g^2(P_\beta e_i, e_j). \quad (20)$$

On the other hand, because $\{e_1, \dots, e_{2s}\}$ is an adapted slant basis of $T_p M$, using (8) and (9), we can see that

$$g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2 \theta, \quad \text{for } i = 1, 3, \dots, 2s-1 \quad (21)$$

and

$$g(P_\beta e_i, e_j) = 0, \quad \text{for } (i, j) \notin \{(2l-1, 2l), (2l, 2l-1) | l \in \{1, 2, \dots, s\}\}. \quad (22)$$

From (20), (21) and (22) we derive

$$n^2 \|H\|^2 = 2\tau(p) + \|h\|^2 - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]. \quad (23)$$

Putting

$$\Theta = 2\tau(p) - \frac{n^2(n-2)}{n-1} \|H\|^2 - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta], \quad (24)$$

we obtain from (23) and (24)

$$n^2 \|H\|^2 = (n-1)(\Theta + \|h\|^2),$$

i.e.

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 = (n-1) \left[\Theta + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right]. \quad (25)$$

Applying Lemma 4.1 for

$$a_i = h_{ii}^{n+1}, \quad \forall i \in \{1, \dots, n\}$$

and

$$b = \Theta + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2,$$

we derive

$$h_{11}^{n+1} h_{22}^{n+1} \geq \frac{1}{2} [\Theta + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2]. \quad (26)$$

From (16) and Gauss equations it also follows

$$K(\pi) = \frac{c}{4} [1 + 3 \sum_{\beta=1}^3 g^2(P_\beta e_1, e_2)] + \sum_{r=n+1}^{4m} [h_{11}^r h_{22}^r - (h_{12}^r)^2];$$

and taking into account (21) we derive

$$K(\pi) = \frac{c}{4} (1 + 9 \cos^2 \theta) + \sum_{r=n+1}^{4m} [h_{11}^r h_{22}^r - (h_{12}^r)^2]. \quad (27)$$

From (26) and (27) we obtain

$$\begin{aligned} K(\pi) &\geq \frac{c}{4} (1 + 9 \cos^2 \theta) + \sum_{r=n+2}^{4m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{4m} (h_{12}^r)^2 \\ &\quad + \frac{1}{2} \Theta + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 \\ &= \frac{c}{4} (1 + 9 \cos^2 \theta) + \frac{1}{2} \Theta + \frac{1}{2} \sum_{i \neq j > 2} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{4m} \sum_{i,j > 2} (h_{ij}^r)^2 \\ &\quad + \frac{1}{2} \sum_{r=n+2}^{4m} (h_{11}^r + h_{22}^r)^2 + \sum_{r=n+1}^{4m} \sum_{j > 2} [(h_{1j}^r)^2 + (h_{2j}^r)^2]; \end{aligned}$$

and so we conclude that

$$K(\pi) \geq \frac{c}{4} (1 + 9 \cos^2 \theta) + \frac{1}{2} \Theta. \quad (28)$$

The case of equality at a point $p \in M$ holds if and only if we have the equality in all the previous inequalities and also in the Lemma 4.1:

$$h_{ij}^{n+1} = 0, \quad i \neq j > 2,$$

$$h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, \quad r \geq n+2, \quad i, j > 2,$$

$$\begin{aligned} h_{1j}^{n+1} &= h_{2j}^{n+1} = 0, \quad j > 2, \\ h_{11}^r + h_{22}^r &= 0, \quad r \geq n + 2, \\ h_{11}^{n+1} + h_{22}^{n+1} &= h_{33}^{n+1} = \dots = h_{nn}^{n+1}. \end{aligned}$$

Finally, since we can choose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$, we obtain the desired form for the shape operators A_r , $r \in \{n + 1, \dots, 4m\}$. \square

5. The second Chen-like inequality for slant submanifolds in quaternionic space forms

Next we prove a generalization of the Theorem 4.2 in terms of Chen's invariant $\delta(n_1, \dots, n_k)$.

Theorem 5.1 *If M^n is a θ -slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$, then we have*

$$\delta(n_1, \dots, n_k) \leq d(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) \frac{c}{4} + \frac{9c}{8} \left(n - \sum_{j=1}^k n_j \right) \cos^2 \theta \quad (29)$$

for any k -tuples $(n_1, \dots, n_k) \in S(n)$.

Equality in (29) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$ and an orthonormal basis $\{e_{n+1}, \dots, e_{4m}\}$ of $T_p^\perp M$ such that the shape operators $A_r \equiv A_{e_r}$, $r \in \{n + 1, \dots, 4m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix} \quad (30)$$

and

$$A_r = \begin{pmatrix} B_1^r & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & B_k^r & 0 & \dots & 0 \\ 0 & \dots & 0 & c_r & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & c_r \end{pmatrix}, \quad r \in \{n + 2, \dots, 4m\}, \quad (31)$$

where a_1, \dots, a_n satisfy the relations

$$a_1 + \dots + a_{n_1} = \dots = a_{n_1 + \dots + n_{k-1} + 1} + \dots + a_{n_1 + \dots + n_k} = a_{n_1 + \dots + n_k + 1} = \dots = a_n$$

and each B_j^r is a symmetric $n_j \times n_j$ submatrix satisfying

$$\text{trace}(B_1^r) = \dots = \text{trace}(B_k^r) = c_r.$$

Proof. We choose $\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\}$ an adapted slant basis of $T_p M$, where $2s = n$, and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis for the normal space $T_p^\perp M$ such that the mean curvature vector H is in the direction of the normal vector e_{n+1} .

Let L_1, \dots, L_k be k mutually orthogonal subspaces of $T_p M$, with $\dim L_j = n_j, \forall j \in \{1, \dots, k\}$, defined by:

$$L_1 = \text{Span}\{e_1, \dots, e_{n+1}\}, L_2 = \text{Span}\{e_{n+1+1}, \dots, e_{n_1+n_2}\}, \dots,$$

$$L_k = \text{Span}\{e_{n_1+\dots+n_{k-1}+1}, \dots, e_{n_1+\dots+n_k}\}.$$

From (16) and Gauss equation it follows

$$\tau(L_j) = \frac{n_j(n_j-1)c}{8} + \frac{3c}{4} \sum_{\beta=1}^3 \sum_{\alpha_j < \beta_j} g^2(P_\beta e_{\alpha_j}, e_{\beta_j}) + \sum_{r=n+1}^{4m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2];$$

and using (21) and (22) we obtain

$$\tau(L_j) = \frac{n_j(n_j-1)c}{8} + \frac{9c}{8} n_j \cos^2 \theta + \sum_{r=n+1}^{4m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2]. \quad (32)$$

If we denote now

$$\Psi = 2\tau(p) - 2d(n_1, \dots, n_k) \|H\|^2 - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta] \quad (33)$$

and

$$\gamma = n + k - \sum_{j=1}^k n_j, \quad (34)$$

we can rewrite(23) as

$$n^2 \|H\|^2 = \gamma(\Psi + \|h\|^2), \quad (35)$$

i.e.

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 = \gamma[\Psi + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2]. \quad (36)$$

From (36) we obtain:

$$\begin{aligned} \left(\sum_{i=1}^{\gamma+1} b_i\right)^2 &= \gamma[\Psi + \sum_{i=1}^{\gamma+1} (b_i)^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 \\ &+ \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2 - 2 \sum_{j=1}^k \sum_{\alpha_j < \beta_j} a_{\alpha_j} a_{\beta_j}], \end{aligned} \quad (37)$$

where

$$a_i = h_{ii}^{n+1}, \forall i \in \{1, \dots, n\},$$

$$b_1 = a_1, b_2 = a_2 + \dots + a_{n_1}, b_3 = a_{n_1+1} + \dots + a_{n_1+n_2}, \dots,$$

$$b_{k+1} = a_{n_1+\dots+n_{k-1}+1} + \dots + a_{n_1+n_2+\dots+n_k},$$

$$b_{k+2} = a_{n_1+\dots+n_k+1}, \dots, b_{\gamma+1} = a_n.$$

Applying Lemma 4.1, we derive

$$\sum_{j=1}^k \sum_{\alpha_j < \beta_j} a_{\alpha_j} a_{\beta_j} \geq \frac{1}{2} [\Psi + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^n (h_{ij}^r)^2] \quad (38)$$

and so we deduce:

$$\begin{aligned} \sum_{j=1}^k \sum_{r=n+1}^{4m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2] &\geq \frac{1}{2} \Psi + \frac{1}{2} \sum_{r=n+1}^{4m} \sum_{(\alpha, \beta) \notin D^2} (h_{\alpha \beta}^r)^2 \\ &+ \sum_{r=n+2}^{4m} \sum_{\alpha_j \in D_j} (h_{\alpha_j \alpha_j}^r)^2, \end{aligned} \quad (39)$$

where D_1, \dots, D_k, D are the sets

$$D_1 = \{1, \dots, n_1\}, D_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \dots,$$

$$D_k = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\},$$

$$D^2 = (D_1 \times D_1) \cup \dots \cup (D_k \times D_k).$$

From (39) we deduce

$$\sum_{j=1}^k \sum_{r=n+1}^{4m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2] \geq \frac{1}{2} \Psi, \quad (40)$$

and so from (32) we obtain

$$\sum_{j=1}^k \tau(L_j) \geq \sum_{j=1}^k [n_j(n_j - 1) + 9n_j \cos^2 \theta] \frac{c}{8} + \frac{1}{2} \Psi. \quad (41)$$

From (33) and (41) we derive the desired inequality.

The case of equality at a point $p \in M$ holds if and only if we have the equality in all the previous inequality and also in the Lemma 4.1. Similarly, as in Theorem 4.2, we obtain the desired form for the shape operators A_r , $r \in \{n+1, \dots, 4m\}$. \square

6. Ricci curvature and squared mean curvature

B.-Y. Chen [10] established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form. The extension of this inequality for slant submanifolds in complex space forms was proved in [25].

A submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic CR-submanifold if there exists two orthogonal complementary distributions D and D^\perp on M such that D is invariant under quaternionic structure and D^\perp is totally real (see [4]). Some recent results concerning quaternionic CR-submanifolds can be found in [21]. An estimation of the Ricci curvature of a quaternionic CR-submanifold in a quaternionic space form has been established in [26]. It is clear that, although quaternionic CR-submanifolds are also the generalization of both quaternionic and totally real submanifolds, there exists no inclusion between the two classes of quaternionic CR-submanifolds and slant submanifolds. Next we find an estimation of the Ricci curvature for slant submanifolds in quaternionic space forms.

Theorem 6.1 *Let M^n be a θ -slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then:*

i) For each unit vector $X \in T_pM$, we have:

$$Ric(X) \leq \frac{(n-1)c}{4} + \frac{n^2}{4} \|H\|^2 + \frac{3c}{8} \cos^2 \theta. \tag{42}$$

ii) If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (42) if and only if X belongs to the relative null space of M at p :

$$N_p = \{X \in T_pM | h(X, Y) = 0, \forall Y \in T_pM\}.$$

iii) The equality case of (42) holds identically for all unit tangent vectors at p if and only either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. i) Let $X \in T_pM$ be a unit tangent vector at p . If we choose an adapted slant basis $\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\}$ of T_pM such that $e_1 = X$, where $2s = n$, then from (23), using (9), we find

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau(p) + \frac{1}{2} \sum_{r=n+1}^{4m} [\sum_{i=1}^n (h_{ii}^r)^2 + (h_{11}^r - \sum_{i=2}^n h_{ii}^r)^2] \\ &\quad + 2 \sum_{r=n+1}^{4m} [\sum_{i < j} (h_{ij}^r)^2 - \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r] \\ &\quad - \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]. \end{aligned} \tag{43}$$

By using the equation of Gauss and (9), we obtain

$$\begin{aligned} \sum_{2 \leq i < j \leq n} K(e_i \wedge e_j) &= \sum_{r=n+1}^{4m} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)c}{8} \\ &\quad + \frac{3(3n-1)c}{8} \cos^2 \theta. \end{aligned} \tag{44}$$

From (43) and (44), we find

$$\begin{aligned} \frac{1}{2}n^2\|H\|^2 \geq & 2\tau(p) + 2 \sum_{r=n+1}^{4m} \sum_{j=2}^n (h_{1j}^r)^2 - 2 \sum_{2 \leq i < j \leq n} K(e_i \wedge e_j) \\ & - \frac{(n-1)c}{2} - \frac{3c}{4} \cos^2 \theta \end{aligned} \tag{45}$$

and therefore we obtain (42).

ii) If $H(p) = 0$, then equality holds in (42) if and only if

$$h_{1j}^r = 0, \quad j \in \{2, \dots, n\},$$

$$h_{11}^r = \sum_{i=2}^n h_{ii}^r, \quad r \in \{n+1, \dots, 4m\}$$

and therefore $e_1 = X$ lies in N_p . The converse is clear.

iii) We have equality in (42) for all unit tangent vectors at p if and only if

$$h_{ij}^r = 0, \quad i \neq j, \quad r \in \{n+1, \dots, 4m\},$$

$$\sum_{i=1}^n h_{ii}^r = 2h_{jj}^r, \quad j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 4m\}.$$

If $n = 2$, it follows that p is a totally umbilical point and if $n \neq 2$ then p is a totally geodesic point. The proof of converse part is straightforward. \square

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