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Extremal Lagrangian submanifolds in a complex space form $N^n(4c)$

Shichang Shu and Annie Yi Han

Abstract

Let $N^n(4c)$ be the complex space form of constant holomorphic sectional curvature $4c$, $\varphi : M \to N^n(4c)$ be an immersion of an n-dimensional Lagrangian manifold M in $N^n(4c)$. Denote by S and H the square of the length of the second fundamental form and the mean curvature of M. Let ρ be the non-negative function on M defined by $\rho^2 = S - nH^2$, Q be the function which assigns to each point of M the infimum of the Ricci curvature at the point. In this paper, we consider the variational problem for non-negative functional $U(\varphi) = \int_M \rho^2 dv = \int_M (S - nH^2)dv$. We call the critical points of $U(\varphi)$ the Extremal submanifold in complex space form $N^n(4c)$. We shall get the new Euler-Lagrange equation of $U(\varphi)$ and prove some integral inequalities of Simons' type for n-dimensional compact Extremal Lagrangian submanifolds $\varphi : M \to N^n(4c)$ in the complex space form $N^n(4c)$ in terms of ρ^2, Q, H and give some rigidity and characterization Theorems.

Key Words: Willmore Lagrangian submanifold, complex hyperbolic space, curvature, totally umbilical.

1. Introduction

Let N^{n+p} be an oriented smooth Riemannian manifold of dimension $n+p$. Let $\varphi : M \to N^{n+p}$ be an n-dimensional compact submanifold of N^{n+p}. Denote by $h^\alpha_{ij}, S, \vec{H}$ and H the second fundamental form, the square of the length of the second fundamental form, the mean curvature vector and the mean curvature of M. We define the non-negative function on M

$$\rho^2 = S - nH^2, \tag{1.1}$$

which vanishes exactly at the umbilical points of M. The Willmore functional is the following non-negative functional (see [1],[11],[12])

$$W(\varphi) = \int_M (S - nH^2)^{\frac{n}{2}} dv, \tag{1.2}$$

where dv is the volume element of M. From [1],[11] and [12], we know that $W(\varphi)$ is an invariant under Moebius (or conformal) transformations of N^{n+p}. Li [7]and Hu-Li [5],[6] used the term Willmore submanifold to call the critical points of Willmore functional (1.2). When $n = 2$, the functional essentially coincides with

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the well-known Willmore functional and its critical points are the Willmore surfaces. In [7] (also see [11],[4]), Li obtained a Euler-Lagrange equation of Willmore functional in terms of Euclidean geometry, which is very important to the study of rigidity and geometry of Willmore submanifold in N^{n+p}.

Let $N^n(4c)$ be the complex space form of constant holomorphic sectional curvature $4c, J$ the standard complex structure on $N^n(4c)$. When $c = 0, N^n(4c) = C^n$; when $c > 0$, $N^n(4c) = CP^n(4c)$; when $c < 0$, $N^n(4c) = CH^n(4c)$. Let $\varphi : M \to N^n(4c)$ be an immersion of an n-dimensional manifold M in $N^n(4c)$. φ is called Lagrangian if $\varphi^*\Omega \equiv 0$. This means that the complex structure J of $N^n(4c)$ carries each tangent space of M into its corresponding normal space. The Euler-Lagrange equation of Willmore functional $W(\varphi)$ of Lagrangian submanifolds in a complex space form $N^n(4c)$ can be found in Hu-Li [6].

In this paper, we consider the non-negative functional

$$U(\varphi) = \int_M \rho^2 dv = \int_M (S - nH^2) dv, \tag{1.3}$$

which vanishes if and only if M is a totally umbilical submanifold, so the functional $U(\varphi)$ measures how derivation $\varphi(M)$ is from totally umbilical submanifold.

Now we give a new definition of so called Extremal Submanifold. We call the critical points of non-negative functional (1.3) the Extremal Submanifold in complex space form $N^n(4c)$. Obviously, when $n = 2$, $U(\varphi)$ reduces to the well known Willmore functional $W(\varphi)$, and its critical points are the Willmore surfaces. Therefore, we know that the Extremal surfaces and the Willmore surfaces are the same surfaces.

Firstly, we can get the new Euler-Lagrange equation of $U(\varphi)$, by calculating the first variation of the functional $U(\varphi)$ as follows.

Theorem 1.1. *A Lagrangian submanifold $\varphi : M \to N^n(4c)$ is Extremal submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

$$(n-1)\Delta^\perp H^{m^*} + \sum_{i,j,k,l^*} h_{ij}^{l^*} h_{ik}^{l^*} h_{kj}^{m^*} - \sum_{i,j,l^*} H^{l^*} h_{ij}^{l^*} h_{ij}^{m^*} - \frac{n}{2}\rho^2 H^{m^*} + 3(n-1)cH^{m^*} = 0, \tag{1.4}$$

where $\Delta^\perp H^{m^*} = \sum_i H_{,ii}^{m^*}$.

Remark 1.1. When $n = 2$, Theorem 1.1 reduces to the following result proved by Hu-Li [6]

Propsition 1.1([6]). *A Lagrangian surface $\varphi : M \to N^2(4c)$ is Willmore(Extremal) surface if and only if for $3 \le m^*, l^* \le 4$*

$$\triangle^\perp H^{m^*} - 2H^2 H^{m^*} + 3cH^{m^*} + \sum_{i,j,l^*} H^{l^*} h_{ij}^{l^*} h_{ij}^{m^*} = 0. \tag{1.5}$$

The typical examples of Extremal Lagrangian submanifolds of $N^n(4c)$ can be stated as follows

Example 1.1([10]). Every minimal Lagrangian surface $\varphi : M \to N^2(4c)$ in a complex space form $N^2(4c)$ is an Extremal(Willmore) Lagrangian surface.

Example 1.2. Every $n(n \geq 3)$-dimensional minimal and Einstein Lagrangian submanifold $\varphi : M \to N^n(4c)$ in a complex space form $N^n(4c)$ is an Extremal Lagrangian submanifold.

In fact, since M is minimal and Einstein, from (1.4), we only need to prove that

$$\sum_{i,j,k,l^*} h_{ij}^{l^*} h_{ik}^{l^*} h_{kj}^{m^*} = 0.$$

From the Gauss equation (2.5) in Section 2, we have

$$\begin{aligned}\sum_{i,j,k,l^*} h_{ij}^{l^*} h_{ik}^{l^*} h_{kj}^{m^*} &= \sum_{j,k}(\sum_{i,l^*} h_{ij}^{l^*} h_{ik}^{l^*}) h_{kj}^{m^*} = \sum_{j,k}[(n-1)c\delta_{jk} - R_{jk}] h_{kj}^{m^*} \\ &= \sum_{j,k}[(n-1)c\delta_{jk} - \tfrac{\tilde{R}}{n}\delta_{jk}] h_{kj}^{m^*} = [(n-1)c - \tfrac{\tilde{R}}{n}] n H^{m^*} = 0,\end{aligned}$$

where $\tilde{R}$ is the scalar curvature of M.

We note that in recent years, due to their backgrounds in mathematics, Willmore submanifolds and Extremal submanifolds in a unit sphere have been extensively studied (see [7] and [3]).

We next study the Extremal Lagrangian submanifolds $\varphi : M \to N^n(4c)$ in a complex space form $N^n(4c)$. By making use of the new Euler-Lagrange equation (1.4), we shall establish some integral inequalities of Simons' type for n-dimensional compact Extremal Lagrangian submanifolds in the complex space form $N^n(4c)$ in terms of the non-negative function ρ^2, the Ricci curvatures and the mean curvatures of the submanifolds and give some rigidity and characterization Theorems of such submanifolds. We state our results as follows.

Theorem 1.2. *Let $\varphi : M \to N^n(4c)$ be an $n(n \geq 2)$-dimensional compact Extremal Lagrangian submanifold in $N^n(4c)$. Then there holds*

$$\int_M \{(\frac{1}{n} - 2)\rho^4 + [(n+1)c + \frac{n(n-2)}{2} H^2]\rho^2 - 4n(n-1)cH^2\} dv \leq 0. \tag{1.6}$$

In particular, If

$$(\frac{1}{n} - 2)\rho^4 + [(n+1)c + \frac{n(n-2)}{2} H^2]\rho^2 - 4n(n-1)cH^2 \geq 0, \tag{1.7}$$

then

(1) If $c = 1 > 0$, then $\varphi : M \to CP^n(4)$ is totally umbilical; or $n = 2$ and $M = S^1 \times S^1$.

(2) If $c = 0$, then $\varphi : M \to C^n$ is totally umbilical.

(3) If $c < 0$, then $\varphi : M \to CH^n(4c)$ is totally umbilical.

Theorem 1.3. *Let $\varphi : M \to N^n(4c)$ be an $n(n \geq 2)$-dimensional compact Extremal Lagrangian submanifold in $N^n(4c)$. Then there holds the follows*

$$\begin{aligned}&\int_M \{(\tfrac{4}{n} - 1)\rho^4 - [(3n-5)c + 4(n-2)H\rho \\ &\quad - \tfrac{1}{2}(n+2)(n-4)H^2 - 4Q]\rho^2 - 4n(n-1)cH^2\} dv \leq 0.\end{aligned} \tag{1.8}$$

In particular, If

$$(\tfrac{4}{n}-1)\rho^4-[(3n-5)c+4(n-2)H\rho \\ -\tfrac{1}{2}(n+2)(n-4)H^2-4Q]\rho^2-4n(n-1)cH^2\geq 0, \tag{1.9}$$

then $\varphi: M\to N^n(4c)$ *is totally umbilical.*

2. Preliminaries

In this section, we review some related facts for Lagrangian submanifolds in $N^n(4c)$ by method of moving frames. We will follow the notation in the first section except agreeing with the convention of indices:

$$A,B,C,\cdots=1,\cdots,n,1^*,\cdots,n^*;\quad 1^*=n+1,\cdots,n^*=2n;$$

$$i,j,k,\cdots=1,\cdots,n.$$

Let $\varphi: M\to N^n(4c)$ be an n-dimensional Lagrangian submanifold. We choose a local field of orthonormal frames $e_1,\cdots,e_n,e_{1^*}=Je_1,\cdots,e_{n^*}=Je_n$ in $N^n(4c)$, such that, restricted to M, the vectors $e_1,\cdots,e_n$ are tangent to M, where J is the complex structure of $N^n(4c)$. Let $\omega_1,\cdots,\omega_{2n}$ is the field of dual frames, θ_A,θ_{AB} be the restriction of ω_A,ω_{AB} to M. Then $\theta_{i^*}=0$, taking its exterior derivative and making use of the structure equations of $N^n(4c)$ and the Cartan lemma we get

$$\theta_{ik^*}=\sum_j h_{ij}^{k^*}\theta_j,\quad h_{ij}^{k^*}=h_{ji}^{k^*}, \tag{2.1}$$

from which we can define the second fundamental form $II=\sum_{i,j,k^*}h_{ij}^{k^*}\omega_i\bigotimes\omega_j e_{k^*}$ and the mean curvature vector $\vec{H}$ of $\varphi: M\to N^n(4c)$ as

$$S=\sum_{i,j,k^*}(h_{ij}^{k^*})^2,\quad \vec{H}=\sum_{k^*}H^{k^*}e_{k^*},\quad H^{k^*}=\frac{1}{n}\sum_i h_{ii}^{k^*},\quad H=|\vec{H}|.$$

Since $\varphi: M\to N^n(4c)$ is Lagrangian, we have for any i,j

$$\langle Je_i,e_j\rangle=0,\quad \langle e_{i^*},Je_j\rangle=\delta_{ij}. \tag{2.2}$$

Taking exterior derivative of (2.2), we get for any i,j,k

$$h_{ij}^{k^*}=h_{jk}^{i^*}=h_{ik}^{j^*}, \tag{2.3}$$

$$\theta_{i^*j^*}=\theta_{ij}, \tag{2.4}$$

If we denote by R_{ijkl} the Riemannian curvature tensor of M, we get the Gauss equations

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_{m^*}(h_{ik}^{m^*}h_{jl}^{m^*}-h_{il}^{m^*}h_{jk}^{m^*}), \tag{2.5}$$

$$R_{ik} = c(n-1)\delta_{ik} + n\sum_{m^*} H^{m^*} h_{ik}^{m^*} - \sum_{j,m^*} h_{ij}^{m^*} h_{jk}^{m^*}, \tag{2.6}$$

$$n(n-1)R = n(n-1)c + n^2H^2 - S, \tag{2.7}$$

where R is the normalized scalar curvature of M.

The first covariant derivative $\{h_{ijk}^{m^*}\}$ and the second covariant derivative $\{h_{ijkl}^{m^*}\}$ of $h_{ij}^{m^*}$ are defined by

$$\sum_k h_{ijk}^{m^*}\theta_k = dh_{ij}^{m^*} + \sum_k h_{kj}^{m^*}\theta_{ki} + \sum_k h_{ik}^{m^*}\theta_{kj} + \sum_{k^*} h_{ij}^{k^*}\theta_{k^*m^*}, \tag{2.8}$$

$$\sum_l h_{ijkl}^{m^*}\theta_l = dh_{ijk}^{m^*} + \sum_l h_{ljk}^{m^*}\theta_{li} + \sum_l h_{ilk}^{m^*}\theta_{lj} + \sum_l h_{ijl}^{m^*}\theta_{lk} + \sum_{l^*} h_{ijk}^{l^*}\theta_{\beta m^*}. \tag{2.9}$$

The Codazzi equations and the Ricci identities are

$$h_{ijk}^{m^*} = h_{ikj}^{m^*}, \tag{2.10}$$

$$h_{ijkl}^{m^*} - h_{ijlk}^{m^*} = \sum_m h_{mj}^{m^*} R_{mikl} + \sum_m h_{im}^{m^*} R_{mjkl} + \sum_{k^*} h_{ij}^{k^*} R_{k^*m^*kl}. \tag{2.11}$$

The Ricci equations are

$$R_{i^*j^*kl} = c(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}) + \sum_m (h_{km}^{i^*} h_{lm}^{j^*} - h_{km}^{j^*} h_{lm}^{i^*}). \tag{2.12}$$

Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum_{m^*} H^{m^*} e_{m^*}$ in the normal bundle $N(M)$ as

$$\sum_i H_{,i}^{m^*}\theta_i = dH^{m^*} + \sum_{k^*} H^{k^*}\theta_{k^*m^*}, \tag{2.13}$$

$$\sum_j H_{,ij}^{m^*}\theta_j = dH_{,i}^{m^*} + \sum_j H_{,j}^{m^*}\theta_{ji} + \sum_{k^*} H_{,i}^{k^*}\theta_{k^*m^*}, \tag{2.14}$$

$$\Delta^\perp H^{m^*} = \sum_i H_{,ii}^{m^*}, \qquad H^{m^*} = \frac{1}{n}\sum_k h_{kk}^{m^*}. \tag{2.15}$$

Let f be a smooth function on M. The first, second covariant derivatives $f_i, f_{i,j}$ and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{i,j}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.16}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \texttt{trace}(A \cdot A^t) = \sum_{i,j} (a_{ij})^2.$$

Clearly, $N(A) = N(T^tAT)$ for any orthogonal matrix T.

We need the following Lemmas due to Chern-Do Carmo-Kobayashi [2] and Li-Vrancken [8].

Lemma 2.1([2]) *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \le 2N(A)N(B), \tag{2.17}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by on orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$, respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and if

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta), 1 \le \alpha, \beta \le 3,$$

then at least one of the matrices A_α must be zero.

Lemma 2.2([8]) *Let $\varphi : M \to N^n(4c)$ be an n-dimensional $(n \ge 2)$ Lagrangian submanifold. Then we have*

$$|\nabla h|^2 \ge \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2, \tag{2.18}$$

where $|\nabla h|^2 = \sum\limits_{i,j,k,m^*} (h_{ijk}^{m^*})^2, \quad |\nabla^\perp \vec{H}|^2 = \sum\limits_{i,m^*} (H_{,i}^{m^*})^2.$

3. Integral equalities

In this section we shall obtain some integral equalities of Extremal Lagrangian submanifolds $\varphi : M \to N^n(4c)$.

Define tensors

$$\tilde{h}_{ij}^{m^*} = h_{ij}^{m^*} - H^{m^*} \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{m^*l^*} = \sum_{i,j} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*}, \qquad \sigma_{m^*l^*} = \sum_{i,j} h_{ij}^{m^*} h_{ij}^{l^*}. \tag{3.2}$$

Then the $(n \times n)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{1^*}, \cdots, e_{n^*}$. We set

$$\tilde{\sigma}_{m^*l^*} = \tilde{\sigma}_{m^*} \delta_{m^*l^*}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^{m^*} = 0, \quad \tilde{\sigma}_{m^*l^*} = \sigma_{m^*l^*} - nH^{m^*}H^{l^*}, \quad \rho^2 = \sum_{m^*} \tilde{\sigma}_{m^*} = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,m^*} h_{kj}^{l^*} h_{ij}^{m^*} h_{ik}^{m^*} = \sum_{i,j,k,m^*} \tilde{h}_{kj}^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{m^*} + 2 \sum_{i,j,m^*} H^{m^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*} + H^{l^*} \rho^2 + nH^2 H^{l^*}. \tag{3.5}$$

From (3.1), (3.4) and (3.5), the new Euler-Lagrange equation(1.4) can be rewritten as

Proposition 3.1. *A Lagrangian submanifold $\varphi : M \to N^n(4c)$ is Extremal submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

$$\sum_{i,j,k,l^*} \tilde{h}_{ij}^{l^*} \tilde{h}_{ik}^{l^*} \tilde{h}_{kj}^{m^*} = -(n-1)\Delta^{\perp} H^{m^*} - \sum_{l^*} H^{l^*} \tilde{\sigma}_{m^* l^*} + \frac{n-2}{2} H^{m^*} \rho^2 - 3(n-1)cH^{m^*}. \tag{3.6}$$

By the definition of $\triangle$ and ρ^2, we have by use of (2.10) and (2.11)

$$\begin{aligned} \tfrac{1}{2}\Delta\rho^2 &= \tfrac{1}{2}\Delta S - \tfrac{1}{2}\Delta(nH^2) = \sum_{i,j,k,m^*} (h_{ijk}^{m^*})^2 + \sum_{i,j,m^*} h_{ij}^{m^*} \Delta h_{ij}^{m^*} - \tfrac{1}{2}\Delta(nH^2) \\ &= |\nabla h|^2 - n^2 |\nabla^{\perp} \vec{H}|^2 + \sum_{i,j,,k,m^*} (h_{ij}^{m^*} h_{kki}^{m^*})_j - \tfrac{1}{2}\Delta(nH^2) \\ &\quad + \sum_{m^*} \sum_{i,j,k,l} h_{ij}^{m^*} (h_{kl}^{m^*} R_{lijk} + h_{li}^{m^*} R_{lkjk}) + \sum_{m^*,l^*} \sum_{i,j,k} h_{ij}^{m^*} h_{ki}^{l^*} R_{l^* m^* jk}. \end{aligned} \tag{3.7}$$

From (2.3), (2.12) and (3.1) we have

$$\begin{aligned} &\sum_{m^*,l^*} \sum_{i,j,k} h_{ij}^{m^*} h_{ki}^{l^*} R_{l^* m^* jk} = \sum_{m^*,l^*} \sum_{i,j,k} h_{ij}^{m^*} h_{ki}^{l^*} c(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) \\ &\quad + \sum_{m^*,l^*} \sum_{i,j,k,p} h_{ij}^{m^*} h_{ki}^{l^*} (h_{jp}^{l^*} h_{pk}^{m^*} - h_{kp}^{l^*} h_{pj}^{m^*}) \\ &\quad = c\rho^2 - n(n-1)cH^2 - \tfrac{1}{2} \sum_{m^*,l^*,j,k} (\sum_p h_{jp}^{l^*} h_{pk}^{m^*} - \sum_p h_{jp}^{m^*} h_{pk}^{l^*})^2 \\ &\quad = c\rho^2 - n(n-1)cH^2 - \tfrac{1}{2} \sum_{m^*,l^*,j,k} (\sum_p \tilde{h}_{jp}^{l^*} \tilde{h}_{pk}^{m^*} - \sum_p \tilde{h}_{jp}^{m^*} \tilde{h}_{pk}^{l^*})^2 \\ &\quad = c\rho^2 - n(n-1)cH^2 - \tfrac{1}{2} \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}), \end{aligned} \tag{3.8}$$

where $\tilde{A}_{m^*} := (\tilde{h}_{ij}^{m^*}) = (h_{ij}^{m^*} - H^{m^*}\delta_{ij})$.

By use of (2.3), (2.5),(3.2),(3.4),(3.5) and (3.8) we can take a simple and direct calculation that

$$\begin{aligned} &\sum_{m^*} \sum_{i,j,k,l} h_{ij}^{m^*} (h_{kl}^{m^*} R_{lijk} + h_{li}^{m^*} R_{lkjk}) = cn\rho^2 - \sum_{m^*,l^*} \sum_{i,j,k,l} h_{ij}^{m^*} h_{ij}^{l^*} h_{lk}^{m^*} h_{lk}^{l^*} \\ &\quad + n \sum_{m^*,l^*} \sum_{i,j,k} H^{l^*} h_{kj}^{l^*} h_{ij}^{m^*} h_{ik}^{m^*} + \sum_{m^*,l^*} \sum_{i,j,k,l} h_{ij}^{m^*} h_{ki}^{l^*} (h_{jl}^{l^*} h_{lk}^{m^*} - h_{kl}^{l^*} h_{lj}^{m^*}) \\ &= cn\rho^2 - \sum_{m^*,l^*} \sigma_{m^*l^*}^2 + n \sum_{m^*,l^*} \sum_{i,j,k} H^{l^*} \tilde{h}_{kj}^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{m^*} + 2n \sum_{m^*,l^*} \sum_{i,j} H^{m^*} H^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*} \\ &\quad + n \sum_{l^*} (H^{l^*})^2 \rho^2 + n^2 H^2 \sum_{l^*} (H^{l^*})^2 - \tfrac{1}{2} \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \\ &= cn\rho^2 - \sum_{m^*,l^*} \tilde{\sigma}_{m^*l^*}^2 + nH^2\rho^2 + n \sum_{m^*,l^*} \sum_{i,j,k} H^{l^*} \tilde{h}_{kj}^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{m^*} \\ &\quad - \tfrac{1}{2} \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}). \end{aligned} \tag{3.9}$$

Putting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 + \sum_{i,j,,k,m^*}(h_{ij}^{m^*}h_{kki}^{m^*})_j - \sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 \\ &\quad - \sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) + n\sum_{m^*,l^*}\sum_{i,j,k}H^{m^*}\tilde{h}_{kj}^{m^*}\tilde{h}_{ij}^{l^*}\tilde{h}_{ik}^{l^*} \\ &\quad +(n+1)c\rho^2 - n(n-1)cH^2 + nH^2\rho^2 - \frac{1}{2}\Delta(nH^2).\end{aligned} \tag{3.10}$$

From (3.6) and (3.10), we have

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 + \sum_{i,j,,k,m^*}(h_{ij}^{m^*}h_{kki}^{m^*})_j + (n+1)c\rho^2 - \sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 \\ &\quad - \sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) + n(H^2\rho^2 - \sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*}) \\ &\quad -n(n-1)\sum_{m^*}(\Delta^\perp H^{m^*})H^{m^*} - 4n(n-1)cH^2 + \frac{n(n-2)}{2}H^2\rho^2 - \frac{1}{2}\Delta(nH^2).\end{aligned} \tag{3.11}$$

Integrating (3.11) and making use of the relation

$$\begin{aligned}\int_M \sum_{m^*} H^{m^*}\Delta^\perp H^{m^*}dv &= \frac{1}{2}\int_M \sum_{m^*}\Delta^\perp (H^{m^*})^2 dv - \int_M \sum_{i,m^*}(H_{,i}^{m^*})^2 dv \\ &= \frac{1}{2}\int_M \Delta H^2 dv - \int_M |\nabla^\perp \vec{H}|^2 dv,\end{aligned}$$

we have the following proposition.

Proposition 3.2. *For any n-dimensional compact Extremal Lagrangian submanifold $\varphi : M \to N^n(4c)$, there holds the following integral equality:*

$$\begin{aligned}&\int_M(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + n\int_M(H^2\rho^2 - \sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv \\ &\quad - \int_M[\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) + \sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2]dv \\ &\quad + \int_M[(n+1)c\rho^2 - 4n(n-1)cH^2 + \frac{n(n-2)}{2}H^2\rho^2]dv = 0.\end{aligned} \tag{3.12}$$

4. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 and note that the calculation of the first variation of the non-negative functional $U(\varphi_0)$ will play an important role in the proof.

Proof of Theorem 1.1. Let $\varphi_0 : M \to N^n(4c)$ be an n-dimensional compact submanifold in $N^n(4c)$ with (possibly empty) boundary ∂M. If otherwise, we will consider the variation with compact support. We calculate the first variation of the non-negative functional $U(\varphi_0)$. Let $\varphi : M \times R \to N^n(4c)$ be a smooth variation of φ_0 such that $\varphi(\cdot,t) = \varphi_0$ on the boundary. Along $\varphi : M \times R \to N^n(4c)$, we choose a local orthonormal basis $\{e_A\}$ for $TN^n(4c)$ with dual basis $\{\omega_A\}$, such that $\{e_i(\cdot,t)\}$ forms a local orthonormal basis for $\varphi_t : M \times \{t\} \to N^n(4c)$. Since $T^*(M \times R) = T^*M \oplus T^*R$, the pullback of $\{\omega_A\}$ and $\{\omega_{AB}\}$ on $N^n(4c)$ through $\varphi : M \times R \to N^n(4c)$ have the decomposition

$$\varphi^*\omega_{m^*} = V_{m^*}dt, \quad \varphi^*\omega_i = \theta_i + V_i dt, \tag{4.1}$$

$$\varphi^*\omega_{ij} = \theta_{ij} + L_{ij}dt, \quad \varphi^*\omega_{im^*} = \theta_{im^*} + M_{im^*}dt, \quad \varphi^*\omega_{m^*l^*} = \theta_{m^*l^*} + N_{m^*l^*}dt, \tag{4.2}$$

where $\{V_i, V_{m^*}, L_{ij}, M_{im^*}, N_{m^*l^*}\}$ are local functions on $M \times R$ with $L_{ij} = -L_{ji}$, $N_{m^*l^*} = -N_{l^*m^*}$ and

$$V = \frac{d}{dt}|_{t=0}\varphi_t = \sum_i V_i d\varphi_0(e_i) + \sum_{m^*} V_{m^*}e_{m^*}, \tag{4.3}$$

is the variation vector field of $\varphi_t : M \to N^n(4c)$. We note that the one forms $\{\theta_i, \theta_{ij}, \theta_{im^*}, \theta_{m^*l^*}\}$ are defined on $M \times \{t\}$, for $t = 0$, and they reduce to the forms with the same notation on M. We denote by d_M the differential operator on T^*M, then $d = d_M + dt\frac{\partial}{\partial t}$ on $T^*(M \times R)$.

Let $\tilde{R}_{ABCD}$ be the components of the Riemannian curvature tensor of $N^n(4c)$. From Hu-Li [6], we have the following lemmas.

Lemma 4.1([6]). *Under the above notations, we have*

$$\frac{\partial\theta_i}{\partial t} = \sum_j (V_{i,j} + L_{ij})\theta_j - \sum_{j,m^*} h_{ij}^{m^*} V_{m^*}\theta_j, \tag{4.4}$$

$$M_{im^*} = V_{m^*,i} + \sum_j h_{ij}^{m^*} V_j, \tag{4.5}$$

$$\frac{\partial\theta_{im^*}}{\partial t} = \sum_j (M_{im^*,j} + \sum_k L_{ik}h_{jk}^{m^*} - \sum_{l^*} N_{l^*m^*}h_{ij}^{l^*} - \sum_k \tilde{R}_{im^*kj}V_k + \sum_{l^*} \tilde{R}_{im^*jl^*}V_{l^*})\theta_j, \tag{4.6}$$

where $h_{ij}^{m^*}$ and the covariant derivatives $V_{i,j}, V_{m^*,i}$ and $M_{im^*,j}$ are defined on $M \times \{t\}$ by

$$\theta_{im^*} = \sum_j h_{ij}^{m^*}\theta_j, \tag{4.7}$$

$$\sum_j V_{i,j}\theta_j = d_M V_i + \sum_j V_j\theta_{ji}, \tag{4.8}$$

$$\sum_i V_{m^*,i}\theta_i = d_M V_{m^*} + \sum_{l^*} V_{l^*}\theta_{l^*m^*}, \tag{4.9}$$

$$\sum_j M_{im^*,j} = d_M M_{im^*} + \sum_j M_{jm^*}\theta_{ji} + \sum_{l^*} M_{il^*}\theta_{l^*m^*}. \tag{4.10}$$

Lemma 4.2([6]).

$$\frac{\partial h_{ij}^{m^*}}{\partial t} = V_{m^*,ij} + \sum_k (L_{ik}h_{kj}^{m^*} + L_{jk}h_{ki}^{m^*} + h_{ijk}^{m^*}V_k) + \sum_{l^*}(N_{m^*l^*}h_{ij}^{l^*} + \tilde{R}_{m^*il^*j}V_{l^*}) + \sum_{k,l^*} h_{ik}^{m^*}h_{kj}^{l^*}V_{l^*}. \tag{4.11}$$

Set $i = j$ in (4.11) and sum over i with using $\sum\limits_{i,k} L_{ik}h_{ki}^{m^*} = 0$, we have

$$\frac{\partial H^{m^*}}{\partial t} = \frac{1}{n}\Delta^\perp V_{m^*} + \sum_k H_{,k}^{m^*}V_k + \sum_{l^*} N_{m^*l^*}H^{l^*} + \frac{1}{n}\sum_{i,k,l^*} h_{ik}^{m^*}h_{ki}^{l^*}V_{l^*} + \frac{1}{n}\sum_{i,l^*}\tilde{R}_{m^*il^*i}V_{l^*}. \tag{4.12}$$

From (4.11) and the fact that

$$S=\sum_{i,j,k^*}(h_{ij}^{k^*})^2,\quad \sum_{i,j,m^*,l^*}N_{m^*l^*}h_{ij}^{m^*}h_{ij}^{l^*}=0,\quad \sum_{i,j,k,m^*}L_{jk}h_{ki}^{m^*}h_{ij}^{m^*}=0,$$

we obtain

$$\frac{1}{2}\frac{\partial S}{\partial t}=\sum_{i,j,m^*}h_{ij}^{m^*}V_{m^*,ij}+\frac{1}{2}\sum_k S_{,k}V_k+\sum_{i,j,m^*,l^*}\tilde{R}_{m^*il^*j}h_{ij}^{m^*}V_{l^*}+\sum_{i,j,k,m^*,l^*}h_{ij}^{m^*}h_{ik}^{m^*}h_{kj}^{l^*}V_{l^*}. \tag{4.13}$$

From (4.12) and $\sum\limits_{m^*,l^*}N_{m^*l^*}H^{m^*}H^{l^*}=0$, we obtain

$$\frac{1}{2}\frac{\partial(nH^2)}{\partial t}=\sum_{m^*}H^{m^*}\Delta^{\perp}V_{m^*}+\frac{n}{2}\sum_k(H^2)_{,k}V_k+\sum_{i,j,m^*,l^*}H^{m^*}h_{ij}^{m^*}h_{ij}^{l^*}V_{l^*}+\sum_{i,m^*,l^*}H^{m^*}\tilde{R}_{m^*il^*i}V_{l^*}. \tag{4.14}$$

For $\varphi_t: M\to N^n(4c)$, we consider the non-negative functional

$$U(\varphi_t)=\int_M\rho^2dv=\int_M(S-nH^2)\theta_1\wedge\cdots\wedge\theta_n, \tag{4.15}$$

From (4.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}(\theta_1\wedge\cdots\wedge\theta_n) &= \sum_i\theta_1\wedge\cdots\wedge\frac{\partial\theta_i}{\partial t}\wedge\cdots\wedge\theta_n\\ &= \sum_i(V_{i,i}+L_{ii}-h_{ii}^{m^*}V_{m^*})\theta_1\wedge\cdots\wedge\theta_n\\ &= (\sum_i V_{i,i}-n\sum_{m^*}H^{m^*}V_{m^*})\theta_1\wedge\cdots\wedge\theta_n.\end{aligned} \tag{4.16}$$

From (4.13) and (4.14), we have

$$\begin{aligned}\frac{\partial\rho^2}{\partial t} &= 2\{\sum_{i,j,m^*}h_{ij}^{m^*}V_{m^*,ij}+\frac{1}{2}\sum_k(\rho^2)_{,k}V_k+\sum_{i,j,m^*,l^*}\tilde{R}_{m^*il^*j}h_{ij}^{m^*}V_{l^*}\\ &\quad-\sum_{m^*}H^{m^*}\Delta^{\perp}V_{m^*}+\sum_{i,j,k,m^*,l^*}h_{ij}^{m^*}h_{ik}^{m^*}h_{kj}^{l^*}V_{l^*}\\ &\quad-\sum_{i,j,m^*,l^*}H^{m^*}h_{ij}^{m^*}h_{ij}^{l^*}V_{l^*}-\sum_{i,m^*,l^*}H^{m^*}\tilde{R}_{m^*il^*i}V_{l^*}\}.\end{aligned} \tag{4.17}$$

From (4.15)–(4.17), we have

$$\begin{aligned}\frac{\partial w(\varphi_t)}{\partial t} &= \int_M\{[2\sum_{i,j,m^*}h_{ij}^{m^*}V_{m^*,ij}-2\sum_{m^*}H^{m^*}\Delta^{\perp}V_{m^*}+\sum_k(\rho^2)_{,k}V_k+\rho^2\sum_k V_{k,k}]\\ &\quad+2\sum_{m^*}[\sum_{i,j,l^*}\tilde{R}_{l^*im^*j}h_{ij}^{l^*}+\sum_{i,j,k,l^*}h_{ij}^{l^*}h_{ik}^{l^*}h_{kj}^{m^*}\\ &\quad-\sum_{i,j,l^*}H^{l^*}h_{ij}^{l^*}h_{ij}^{m^*}-\sum_{i,l^*}H^{l^*}\tilde{R}_{l^*im^*i}-\frac{n}{2}\rho^2H^{m^*}]V_{m^*}\}dv.\end{aligned} \tag{4.18}$$

We know that

$$\sum_k(\rho^2)_{,k}V_k+\rho^2\sum_k V_{k,k}=\sum_k(\rho^2V_k)_{,k}, \tag{4.19}$$

and M is compact (without boundary), also noting

$$\sum_j h^{m^*}_{ijj} = nH^{m^*}_{,i}, \quad \sum_{i,j} h^{m^*}_{ijji} = n\Delta^\perp H^{m^*}. \tag{4.20}$$

It follows from (4.18), (4.19) and Green's formula that

$$\begin{aligned}\frac{\partial w(\varphi_t)}{\partial t} = & 2\int_M \sum_{m^*}\{[\sum_{i,j,k,l^*} h^{l^*}_{ij}h^{l^*}_{ik}h^{m^*}_{kj} + \sum_{i,j,l^*} \tilde{R}_{l^*im^*j}h^{l^*}_{ij} \\ & - \sum_{i,j,l^*} H^{l^*}h^{l^*}_{ij}h^{m^*}_{ij} - \sum_{i,l^*} H^{l^*}\tilde{R}_{l^*im^*i} - \frac{n}{2}\rho^2 H^{m^*}] + \sum_{i,j}(h^{m^*}_{ij})_{ij} - \Delta^\perp H^{m^*}\}V_{m^*}dv\end{aligned} \tag{4.21}$$

Since for $n+1 \leq m^*, l^*, \cdots \leq 2n$, we have (see (6.4) of Hu-Li [6])

$$\sum_{i,j,l^*} \tilde{R}_{l^*im^*j}h^{l^*}_{ij} = 4ncH^{m^*}, \quad \sum_{i,l^*} H^{l^*}\tilde{R}_{l^*im^*i} = (n+3)cH^{m^*}. \tag{4.22}$$

From (4.3), (4.20), (4.21) and (4.22) with restriction to $t=0$, we have proved Theorem 1.1. □

5. Proofs of Theorem 1.2 and Theorem 1.3

Now we shall prove Theorem 1.2 and Theorem 1.3 as follows. We should note that the integral equalities (3.12) will play an important role in the proofs of Theorem 1.2 and Theorem 1.3. The following result due to Ludden, Okumura and Yano [9] is very important to us.

Theorem 5.1([9]). *Let M be a compact $n(n \geq 2)$-dimensional minimal Lagrangian submanifold immersed in $CP^n(4)$. If*

$$S = \frac{n+1}{2-\frac{1}{n}},$$

then $n=2$ and $M = S^1 \times S^1$.

Proof of Theorem 1.2 From lemma2.1, (3.2) and (3.3), we have

$$\begin{aligned} & -\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) - \sum_{m^*,l^*} \tilde{\sigma}^2_{m^*l^*} \\ & \geq -\sum_{m^*} \tilde{\sigma}^2_{m^*} - 2\sum_{m^* \neq l^*} \tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = -2(\sum_{m^*} \tilde{\sigma}_{m^*})^2 + \sum_{m^*} \tilde{\sigma}^2_{m^*} \\ & \geq -2\rho^4 + \frac{1}{n}(\sum_{m^*} \tilde{\sigma}_{m^*})^2 = -(2-\frac{1}{n})\rho^4,\end{aligned} \tag{5.1}$$

where, we used

$$\sum_{m^*} \tilde{\sigma}^2_{m^*} \geq \frac{1}{n}(\sum_{m^*} \tilde{\sigma}_{m^*})^2. \tag{5.2}$$

We also have

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = \sum_{m^*}(H^{m^*})^2\tilde{\sigma}_{m^*} \leq \sum_{m^*}(H^{m^*})^2\sum_{l^*}\tilde{\sigma}_{l^*} = H^2\rho^2. \tag{5.3}$$

By making use of lemma2.2, (3.12),(5.1) and (5.3), we have

$$\begin{aligned}0 &\geq \int_M(|\nabla h|^2-\tfrac{3n^2}{n+2}|\nabla^\perp\vec{H}|^2)dv+\int_M(\tfrac{3n^2}{n+2}-n)|\nabla^\perp\vec{H}|^2dv\\&-\int_M(2-\tfrac{1}{n})\rho^4dv+\int_M[(n+1)c\rho^2-4n(n-1)cH^2+\tfrac{n(n-2)}{2}H^2\rho^2]dv\\&\geq\int_M\{(\tfrac{1}{n}-2)\rho^4+[(n+1)c+\tfrac{n(n-2)}{2}H^2]\rho^2-4n(n-1)cH^2\}dv.\end{aligned}\tag{5.4}$$

From (1.7) and (5.4), we have

$$(\frac{1}{n}-2)\rho^4+[(n+1)c+\frac{n(n-2)}{2}H^2]\rho^2-4n(n-1)cH^2=0.\tag{5.5}$$

Case 1. If $\rho^2=0$ on M, we have M is totally umbilical.

Case 2. If $\rho^2\neq 0$ on M, we know that the equalities in (5.4) hold. Therefore, we have

$$\begin{aligned}&\nabla^\perp\vec{H}=0,\quad \nabla h=0,\\&N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})=2N(\tilde{A}_{m^*})N(\tilde{A}_{l^*}),\quad m^*\neq l^*,\\&n\sum_{m^*}\tilde{\sigma}_{m^*}^2=(\sum_{m^*}\tilde{\sigma}_{m^*})^2,\end{aligned}\tag{5.6}$$

that is

$$\tilde{\sigma}_{n+1}=\cdots=\tilde{\sigma}_{2n}.\tag{5.7}$$

We also have

$$\sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*}=H^2\rho^2.\tag{5.8}$$

From lemma 2.1, we know that at most two of $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*}), m^*=n+1,\cdots,2n$, are different from zero. If all of $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*})$ are zero, which is contradiction with M is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, which is contradiction with (5.7). Therefore, we may assume that

$$\begin{gathered}\tilde{A}_{n+1}=\lambda\tilde{A},\quad \tilde{A}_{n+2}=\mu\tilde{B},\quad \lambda,\mu\neq 0,\\ \tilde{A}_{m^*}=0,\qquad m^*\geq n+3,\end{gathered}$$

where $\tilde{A}$ and $\tilde{B}$ are defined in lemma 2.1.

From (5.8), we have

$$\lambda^2(H^{n+1})^2+\mu^2(H^{n+2})^2=(\lambda^2+\mu^2)\sum_{m^*}(H^{m^*})^2.$$

Since $\lambda,\mu\neq 0$, we infer that $H^{m^*}=0, n+1\leq m^*\leq 2n$, that is, $\vec{H}=0$, i.e, $\varphi: M\to N^n(4c)$ is a minimal Lagrangian submanifold in $N^n(4c)$. We consider the folloing three cases:

(1) If $c = 1 > 0$, from the assertion as above and (5.5), we know that $\varphi : M \to CP^n(4)$ is a minimal Lagrangian submanifold in $CP^n(4)$ with $\rho^2 = \frac{n+1}{2-\frac{1}{n}}$, that is, $S = \frac{n+1}{2-\frac{1}{n}}$. From the Theorem 5.1 of Ludden, Okumura and Yano[9], we know that $n = 2$ and $M = S^1 \times S^1$.

(2) If $c = 0$, from the assertion as above, we know that $\varphi : M \to C^n$ is a minimal Lagrangian submanifold in C^n. This is contradiction with the well known fact that there are not compact minimal submanifolds in the complex Euclidean space C^n.

(3) If $c < 0$, from the assertion as above and (5.5), we know that $\varphi : M \to CH^n(4c)$ is a minimal Lagrangian submanifold in $CH^n(4c)$ with $(2 - \frac{1}{n})\rho^4 + (n+1)\rho^2 = 0$. This is contradiction with $\rho^2 \neq 0$ on M. Therefore, we complete the proof of Theorem 1.2. □

Now, we consider the rigidity of Extremal Lagrangian submanifolds in terms of Ricci curvatures. Denote by Q the function which assigns to each point of M the infimum of the Ricci curvature at that point, we have the following lemma.

Lemma 4.1. *For any n-dimensional Lagrangian submanifold in $N^n(4c)$, there holds*

$$\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \le 4\{(n-1)c + (n-2)H\rho + H^2 - Q\}\rho^2 - \frac{4}{n}\rho^4. \tag{5.9}$$

Proof. From Gauss equation (2.6) and (3.1), we have

$$R_{ik} = (n-1)c\delta_{ik} + (n-2)\sum_{m^*} H^{m^*}\tilde{h}^{m^*}_{ik} + (n-1)H^2\delta_{ik} - \sum_{m^*,j}\tilde{h}^{m^*}_{ij}\tilde{h}^{m^*}_{jk}.$$

Thus, we get

$$R_{ii} = (n-1)c + (n-2)\sum_{m^*} H^{m^*}h^{m^*}_{ii} + H^2 - \sum_{m^*,j}(\tilde{h}^{m^*}_{ij})^2. \tag{5.10}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{m^*} H^{m^*}h^{m^*}_{ii} \le \sqrt{\sum_{m^*}(H^{m^*})^2}\sqrt{\sum_{m^*}(h^{m^*}_{ii})^2} \le H\rho. \tag{5.11}$$

(5.10) and (5.11) infer that

$$Q \le (n-1)c + (n-2)H\rho + H^2 - \sum_{m^*,j}(\tilde{h}^{m^*}_{ij})^2. \tag{5.12}$$

Therefore, we have

$$\sum_{m^*\neq l^*,i}(\tilde{h}^{m^*}_{il})^2 \le (n-1)c + (n-2)H\rho + H^2 - Q - (\tilde{h}^{m^*}_{il})^2. \tag{5.13}$$

From (5.13) and $\tilde{h}_{ij}^{m^*}=\mu_i^{m^*}\delta_{ij}$, it is easy to see

$$\begin{aligned}&\sum_{l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})=\sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_i^{m^*}-\mu_l^{m^*})^2\leq 4\sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_l^{m^*})^2\\&\quad\leq 4\sum_l\{(n-1)c+(n-2)H\rho+H^2-Q-(\mu_l^{m^*})^2\}(\mu_l^{m^*})^2\\&\quad=4\{(n-1)c+(n-2)H\rho+H^2-Q\}\sum_l(\mu_l^{m^*})^2-4\sum_l(\mu_l^{m^*})^4\\&\quad\leq 4\{(n-1)c+(n-2)H\rho+H^2-Q\}\sum_l(\mu_l^{m^*})^2-\tfrac{4}{n}(\sum_l(\mu_l^{m^*})^2)^2.\end{aligned}\tag{5.14}$$

Therefore, we know that (5.9) holds. This completes the proof of lemma 4.1. □

Proof of Theorem 1.3. From (3.12), lemma 2.2, (3.3), (5.3) and lemma 4.1, we have

$$\begin{aligned}0&\geq\int_M[(n+1)c\rho^2-4n(n-1)cH^2+\tfrac{n(n-2)}{2}H^2\rho^2]dv\\&\quad-\int_M\{4[(n-1)c+(n-2)H\rho+H^2-Q]\rho^2-\tfrac{4}{n}\rho^4\}dv-\int_M\rho^4dv\\&=\int_M\{(\tfrac{4}{n}-1)\rho^4-[(3n-5)c+4(n-2)H\rho\\&\quad-\tfrac{1}{2}(n+2)(n-4)H^2-4Q]\rho^2-4n(n-1)cH^2\}dv,\end{aligned}\tag{5.15}$$

where we used

$$\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2=\sum_{m^*}\tilde{\sigma}_{m^*}^2\leq(\sum_{m^*}\tilde{\sigma}_{m^*})^2=\rho^4.\tag{5.16}$$

From (1.9) and (5.15), we conclude

$$(\frac{4}{n}-1)\rho^4-[(3n-5)c+4(n-2)H\rho-\frac{1}{2}(n+2)(n-4)H^2-4Q]\rho^2-4n(n-1)cH^2=0,$$

If $\rho^2=0$, then M is totally umbilical; if $\rho^2\neq 0$, we have the equalities in (5.15) and (5.16) hold. From $\sum_{m^*}\tilde{\sigma}_{m^*}^2=(\sum_{m^*}\tilde{\sigma}_{m^*})^2$, we have $\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*}=0$. This implies that $(n-1)$ of the $\tilde{\sigma}_{m^*}$ must be zero. Since $\rho^2=\sum_{m^*,i,j}(\tilde{h}_{ij}^{m^*})^2\neq 0$ and $\tilde{\sigma}_{m^*}=\sum_{i,j}(\tilde{h}_{ij}^{m^*})^2$, we infer that $(n-1)$ of the $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*})$ must be zero so that $n=1$. This is a contradiction for we assume that $n\geq 2$. We complete the proof of Theorem 1.3.

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Shichang SHU
Department of Mathematics
Xianyang Normal University
Xianyang 712000 Shaanxi
P. R. CHINA
e-mail: shushichang@126.com
Annie Yi HAN
Department of Mathematics
Borough of Manhattan Community College
CUNY 10007 N.Y. USA
e-mail: DrHan@nyc.rr.com

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