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Nontrivial periodic solutions of nonlinear functional differential systems with feedback control

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Abstract

This paper examines the existence of nontrivial periodic solutions for the nonlinear functional differential system with feedback control:

$$\begin{cases} x'(t) = x(t)a(t) - \Big[\sum_{i=1}^{n} a_i(t) \int_0^{+\infty} f(t, x(t-\theta))\, \mathrm{d}\varphi_i(\theta) \\ \qquad + \sum_{j=1}^{m} b_j(t) \int_0^{+\infty} f(t, x'(t-\theta))\, \mathrm{d}\phi_j(\theta) + \sum_{\mu=1}^{p} c_\mu(t) \int_0^{\infty} u(t-\theta)\, \mathrm{d}\delta_\mu(\theta)\Big], \\ u'(t) = -\rho(t)u(t) + \sum_{\nu=1}^{q} \beta_\nu(t) \int_0^{\infty} f(t, x(t-\theta))\, \mathrm{d}\psi_\nu(\theta). \end{cases}$$

Under certain growth conditions on the nonlinearity f, several sufficient conditions for the existence of nontrivial solution are obtained by using Leray-Schauder nonlinear alternative.

Key Words: Nonlinear functional differential equations with feedback control; Nontrivial periodic solutions; Leray-Schauder nonlinear alternative; Fixed point.

1. Introduction

In this paper, we are concerned with the existence of nontrivial periodic solutions for the delay functional differential equations with feedback control:

$$\begin{cases} x'(t) = x(t)a(t) - \Big[\sum_{i=1}^{n} a_i(t) \int_0^{+\infty} f(t, x(t-\theta))\, \mathrm{d}\varphi_i(\theta) \\ \qquad + \sum_{j=1}^{m} b_j(t) \int_0^{+\infty} f(t, x'(t-\theta))\, \mathrm{d}\phi_j(\theta) + \sum_{\mu=1}^{p} c_\mu(t) \int_0^{\infty} u(t-\theta)\, \mathrm{d}\delta_\mu(\theta)\Big], \\ u'(t) = -\rho(t)u(t) + \sum_{\nu=1}^{q} \beta_\nu(t) \int_0^{\infty} f(t, x(t-\theta))\, \mathrm{d}\psi_\nu(\theta). \end{cases} \tag{1.1}$$

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where $f \in C(\mathbb{R}^2, \mathbb{R})$ with $f(t+\omega, z) = f(t, z)$, the coefficients $a, a_i, b_j, c_\mu, \rho, \beta_\nu \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic functions, $\varphi_i, \phi_j, \delta_\mu, \psi_\nu \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing with $\int_0^\infty \mathrm{d}\varphi_i(\theta) = 1, \int_0^\infty \mathrm{d}\phi_i(\theta) = 1, \int_0^\infty \mathrm{d}\delta_\mu(\theta) = 1$, and $\int_0^\infty \mathrm{d}\psi_\nu(\theta) = 1, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; \mu = 1, 2, \ldots, p\ ; \nu = 1, 2, \ldots, q$, where $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = (0, +\infty)$.

In the real world, the variation of the environment plays an important role in many biological and ecological system. Thus, the assumption of periodicity of the parameters in the way incorporates the periodicity of the environment. An ecological justification of model (1.1) can be found in [3, 4, 6, 10]. Using continuation theory for k-set-contractions, Lu [8], Lu and Ge [9] studied the existence of positive periodic solutions for the following system of functional differential equations

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)\Big[a(t) - \beta(t)N(t) - \sum_{i=1}^{n} b_i(t)N_i(t-\tau_i(t)) - \sum_{j=1}^{m} c_j(t)N_j'(t-\sigma_j(t))\Big], \tag{1.2}$$

where the functions $a(t), \beta(t), b_i(t), c_j(t), \tau_i(t), \sigma_j(t)$ are continuous ω-periodic, and $a(t) \geq 0$, $\beta(t) \geq 0$, $b_i(t) \geq 0$, $c_j(t) \geq 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$. Yang and Cao [11] used Mawhin's continuation theorem [2] to investigated the existence of positive periodic solutions of (1.2). The main results obtained in [6, 9] required $c_j \in C^1, \sigma_j \in C^2$ and $\sigma_j' < 1$ $(j = 1, 2, \ldots, n)$. As far as we known, there has few papers which deal with the existence of periodic solutions of neutral equations with distributed delays (1.1).

Recently, Chen et al. [13], by using Mawhin's continuation theorem [2], investigated the existence of positive periodic solutions of the following single species neutral logistic model with several discrete delays

$$\begin{cases} n'(t) = n(t)\Big[a(t) - \beta(t)n(t) - \sum_{i=1}^{n} b_i(t)n(t, x'(t-\tau_i(t))) - \sum_{i=1}^{n} c_i n'(t-\gamma_i(t)) \\ -\ \delta(t)u(t) - \sum_{i=1}^{n} d_i(t)u(t-\sigma_i(t))), \\ u'(t) = -e(t)u(t) + f(t)n(t) + \sum_{i=1}^{n} g_i(t)n(t-\eta_i(t))). \end{cases} \tag{1.3}$$

where the functions $a(t), \beta(t), b_i(t), c_i(t), \delta, d_i(t), e, f, g_i, \tau_i, \gamma_i, \sigma_i, \eta_i$ are nonnegative continuous ω-periodic, and $i = 1, 2, \ldots, n$. Liu and Li [7], by using Avery-Henderson fixed point theorem, study the existence of positive periodic solutions of the following nonlinear nonautonomous functional differential system with feedback control

$$\begin{cases} x'(t) = -r(t)x(t) + F(t, x_t, u(t-\delta(t))), \\ u'(t) = -h(t)u(t) + g(t)x(t-\sigma(t)), \end{cases} \tag{1.4}$$

where the functions $\delta(t), \sigma(t) \in C(\mathbb{R}, \mathbb{R}), r(t), h(t), g(t) \in C(\mathbb{R}, (0, +\infty))$, all of the above functions are ω-periodic, and ω is a constant, $F(t+\omega, x_{t+\omega}, z) = F(t, x_t, z)$.

In [1, 5], Li, by using the abstract continuous theory for k-contractions and a fixed point theorem of strict-set-contraction, established criteria for the existence of positive periodic solutions for the periodic functional differential equations with feedback control,

$$\begin{cases} N'(t) = r(t)N(t)\Big[1 - \dfrac{N(t-\tau(t)) + c(t)N'(t-\tau(t))}{K(t)} - \beta(t)u(t-\delta(t))\Big], \\ u'(t) = -a(t)u(t) + b(t)N(t-\sigma(t)). \end{cases} \tag{1.5}$$

and

$$\begin{cases} x'(t) = x(t)\Big[a(t) - \sum_{i=1}^{n} a_i(t)\int_0^{+\infty} x(t-\theta)\,\mathrm{d}\varphi_i(\theta) \\ + \sum_{j=1}^{m} b_j(t)\int_0^{+\infty} x'(t-\theta)\,\mathrm{d}\phi_j(\theta) + \sum_{\mu=1}^{p} c_\mu(t)\int_0^{\infty} u(t-\theta)\,\mathrm{d}\delta_\mu(\theta)\big], \\ u'(t) = -\rho(t)u(t) + \sum_{\nu=1}^{q} \beta_\nu(t)\int_0^{\infty} x(t-\theta)\,\mathrm{d}\psi_\nu(\theta). \end{cases} \tag{1.6}$$

where $r, K, a, b, c, \beta, \tau, \delta, \sigma \in C(\mathbb{R}, (0, +\infty))$ are ω-periodic functions in (1.5); the coefficients $a, a_i, b_j, c_\mu, \rho, \beta_\nu \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic functions, $\varphi_i, \phi_j, \delta_\mu, \psi_\nu \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing with $\int_0^\infty \mathrm{d}\varphi_i(\theta) = 1, \int_0^\infty \mathrm{d}\phi_i(\theta) = 1, \int_0^\infty \mathrm{d}\delta_\mu(\theta) = 1,$ and $\int_0^\infty \mathrm{d}\psi_\nu(\theta) = 1, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, \mu = 1, 2, \ldots, p$, $\nu = 1, 2, \ldots, q$ in (1.6).

Motivated by the above paper, we are concerned with (1.1). The main purpose of this paper is to establish some simple criteria for the existence of nontrivial solution of the delay functional differential equations with feedback control (1.1). Note that we do not require any monotonicity and nonnegative on f. A periodic solution $u(t)$ of (1.1) is called nontrivial periodic solution if $u(t) \not\equiv const, t \in [0, \omega]$.

For convenience, we introduce the following notation:

$$\begin{aligned} k^M &= \max_{t\in[0,\omega]} \big[k(t)\sum_{i=1}^{n} a_i(t) + k(t)\sum_{j=1}^{m} b_j(t) \\ &+ \sum_{\mu=1}^{p} c_\mu(t)\int_0^\infty \int_{t-\theta}^{t-\theta+\omega} H(t-\theta, s)\sum_{\nu=1}^{q} \beta_\nu(s)k(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big], \\ K &= \int_0^\omega \big[k(\varsigma)\sum_{i=1}^{n} a_i(\varsigma) + k(\varsigma)\sum_{j=1}^{m} b_j(t) \\ &+ \sum_{\mu=1}^{p} c_\mu(\varsigma)\int_0^\infty \int_{\varsigma-\theta}^{\varsigma-\theta+\omega} H(\varsigma-\theta, s)\sum_{\nu=1}^{q} \beta_\nu(s)k(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\,d\varsigma, \\ r^M &= \max_{t\in[0,\omega]} \big[r(t)\sum_{i=1}^{n} a_i(t) + r(t)\sum_{j=1}^{m} b_j(t) \end{aligned}$$

$$+\sum_{\mu=1}^{p} c_\mu(t)\int_0^\infty \int_{t-\theta}^{t-\theta+\omega} H(t-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)r(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big],$$

$$R=\int_0^\omega \big[r(\varsigma)\sum_{i=1}^{n}a_i(\varsigma)+r(\varsigma)\sum_{j=1}^{m}b_j(t)$$

$$+\sum_{\mu=1}^{p} c_\mu(\varsigma)\int_0^\infty \int_{\varsigma-\theta}^{\varsigma-\theta+\omega} H(\varsigma-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)r(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]d\varsigma,$$

$$\Pi=\max_{t\in[0,\omega]}\{a(t)\},$$

$$H(t,s)=\frac{e^{\int_t^s \rho(\theta)\,\mathrm{d}\theta}}{e^{\int_0^\omega \rho(\theta)\,\mathrm{d}\theta}-1},\quad s\in[t,t+\omega],$$

$$G(t,s)=\frac{e^{-\int_t^s a(\theta)\,\mathrm{d}\theta}}{1-e^{-\int_0^\omega a(\theta)\,\mathrm{d}\theta}},\quad s\in[t,t+\omega].$$

It is easy to see that $H(t+\omega,s+\omega)=H(t,s), G(t+\omega,s+\omega)=G(t,s)$ and

$$0<\frac{1}{e^{\int_0^\omega \rho(\theta)\,\mathrm{d}\theta}-1}\le H(t,s)\le \frac{e^{\int_0^\omega \rho(\theta)\,\mathrm{d}\theta}}{e^{\int_0^\omega \rho(\theta)\,\mathrm{d}\theta}-1},\quad s\in[t,t+\omega].$$

$$0<\frac{e^{-\int_0^\omega a(\theta)\,\mathrm{d}\theta}}{1-e^{-\int_0^\omega a(\theta)\,\mathrm{d}\theta}}\le G(t,s)\le \frac{1}{1-e^{-\int_0^\omega a(\theta)\,\mathrm{d}\theta}}:=\Lambda,\quad s\in[t,t+\omega].$$

2. Preliminaries and lemmas

In this section, we give some preliminaries and lemmas which will be used in the proof of the main results. Since each ω-periodic solution of the equation

$$u'(t)=-\rho(t)u(t)+\sum_{\nu=1}^{q}\beta_\nu(t)\int_0^\infty f(t,x(t-\theta))\,\mathrm{d}\psi_\nu(\theta)$$

is equivalent to that of the equation

$$u(t)=\int_t^{t+\omega} H(t,s)\sum_{\nu=1}^{q}\beta_\nu(s)\int_0^\infty f(s,x(s-\theta))\,\mathrm{d}\psi_\nu(\theta)\,\mathrm{d}s:=(\Phi x)(t), \tag{2.1}$$

and vice versa, where $x(t)$ is ω-periodic, therefore, the existence problem of ω-periodic solution of system (1.1) is equivalent to that of ω-periodic solution of the equation

$$\begin{aligned} x'(t) = x(t)a(t) - \big[\sum_{i=1}^{n} a_i(t) \int_0^{+\infty} f(t, x(t-\theta)) \,\mathrm{d}\varphi_i(\theta) \\ + \sum_{j=1}^{m} b_j(t) \int_0^{+\infty} f(t, x'(t-\theta)) \,\mathrm{d}\phi_j(\theta) \\ + \sum_{\mu=1}^{p} c_\mu(t) \int_0^{\infty} \int_{t-\theta}^{t-\theta+\omega} H(t-\theta, s) \sum_{\nu=1}^{q} \beta_\nu(s) \int_0^{\infty} f(s, x(s-\xi)) \,\mathrm{d}\psi_\nu(\xi) \,\mathrm{d}s \,\mathrm{d}\delta_\mu(\theta)\big]. \end{aligned} \tag{2.2}$$

Remark 2.1 It is easy to see that the function $u(t)$ defined by (2.1) is ω-periodic function. In fact, for any ω-periodic function $x(t)$,

$$\begin{aligned} u(t+\omega) &= \int_{t+\omega}^{t+2\omega} H(t+\omega, s) \sum_{\nu=1}^{q} \beta_\nu(s) \int_0^{\infty} f(s, x(s-\theta)) \,\mathrm{d}\psi_\nu(\theta) \,\mathrm{d}s \\ &= \int_{t}^{t+\omega} H(t+\omega, h+\omega) \sum_{\nu=1}^{q} \beta_\nu(h+\omega) \int_0^{\infty} f(h+\omega, x(h+\omega-\theta)) \,\mathrm{d}\psi_\nu(\theta) \,\mathrm{d}h \\ &= \int_{t}^{t+\omega} H(t, s) \sum_{\nu=1}^{q} \beta_\nu(s) \int_0^{\infty} f(s, x(s-\theta)) \,\mathrm{d}\psi_\nu(\theta) \,\mathrm{d}s \\ &= u(t). \end{aligned}$$

To obtain the existence of periodic solutions to (1.1), we make the following preparations:

Lemma 2.2 [12] *Let X be a real Banach space, Ω be a bounded open subset of X, $0 \in \Omega$, $T : \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exists $x \in \partial\Omega, \mu > 1$ such that $T(x) = \mu x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

To apply Lemma 2.2 to the equation (2.2), we set

$$C_\omega^0 = \{x \in C^0(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t)\}$$

with the norm $|x|_0 = \max_{t\in[0,\omega]}\{|x(t)|\}$, and

$$C_\omega^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t)\}$$

with the norm $|x|_1 = |x|_0 + |x'|_0$. Then C_ω^0 and C_ω^1 are all Banach spaces.

Let T be defined by

$$(Tx)(t)=\int_t^{t+\omega}G(t,\varsigma)\big[\sum_{i=1}^{n}a_i(\varsigma)\int_0^{+\infty}f(\varsigma,x(\varsigma-\theta))\,\mathrm{d}\varphi_i(\theta)$$

$$+\sum_{j=1}^{m}b_j(\varsigma)\int_0^{+\infty}f(\varsigma,x'(\varsigma-\theta))\,\mathrm{d}\phi_j(\theta)$$

$$+\sum_{\mu=1}^{p}c_\mu(\varsigma)\int_0^{\infty}\int_{\varsigma-\theta}^{\varsigma-\theta+\omega}H(\varsigma-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)\int_0^{\infty}f(s,x(s-\xi))\,\mathrm{d}\psi_\nu(\xi)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]d\varsigma,$$

Then, when $x\in C^1_\omega$, we have

$$(Tx)(t+\omega)=\int_{t+\omega}^{t+2\omega}G(t+\omega,\varsigma)\big[\sum_{i=1}^{n}a_i(\varsigma)\int_0^{+\infty}f(\varsigma,x(\varsigma-\theta))\,\mathrm{d}\varphi_i(\theta)$$

$$+\sum_{j=1}^{m}b_j(\varsigma)\int_0^{+\infty}f(\varsigma,x'(\varsigma-\theta))\,\mathrm{d}\phi_j(\theta)$$

$$+\sum_{\mu=1}^{p}c_\mu(\varsigma)\int_0^{\infty}(\Phi x)(\varsigma-\theta)\,\mathrm{d}\delta_\mu(\theta)\big]d\varsigma$$

$$=\int_t^{t+\omega}G(t+\omega,v+\omega)\big[\sum_{i=1}^{n}a_i(v+\omega)\int_0^{+\infty}f(v+\omega,x(v+\omega-\theta))\,\mathrm{d}\varphi_i(\theta)$$

$$+\sum_{j=1}^{m}b_j(v+\omega)\int_0^{+\infty}f(v+\omega,x'(v+\omega-\theta))\,\mathrm{d}\phi_j(\theta)$$

$$+\sum_{\mu=1}^{p}c_\mu(v+\omega)\int_0^{\infty}(\Phi x)(v+\omega-\theta)\,\mathrm{d}\delta_\mu(\theta)\big]dv$$

$$=\int_t^{t+\omega}G(t,v)\big[\sum_{i=1}^{n}a_i(v)\int_0^{+\infty}f(v,x(v-\theta))\,\mathrm{d}\varphi_i(\theta)$$

$$+\sum_{j=1}^{m}b_j(v)\int_0^{+\infty}f(v,x'(v-\theta))\,\mathrm{d}\phi_j(\theta)+\sum_{\mu=1}^{p}c_\mu(v)\int_0^{\infty}(\Phi x)(v-\theta)\,\mathrm{d}\delta_\mu(\theta)\big]dv$$

$$=(Tx)(t).$$

Using the Arzela-Ascoli theorem, we can conclude that $T:C^1[0,\omega]\to C^1[0,\omega]$ is a completely continuous operator, and (2.2) has a solution $x^*\in C^1[0,\omega]$ if and only if u^* is a fixed point of T in $C^1[0,\omega]$.

3. Main results

We are now in a position to state and show that our main result of this paper.

Theorem 3.1 *Suppose that $f(t,0)\not\equiv 0$, and there exist nonnegative functions $k,r\in L^1[0,\omega]$ such that*

$$\begin{cases} |f(t,x)|\le k(t)|x|+r(t), \quad a.e. \ \ (t,x)\in\mathbb{R}\times\mathbb{R}, \\ \Pi+k^M+\Lambda K<1. \end{cases} \tag{3.1}$$

Then system (1.1) has at least one nontrivial ω-periodic solution.

Proof. Since $f(t,0)\not\equiv 0$, there exists $[\sigma,\tau]\subset[0,\omega]$ such that

$$\min_{\sigma\le t\le\tau}|f(t,0)|>0.$$

On the other hand, from the conditions $r(t)\ge|f(t,0)|$, a.e. $t\in\mathbb{R}$, we know that $r^M+\Lambda R>0$. Let $\alpha=(r^M+\Lambda R)[1-(\Pi+k^M+\Lambda K)]^{-1}, \Omega_\alpha=\{x\in C^1_\omega:|x|_1<\alpha, t\in[0,\omega]\}$.

Suppose $x\in\partial\Omega_\alpha,\lambda>1$ such that $Tx=\lambda x$; then

$$\lambda\alpha=\lambda|x|_1=|Tx|_1=|Tx|_0+|(Tx)'|_0.$$

Due to

$$\begin{aligned}
|(Tx)'(t)|_0 &= \max_{t\in[0,\omega]}\Big|x(t)a(t)-\big[\sum_{i=1}^{n}a_i(t)\int_0^{+\infty}f(t,x(t-\theta))\,\mathrm{d}\varphi_i(\theta) \\
&\quad+\sum_{j=1}^{m}b_j(t)\int_0^{+\infty}f(t,x'(t-\theta))\,\mathrm{d}\phi_j(\theta) \\
&\quad+\sum_{\mu=1}^{p}c_\mu(t)\int_0^{\infty}\int_{t-\theta}^{t-\theta+\omega}H(t-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)\int_0^{\infty}f(s,x(s-\xi))\,\mathrm{d}\psi_\nu(\xi)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\Big| \\
&\le |x(t)|_1\max_{t\in[0,\omega]}\big\{a(t)+\big[k(t)\sum_{i=1}^{n}a_i(t)+k(t)\sum_{j=1}^{m}b_j(t) \\
&\quad+\sum_{\mu=1}^{p}c_\mu(t)\int_0^{\infty}\int_{t-\theta}^{t-\theta+\omega}H(t-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)k(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\big\} \\
&\quad+\max_{t\in[0,\omega]}\big[r(t)\sum_{i=1}^{n}a_i(t)+r(t)\sum_{j=1}^{m}b_j(t) \\
&\quad+\sum_{\mu=1}^{p}c_\mu(t)\int_0^{\infty}\int_{t-\theta}^{t-\theta+\omega}H(t-\theta,s)\sum_{\nu=1}^{q}\beta_\nu(s)r(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big] \\
&= (\Pi+k^M)|x(t)|_1+r^M,
\end{aligned}$$

and

$$|(Tx)(t)|_0 = \max_{t\in[0,\omega]} \Big| \int_t^{t+\omega} G(t,\varsigma)\big[\sum_{i=1}^n a_i(\varsigma)\int_0^{+\infty} f(\varsigma, x(\varsigma-\theta))\,\mathrm{d}\varphi_i(\theta)$$

$$+\sum_{j=1}^m b_j(\varsigma)\int_0^{+\infty} f(\varsigma, x'(\varsigma-\theta))\,\mathrm{d}\phi_j(\theta)$$

$$+\sum_{\mu=1}^p c_\mu(\varsigma)\int_0^\infty \int_{\varsigma-\theta}^{\varsigma-\theta+\omega} H(\varsigma-\theta, s)\sum_{\nu=1}^q \beta_\nu(s)\int_0^\infty f(s, x(s-\xi))\,\mathrm{d}\psi_\nu(\xi)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\,d\varsigma\Big|$$

$$\leq \Lambda|x(t)|_1\int_0^\omega \big[k(\varsigma)\sum_{i=1}^n a_i(\varsigma) + k(\varsigma)\sum_{j=1}^m b_j(t)$$

$$+\sum_{\mu=1}^p c_\mu(\varsigma)\int_0^\infty\int_{\varsigma-\theta}^{\varsigma-\theta+\omega} H(\varsigma-\theta, s)\sum_{\nu=1}^q \beta_\nu(s)k(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\,d\varsigma$$

$$+\Lambda\int_0^\omega \big[r(\varsigma)\sum_{i=1}^n a_i(\varsigma) + r(\varsigma)\sum_{j=1}^m b_j(\varsigma)$$

$$+\sum_{\mu=1}^p c_\mu(\varsigma)\int_0^\infty\int_{\varsigma-\theta}^{\varsigma-\theta+\omega} H(\varsigma-\theta, s)\sum_{\nu=1}^q \beta_\nu(s)r(s)\,\mathrm{d}s\,\mathrm{d}\delta_\mu(\theta)\big]\,d\varsigma = \Lambda K|x(t)|_1 + \Lambda R,$$

then

$$\begin{aligned}\lambda\alpha = \lambda|x|_1 = |Tx|_1 &= |Tx|_0 + |(Tx)'|_0\\ &\leq (\Pi + k^M + \Lambda K)|x(t)|_1 + (r^M + \Lambda R)\\ &= (\Pi + k^M + \Lambda K)\alpha + (r^M + \Lambda R).\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda &\leq (\Pi + k^M + \Lambda K) + \frac{(r^M + \Lambda R)}{\alpha}\\ &= (\Pi + k^M + \Lambda K) + [1 - (\Pi + k^M + \Lambda K)] = 1.\end{aligned}$$

This contradicts $\lambda > 1$. By Lemma 2.2, T has a fixed point $x^* \in \overline{\Omega}_\alpha$. Since $f(t,0) \not\equiv 0$, (2.2) has at least one nontrivial solution $u^* \in C^1[0,\omega]$. Thus, system (1.1) has at least one nontrivial ω-periodic solution. This completes the proof. □

Example. Consider the following system:

$$\begin{cases} x'(t) = \dfrac{(1+\cos t)x(t)}{10\pi} - \Big\{\dfrac{1-\cos t}{400}\displaystyle\int_0^{+\infty}[\frac{2e^{-t}x^3(t-\theta)}{1+x^4(t-\theta)} - \sin^2 t]\,\mathrm{d}\varphi_i(\theta)\\ +\dfrac{1+\sin t}{400}\displaystyle\int_0^{+\infty}[\frac{2e^{-t}x'^3(t-\theta)}{1+x'^4(t-\theta)} - \sin^2 t]\,\mathrm{d}\phi_j(\theta) + \frac{1-\sin t}{320}\int_0^\infty u(t-\theta)\,\mathrm{d}\delta_\mu(\theta)\Big\}, \\ u'(t) = -(1-\cos t)u(t) + \displaystyle\int_0^\infty [\frac{2e^{-t}x^3(t-\theta)}{1+x^4(t-\theta)} - \sin^2 t]\,\mathrm{d}\psi_\nu(\theta). \end{cases} \tag{3.2}$$

Obviously, $a(t)=\frac{1+\cos t}{10\pi}, \sum_{i=1}^{n} a_i(t)=\frac{1-\cos t}{400}, \sum_{j=1}^{m} b_j(t)=\frac{1+\sin t}{400}, \sum_{\mu=1}^{p} c_\mu(t)=\frac{1-\sin t}{320}$, $\rho(t)=1-\cos t, \sum_{\nu=1}^{q}\beta_\nu(t)=1, f(t,x)=\frac{2e^{-t}x^3}{1+x^4}-\sin^2 t, k(t)=e^{-t}, r(t)=\sin^2 t$. Then, it is easy to prove that $|f(t,x)|\le k(t)|x|+r(t)$, a.e.$(t,x)\in\mathbb{R}\times\mathbb{R}$, and $e^{-2\pi}\le k(s)\le 1, s\in[0,2\pi], \Lambda=\frac{1}{1-e^{-\frac{1}{5}}}, \frac{1}{e^{2\pi}-1}\le H(t,s)\le\frac{e^{2\pi}}{e^{2\pi}-1}$,

$$
\begin{aligned}
k^M &\le \max_{t\in[0,2\pi]}\Big[\frac{1-\cos t}{400}+\frac{1+\sin t}{400}+\frac{2\pi e^{2\pi}(1-\sin t)}{320(e^{2\pi}-1)}\Big]\\
&\le \frac{4}{400}+\frac{4\pi e^{2\pi}}{320(e^{2\pi}-1)} < 0.0495,\\
K &\le \int_0^{2\pi}\Big[\frac{1-\cos t}{400}+\frac{1+\sin t}{400}+\frac{2\pi e^{2\pi}(1-\sin t)}{320(e^{2\pi}-1)}\Big]dt\\
&\le \frac{\pi}{100}+\frac{4\pi^2 e^{2\pi}}{320(e^{2\pi}-1)},\\
\Lambda K &\le \Big[\frac{\pi}{100}+\frac{4\pi^2 e^{2\pi}}{320(e^{2\pi}-1)}\Big]\cdot\frac{1}{1-e^{-\frac{1}{5}}} < 0.8561,\\
\Pi &= \frac{1}{5\pi} < 0.0637.
\end{aligned}
$$

So $\Pi + k^M + \Lambda K < 0.9693 < 1$. Hence, by theorem 3.1, system (3.2) has at least one nontrivial 2π-periodic solution.

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