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Uniqueness of derivatives of meromorphic functions sharing two or three sets

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Abstract

In the paper we consider the problem of uniqueness of derivatives of meromorphic functions when they share two or three sets and obtained five results which will improve all the existing results.

Key word and phrases: Meromorphic functions, uniqueness, weighted sharing, derivative, shared set.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM. We denote by $T(r)$ the maximum of $T\left(r, f^{(k)}\right)$ and $T\left(r, g^{(k)}\right)$. The notation $S(r)$ denotes any quantity satisfying

$$S(r) = o(T(r)) \quad (r \longrightarrow \infty, r \notin E).$$

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

F. Gross first considered the uniqueness of meromorphic functions that share sets of distinct elements instead of values and in 1976 he posed the following question in [8]:

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Question A *Can one find two finite sets* S_j $(j=1,2)$ *such that any two non-constant entire functions* f *and* g *satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2$ *must be identical?*

Now it is natural to ask the following question.

Question B [19]*Can one find two finite sets* S_j $(j=1,2)$ *such that any two non-constant meromorphic functions* f *and* g *satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2$ *must be identical ?*

Also for meromorphic functions in [22] the following question was asked.

Question C *Can one find three finite sets* S_j $(j=1,2,3)$ *such that any two non-constant meromorphic functions* f *and* g *satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2,3$ *must be identical ?*

The possible answer to *Question B* and *Question C* has been investigated by many authors and naturally a substantial number of remarkable as well as elegant results have been obtained in this aspect (see [3]–[5], [7], [10], [14], [17], [19], [22], [25], [28]–[29]} and {[1]–[2], [6], [15], [18], [21]–[23], [27]). But the uniqueness of derivatives of two meromorphic functions when they share two or three sets is seldom studied. To the knowledge of the authors, perhaps the following result is the only known result on the uniqueness of the derivatives of meromorphic functions in the direction of *Question B*.

Theorem A *[7] Let* $S_1=\{z:z^n+az^{n-1}+b=0\}$ *and* $S_2=\{\infty\}$, *where* a, b *are nonzero constants such that* $z^n+az^{n-1}+b=0$ *has no repeated root and* $n\ (\geq 7)$, k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_1)=E_{g^{(k)}}(S_1)$ *and* $E_f(S_2)=E_g(S_2)$, *then* $f^{(k)}\equiv g^{(k)}$.

In 2003, in the direction of *Question C*, concerning the uniqueness of derivatives of meromorphic functions, Qiu and Fang obtained the following result.

Theorem B *[21] Let* $S_1=\{z:z^n-z^{n-1}-1=0\}$, $S_2=\{\infty\}$ *and* $S_3=\{0\}$ *and* $n\ (\geq 3)$, k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_j)=E_{g^{(k)}}(S_j)$ *for* $j=1,3$ *and* $E_f(S_2)=E_g(S_2)$ *then* $f^{(k)}\equiv g^{(k)}$.

In 2004 Yi and Lin [27] independently proved the following theorem.

Theorem C *[27] Let* $S_1=\{z:z^n+az^{n-1}+b=0\}$, $S_2=\{\infty\}$ *and* $S_3=\{0\}$, *where* a, b *are nonzero constants such that* $z^n+az^{n-1}+b=0$ *has no repeated root and* $n\ (\geq 3)$, k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_j)=E_{g^{(k)}}(S_j)$ *for* $j=1,2,3$, *then* $f^{(k)}\equiv g^{(k)}$.

The following two examples show that the condition $n(\geq 3)$ in *Theorems B–C* is the best possible.

Example 1.1 *Let* $f(z)=1+e^z$ *and* $g(z)=1+(-1)^{k+1}e^{-z}$ *and* $S_1=\{1,-1\}$, $S_2=\{\infty\}$, $S_3=\{0\}$. *Clearly* $E_{f^{(k)}}(S_j)=E_{g^{(k)}}(S_j)$ *for* $j=1,2,3$, *but* $f^{(k)}\not\equiv g^{(k)}$.

Example 1.2 *Let* $f(z)=\sqrt{a}\sqrt{b}\ e^z$ *and* $g(z)=(-1)^k\sqrt{a}\sqrt{b}\ e^{-z}$ *and* $S_1=\{a,b\}$, $S_2=\{\infty\}$, $S_3=\{0\}$, *where* a *and* b *be two arbitrary non zero constants. Clearly* $E_{f^{(k)}}(S_j)=E_{g^{(k)}}(S_j)$ *for* $j=1,2,3$, *but* $f^{(k)}\not\equiv g^{(k)}$.

In view of the above two examples one perhaps will not try to reduce the lower bound of n in *Theorems B–C*. So in order to improve the theorems one can only try to relax the nature of sharing of the sets. Relaxation of the nature of sharing of sets may be done with the aid of the notion of a gradation of sharing of values and sets known as weighted sharing introduced in [12, 13] which measures how close a shared value is to being shared IM or to being shared CM. We now give the following definition.

Definition 1.1 *[12, 13] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.2 *[12] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

We now state the following five theorems which are the main results of the paper.

Theorem 1.1 *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7), k be two positive integers. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 2) = E_{g^{(k)}}(S_1, 2)$, $E_f(S_2, 1) = E_g(S_2, 1)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.2 *Let S_i, $i = 1, 2$ be defined as in* Theorem 1.1 *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 3) = E_{g^{(k)}}(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.3 *Let S_i, $i = 1, 2, 3$ be defined as in* Theorem C *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 4) = E_{g^{(k)}}(S_1, 4)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 7) = E_{g^{(k)}}(S_3, 7)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.4 *Let S_i, $i = 1, 2, 3$ be defined as in* Theorem C *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 5) = E_{g^{(k)}}(S_1, 5)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 1) = E_{g^{(k)}}(S_3, 1)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.5 *Let S_i, $i = 1, 2, 3$ be defined as in* Theorem C *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 6) = E_{g^{(k)}}(S_1, 6)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 0) = E_{g^{(k)}}(S_3, 0)$, then $f^{(k)} \equiv g^{(k)}$.*

Though we follow the standard definitions and notations of the value distribution theory available in [9], we explain some notations which are used in the paper.

Definition 1.3 *[11] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.4 *We denote by $\overline{N}(r, a; f \mid= k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k \geq 2$ is an integer.*

Definition 1.5 *Let f and g be two non-constant meromorphic functions such that f and g share (a, k), where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q$, by $\overline{N}_E^{(k+1}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq k+1$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$ and $\overline{N}_E^{(k+1}(r, a; g)$. Clearly $\overline{N}_E^{(k+1}(r, a; f) = \overline{N}_E^{(k+1}(r, a; g)$.*

Definition 1.6 *[13] We denote $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.*

Definition 1.7 *[12, 13] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 1.8 *[16] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid\ g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.9 *[16] Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid\ g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F = \frac{\left(f^{(k)}\right)^{n-1}\left(f^{(k)} + a\right)}{-b}, \qquad G = \frac{\left(g^{(k)}\right)^{n-1}\left(g^{(k)} + a\right)}{-b}, \tag{2.1}$$

where $n(\geq 2)$ and k be two positive integers.

Henceforth we shall denote by H, Φ and V the following four functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

$$\Phi = \frac{F^{'}}{F-1} - \frac{G^{'}}{G-1}$$

and

$$V = \left(\frac{F^{'}}{F-1} - \frac{F'}{F}\right) - \left(\frac{G^{'}}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 *([13], Lemma 1) Let F, G share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F\mid=1) = N(r,1;G\mid=1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2 *([16], Lemma 4) If two non-constant meromorphic functions F and G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \quad \leq \quad & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}_{*}(r,1;F,G) + \overline{N}_{*}(r,\infty;F,G) \\ & + \overline{N}_{0}(r,0;F^{'}) + \overline{N}_{0}(r,0;G^{'}), \end{aligned}$$

where $\overline{N}_{0}(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_{0}(r,0;G^{'})$ is similarly defined.

Lemma 2.3 *[20] Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

Lemma 2.4 *Let F and G be given by (2.1). If $f^{(k)}$, $g^{(k)}$ share $(0,0)$ and 0 is not a Picard exceptional value of $f^{(k)}$ and $g^{(k)}$. Then $\Phi \equiv 0$ implies $F \equiv G$.*

Proof. We omit the proof since proceeding in the same way as done in *Lemma 2.4* [2] we can prove the lemma. □

Lemma 2.5 *Let F and G be given by (2.1), $n \geq 3$ an integer and $\Phi \not\equiv 0$. If F, G share $(1,m)$; f, g share (∞,l), and $f^{(k)}$, $g^{(k)}$ share $(0,p)$, where $0 \leq p < \infty$ then*

$$\begin{aligned} [(n-1)p + n - 2]\,\overline{N}(r,0;f^{(k)}\mid\geq p+1) \quad \leq \quad & \overline{N}_{*}(r,1;F,G) + \overline{N}_{*}(r,\infty;F,G) \\ & + S(r). \end{aligned}$$

Proof. The lemma can be proved in the line of proof of *Lemma 2.5* [2]. □

Lemma 2.6 *Let F and G be given by (2.1) and f, g share $(\infty,0)$ and ∞ is not a Picard exceptional value of f and g. Then $V\equiv 0$ implies $F\equiv G$.*

Proof. We omit the proof since it can be proved in the line of proof of *Lemma 2.6* [2]. □

Lemma 2.7 *Let F, G be given by (2.1) and $H\not\equiv 0$. If $f^{(k)}$, $g^{(k)}$ share $(0,p)$; f and g share (∞,l), where $0\leq l<\infty$ and F, G share $(1,m)$, where $1\leq m\leq\infty$ then*

$$\begin{aligned}&\{(nl+nk+n)-1\}\,\overline{N}(r,\infty;f\mid\geq l+1)\\ \leq\;&\overline{N}_{*}\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}\left(r,0;f^{(k)}+a\right)+\overline{N}\left(r,0;g^{(k)}+a\right)\\ &+\overline{N}_{*}(r,1;F,G)+S(r).\end{aligned}$$

Similar expressions hold for g, also.

Proof. Suppose ∞ is not an e.v.P. of f and g. Since $H\not\equiv 0$, it follows that $F\not\equiv G$. So from *Lemma 2.6* we know that $V\not\equiv 0$. Since f, g share $(\infty;l)$, it follows that F, G share $(\infty;n(k+l))$. Clearly a pole of F with multiplicity $s(\geq n(k+l)+1)$ is a pole of G with multiplicity $r(\geq n(k+l)+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $n(k+l)<q<n(k+l+1)$. Also since any common pole of F and G of multiplicity $s\leq n(k+l)$ is a zero of V of multiplicity $s-1$, using *Lemma 2.3* we get from the definition of V

$$\begin{aligned}&\{n(l+k+1)-1\}\,\overline{N}(r,\infty;f\mid\geq l+1)\\ \leq\;&N(r,0;V)\\ \leq\;&N(r,\infty;V)+S(r,f^{(k)})+S(r,g^{(k)})\\ \leq\;&\overline{N}_{*}(r,0;f^{(k)},g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)\\ &+\overline{N}_{*}(r,1;F,G)+S(r).\end{aligned}$$

If ∞ is an e.v.P. of f and g and F and G respectively then the lemma follows immediately. □

Lemma 2.8 *Let F, G be given by (2.1) and $V\not\equiv 0$. If f, g share (∞,l), where $0\leq l<\infty$ and F, G share $(1,m)$ then the poles of F and G are the zeros of V and*

$$\begin{aligned}&\{n(k+l+1)-1\}\,\overline{N}(r,\infty;f\mid\geq l+1)\\ \leq\;&\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)\\ &+\overline{N}_{*}(r,1;F,G)+S(r).\end{aligned}$$

Similar expressions hold for g also.

Proof. Suppose ∞ is an e.v.P. of f and g then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and g. Now using the same argument as in *Lemma 2.7*, we can deduce from the definition of V that

$$\begin{aligned}
& \{n(k+1)-1\}\ N(r,\infty;f\mid=1)+\{n(k+2)-1\}\overline{N}(r,\infty;f\mid=2)+\ldots+ \\
& \{n(k+l)-1\}\overline{N}(r,\infty;f\mid=l)+\{n(k+l+1)-1\}\ \overline{N}(r,\infty;f\mid\geq l+1) \\
\leq\ & N(r,0;V) \\
\leq\ & T(r,V) \\
\leq\ & N(r,\infty;V)+S(r,f^{(k)})+S(r,g^{(k)}) \\
\leq\ & \overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a) \\
& +\overline{N}_*(r,1;F,G)+S(r).
\end{aligned}$$

□

Lemma 2.9 *([1], Lemma 3) Let f and g be two meromorphic functions sharing $(1,m)$, where $2\leq m<\infty$. Then*

$$\begin{aligned}
& \overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\ldots+(m-1)\overline{N}(r,1;f\mid=m)+m\overline{N}_L(r,1;f) \\
& +(m+1)\overline{N}_L(r,1;g)+m\overline{N}_E^{(m+1}(r,1;f)\leq N(r,1;g)-\overline{N}(r,1;g).
\end{aligned}$$

Lemma 2.10 *Let F, G be given by (2.1) and they share $(1,m)$. If f, g share $(0,p)$, (∞,l) where $2\leq m<\infty$ and $H\not\equiv 0$. Then*

$$\begin{aligned}
T(r,F)\ \leq\ & \overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}_*(r,0;f^{(k)},g^{(k)})+N_2(r,0;f^{(k)}+a) \\
& +N_2(r,0;g^{(k)}+a)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g) \\
& -m(r,1;G)-\overline{N}(r,1;F\mid=3)--(m-2)\overline{N}(r,1;F\mid=m)\ldots \\
& -(m-2)\ \overline{N}_L(r,1;F)-(m-1)\overline{N}_L(r,1;G)-(m-1)\overline{N}_E^{(m+1}(r,1;F) \\
& +S(r).
\end{aligned}$$

Proof. We omit the proof since it can be carried out in the line of proof of *Lemma 2.9* [2]. □

Lemma 2.11 *Let F, G be given by (2.1) and they share $(1,m)$. If f, g share (∞,k) where $2\leq m<\infty$ and $H\not\equiv 0$. Then*

$$\begin{aligned}
T(r,F)\ \leq\ & 2\overline{N}(r,0;f^{(k)})+2\overline{N}(r,0;g^{(k)})+N_2(r,0;f^{(k)}+a)+N_2(r,0;g^{(k)}+a) \\
& +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-m(r,1;G) \\
& -\overline{N}(r,1;F\mid=3)-\ldots-(m-2)\overline{N}(r,1;F\mid=m)-(m-2)\ \overline{N}_L(r,1;F) \\
& -(m-1)\overline{N}_L(r,1;G)-(m-1)\overline{N}_E^{(m+1}(r,1;F)+S(r).
\end{aligned}$$

Proof. We omit the proof since using *Lemmas 2.1*, *2.2* and *2.9* the proof of the lemma can be carried out in the line of proof of *Lemma 2.10.* □

Lemma 2.12 *Let $f^{(k)}$, $g^{(k)}$ be two non-constant meromorphic functions sharing $(0,\infty)$, (∞,∞). Then $\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right) \equiv \left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)$ implies $f^{(k)} \equiv g^{(k)}$, where $n\ (\geq 2)$ is an integer, k be a positive integer and a is a nonzero finite constant.*

Proof. We first note that $\Theta\left(\infty; f^{(k)}\right) + \Theta\left(\infty; g^{(k)}\right) > 2 - \frac{2}{k+1} = \frac{2k}{k+1} > 0$. Now since the given condition implies $f^{(k)}$, $g^{(k)}$ share $(0;\infty)$, the lemma can be proved in the line of proof of *Lemma 3* [15]. □

Lemma 2.13 *If two meromorphic functions f, g share $(\infty, 0)$ then for $n \geq 2$*

$$\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right) \not\equiv b^2,$$

where a, b are finite nonzero constants and k be a positive integer.

Proof. Noting that according to the lemma $f^{(k)}$, $g^{(k)}$ share (∞, k), we omit the proof since the proof of the lemma can be carried out in the line of proof of *Lemma 5* [14]. □

Lemma 2.14 *([26], Lemma 6) If $H \equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty, 0)$ then F, G share (∞,∞).*

Lemma 2.15 *Let F, G be given by (2.1) and they share $(1, m)$. Also let $\omega_1, \omega_2 \ldots \omega_n$ are the members of the set $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and $n\ (\geq 3)$ is an integer. Then*

$$\overline{N}_L(r,1;F) \leq \frac{1}{m+1}\left[\overline{N}\left(r,0;f^{(k)}\right) + \overline{N}(r,\infty;f) - N_{\otimes}\left(r,0;f^{(k+1)}\right)\right] + S(r),$$

where $N_{\otimes}\left(r,0;f^{(k+1)}\right) = N\left(r,0;f^{(k+1)} \mid f^{(k)} \neq 0, \omega_1, \omega_2 \ldots \omega_n\right)$.

Proof. The proof can be carried out along the lines of the proof of *Lemma 2.14* [2]. □

Lemma 2.16 *[24] Let F, G be two meromorphic meromorphic functions sharing $(1,\infty)$ and (∞,∞). If*

$$N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) < \lambda T_1(r) + S_1(r),$$

where $\lambda < 1$ and $T_1(r) = \max\{T(r,F), T(r,G)\}$ and $S_1(r) = o(T_1(r))$, $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 2.17 *Let F, G be given by (2.1) $n \geq 3$ and they share $(1, m)$. If $f^{(k)}$, $g^{(k)}$ share $(0,0)$, and f, g share (∞, l) and $H \equiv 0$. Then $f^{(k)} \equiv g^{(k)}$.*

Proof. Since $H \equiv 0$ we get from *Lemma 2.14* F and G share $(1, \infty)$ and (∞, ∞). If possible let us suppose $F \not\equiv G$. Then from *Lemma 2.4* and *Lemma 2.5* we have

$$\overline{N}(r, 0; f^{(k)}) = \overline{N}(r, 0; g^{(k)}) = S(r).$$

Again from *Lemma 2.6* we get $V \not\equiv 0$ and so in view of *Lemma 2.7* we have

$$\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \leq \frac{4}{n(k+1)-1}\, T(r) + S(r).$$

Therefore we see that

$$\begin{aligned} & N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \qquad (2.2)\\ \leq\; & 2\overline{N}(r, 0; f^{(k)}) + 2\overline{N}(r, 0; g^{(k)}) + N_2(r, 0; f^{(k)} + a) + N_2(r, 0; g^{(k)} + a) \\ & +2\overline{N}(r, \infty; f) \\ \leq\; & N_2\left(r, 0; f^{(k)} + a\right) + N_2\left(r, 0; g^{(k)} + a\right) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r). \end{aligned}$$

Using *Lemma 2.3*, we obtain

$$T_1(r) = n\ \max\left\{T\left(r, f^{(k)}\right), T\left(r, g^{(k)}\right)\right\} + O(1) = n\, T(r) + O(1). \qquad (2.3)$$

So again using *Lemma 2.3* we get from (2.2) and (2.3)

$$\begin{aligned} & N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\ \leq\; & \frac{\left[2 + \frac{4}{n(k+1)-1}\right]}{n} T_1(r) + S(r). \end{aligned}$$

Since $k \geq 1$ and $n \geq 3$ we have by *Lemma 2.16* $FG \equiv 1$, which is impossible by *Lemma 2.13*. Hence $F \equiv G$ i.e. $\left(f^{(k)}\right)^{n-1}(f^{(k)} + a) \equiv \left(g^{(k)}\right)^{n-1}(g^{(k)} + a)$. From this condition it is clear that $f^{(k)}$ and $g^{(k)}$ share $(0, \infty)$. Now the lemma follows from *Lemma 2.12*. □

Lemma 2.18 *Suppose F and G be defined as in (2.1) and $n \geq 7$ be an integer. Then $F \equiv G$ implies $f^{(k)} \equiv g^{(k)}$.*

Proof. We note that $\Theta\left(\infty; f^{(k)}\right) > 1 - \frac{1}{k+1} = \frac{k}{k+1} \geq \frac{1}{2} > \frac{2}{n-1}$, for $n \geq 7$. So the proof of the lemma can be carried out along the lines of proof of *Lemma 2* [28]. □

Lemma 2.19 *Suppose F and G be defined as in (2.1) and $n \geq 7$ be an integer. If f, g share (∞, k) and $H \equiv 0$. Then $f^{(k)} \equiv g^{(k)}$.*

Proof. Since $H\equiv 0$ we get from *Lemma 2.14* F and G share $(1,\infty)$ and (∞,∞). If possible let us suppose $F\not\equiv G$. Using *Lemma 2.6* and *Lemma 2.8* with $l=0$ we have

$$\begin{aligned}
& N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)\\
\leq\ & 2\overline{N}(r,0;f^{(k)})+2\overline{N}(r,0;g^{(k)})+N_2(r,0;f^{(k)}+a)\\
& +N_2(r,0;g^{(k)}+a)+2\overline{N}(r,\infty;f)\\
\leq\ & \frac{\left[6+\frac{8}{nk+n-1}\right]}{n}\,T_1(r)+S(r).
\end{aligned}$$

So respectively using *Lemmas 2.16*, *2.13* we can deduce a contradiction. Hence $F\equiv G$. Now the lemma follows from *Lemma 2.18*. □

3. Proofs of the theorems

Proof. [Proof of Theorem 1.1] Let F, G be given by (2.1). Then F and G share $(1,2)$, $(\infty;k+1)$. We consider the following cases.

Case 1. Let $H\not\equiv 0$. Then $F\not\equiv G$. Suppose ∞ is not an e.v.P. of f and g. Then by *Lemma 2.6* we get $V\not\equiv 0$. Hence from *Lemmas 2.3*, *2.8*, *2.11* and *2.15* we obtain

$$\begin{aligned}
nT\left(r,f^{(k)}\right) \leq\ & 2\overline{N}\left(r,0;f^{(k)}\right)+2\overline{N}\left(r,0;g^{(k)}\right)+N_2\left(r,0;f^{(k)}+a\right)\\
& +N_2\left(r,0;g^{(k)}+a\right)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
& +\overline{N}(r,\infty;f\mid\geq 2)-\overline{N}_L(r,1;G)+S(r)\\
\leq\ & 3T\left(r,f^{(k)}\right)+3T\left(r,g^{(k)}\right)+\left[\frac{2}{nk+n-1}\right.\\
& \left.+\frac{1}{nk+2n-1}\right]\left\{2T\left(r,f^{(k)}\right)+2T\left(r,g^{(k)}\right)\right.\\
& \left.+\frac{1}{3}\left(\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}(r,\infty;f)\right)\right\}+S(r)\\
\leq\ & \left[6+\frac{28}{3(nk+n-1)}+\frac{14}{3(nk+2n-1)}\right]T(r)+S(r).
\end{aligned}\tag{3.1}$$

If ∞ is an e.v.P. of f and g, then (3.1) automatically holds.
In the same way we can obtain

$$nT\left(r,g^{(k)}\right)\leq\left[6+\frac{28}{3(nk+n-1)}+\frac{14}{3(nk+2n-1)}\right]T(r)+S(r).\tag{3.2}$$

Combining (3.1) and (3.2) we see that

$$\left[n-6-\frac{28}{3(nk+n-1)}-\frac{14}{3(nk+2n-1)}\right]\,T(r)\leq S(r),$$

which leads to a contradiction for $n \geq 7$.
Case 2. Let $H \equiv 0$. Now the theorem follows from *Lemma 2.19*. □

Proof. [Proof of Theorem 1.2] Let F, G be given by (2.1). Then F and G share $(1,3)$, $(\infty;0)$. We omit the proof as it can be demonstrated by proceeding in the same way as in *Theorem 1.1*. □

Proof. [Proof of Theorem 1.3] Let F, G be given by (2.1). Then F and G share $(1,4)$, $(\infty;\infty)$; $f^{(k)}$, $g^{(k)}$ share $(0,7)$. We consider the following cases.
Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of $f^{(k)}$ and $g^{(k)}$. Then by *Lemma 2.4* and *Lemma 2.6* we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from *Lemmas 2.3*, *2.5*, *2.7*, *2.10* and *2.15* we obtain

$$
\begin{aligned}
& nT\left(r, f^{(k)}\right) && (3.3)\\
\leq\ & \overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,0;g^{(k)}\right) + \overline{N}\left(r,0;f^{(k)} \mid\geq 8\right) \\
& +N_2\left(r,0;f^{(k)}+a\right) + N_2\left(r,0;g^{(k)}+a\right) + \overline{N}(r,\infty;f) \\
& +\overline{N}(r,\infty;g) - 2\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G) + S(r) \\
\leq\ & \left(\frac{2}{n-2} + \frac{2}{nk+n-1}\right) \overline{N}_*(r,1;F,G) + \left(1 + \frac{2}{nk+n-1}\right) \\
& \overline{N}\left(r,0;f^{(k)} \mid\geq 8\right) + \left(2 + \frac{4}{nk+n-1}\right) T(r) - 2\overline{N}_*(r,1;F,G) \\
& -\overline{N}_L(r,1;G) + S(r) \\
\leq\ & \left(2 + \frac{4}{nk+n-1}\right) T(r) + \frac{nk+17n-17}{(nk+n-1)(8n-9)} \overline{N}_L(r,1;F) + S(r) \\
\leq\ & \left[2 + \frac{4}{nk+n-1} + \frac{2(nk+17n-17)}{5(nk+n-1)(8n-9)}\right] T(r) + S(r).
\end{aligned}
$$

If 0, ∞ are e.v.P. of f and g, then (3.3) automatically holds. In the same way we can obtain

$$
nT\left(r, g^{(k)}\right) \leq \left[2 + \frac{4}{nk+n-1} + \frac{2(nk+17n-17)}{5(nk+n-1)(8n-9)}\right] T(r) + S(r). \tag{3.4}
$$

Combining (3.3) and (3.4) we see that

$$
\left[n - 2 - \frac{4}{(nk+n-1)} - \frac{2(nk+17n-17)}{5(nk+n-1)(8n-9)}\right] T(r) \leq S(r),
$$

which is a contradiction for $n \geq 3$.
Case 2. Let $H \equiv 0$. Now the theorem follows from *Lemma 2.17*. □

Proof. [Proof of Theorem 1.4] Let F, G be given by (2.1). Then F and G share $(1,5)$, $(\infty;\infty)$; $f^{(k)}$, $g^{(k)}$ share $(0,1)$. We omit the proof since proceeding in the same way as done in *Theorem 1.3* the proof of the theorem carried out. □

Proof. [Proof of Theorem 1.5] Let F, G be given by (2.1). Then F and G share $(1,6)$, $(\infty;\infty)$. Since $f^{(k)}$ and $g^{(k)}$ share we observe that $\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)\leq \overline{N}\left(r,0;f^{(k)}\right)$. We consider the following cases.
Case 1. Let $H\not\equiv 0$. Then $F\not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by *Lemma 2.4* and *Lemma 2.6* we get $\Phi\not\equiv 0$ and $V\not\equiv 0$. Hence from *Lemmas 2.3*, *2.5*, *2.7* and *2.10* we see that

$$\begin{aligned}
& nT\left(r,f^{(k)}\right) \qquad (3.5)\\
\leq\ & 3\,\overline{N}\left(r,0;f^{(k)}\right)+N_2\left(r,0;f^{(k)}+a\right)+N_2\left(r,0;g^{(k)}+a\right)\\
& +2\overline{N}(r,\infty;f)-4\overline{N}_*(r,1;F,G)+S(r)\\
\leq\ & \left(3+\frac{2}{nk+n-1}\right)\overline{N}\left(r,0;f^{(k)}\right)+\frac{2}{nk+n-1}\,\overline{N}_*(r,1;F,G)\\
& +\left(2+\frac{4}{nk+n-1}\right)T(r)-4\overline{N}_*(r,1;F,G)+S(r)\\
\leq\ & \left[\frac{3}{n-2}+\frac{2(n-1)}{(n-2)(nk+n-1)}\right]\overline{N}_*(r,1;F,G)\\
& +\left(2+\frac{4}{nk+n-1}\right)T(r)-4\overline{N}_*(r,1;F,G)+S(r)\\
\leq\ & \left[2+\frac{4}{nk+n-1}\right]T(r)+S(r).
\end{aligned}$$

If 0, ∞ are e.v.P. of f and g, then (3.5) automatically holds.
In the same way we can obtain

$$nT\left(r,g^{(k)}\right)\leq\left[2+\frac{4}{nk+n-1}\right]T(r)+S(r). \qquad (3.6)$$

Combining (3.5) and (3.6) we see that

$$\left[n-2-\frac{4}{nk+n-1}\right]T(r)\leq S(r).$$

which is a contradiction for $n\geq 3$.
Case 2. Let $H\equiv 0$. Now the theorem follows from *Lemma 2.17*. □

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