Turk J Math
34 (2010) , 45 – 58.

doi:10.3906/mat-0806-3

Existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay in $L^p(\Omega, C_h)$

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Abstract

In this paper, we shall consider the existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay in $L^p(\Omega, C_h)$ space:

$$d[x(t) - G(x_t)] = f(t, x_t)dt + g(t, x_t)dB(t),$$

where we assume $f : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, R^n)$, $g : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, L(R^m, R^n))$, $G : L^p(\Omega, C_h) \to L^p(\Omega, R^n)$, $p > 2$, and $B(t)$ is a given m-dimensional Brownian motion.

Key Words: Neutral stochastic functional differential equations; existence; uniqueness; infinite delay.

1. Introduction

In recent years, the existence, uniqueness and stability of stochastic differential equations have been extensively investigated by many authors (for example, see L. Arnold [1], A. Friedman [4], R. Z. Has'minskii [6], N. Ikeda and S. Watanabe [7], X. Mao [12, 13]). It is well known that these topics have been developed mainly by using two different methods, that is, the iterative method and the Banach fixed point theorem. As a matter of fact, there exist extensive literatures on the related topics for stochastic functional differential equations and stochastic partial functional differential equations with finite delay (for example, see K. Liu and X. Mao [10], K. Liu and X. Xia [11], S. E. A. Mohammed [14], T. Taniguchi *et al* [15], T. Taniguchi [16] and the references therein). However, as far as the present authors know, there seems to be no work on neutral stochastic functional differential equations with infinite delay. We would also like to mention that some similar topics to the above for functional differential equations with infinite delay have already been investigated by various authors (cf. [2], [3], [5], [9], [17], [18]).

In this paper, we adopt the symbols as follow: R^n denotes the usual n-dimensional Euclidean space, R^-, R^+ and R denote the interval $(-\infty, 0]$, $[0, \infty)$ and $(-\infty, +\infty)$ respectively. Suppose $x \in R^n$, let $|x| = \sum_{i=1}^n |x_i|, x = (x_1, ..., x_n)$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined. Suppose $x(t) : \Omega \to R^n$ is a continuous $\mathcal{F}_t$-adapted stochastic process, we can associate with another process $x_t : \Omega \to L^p(\Omega, C_h)$, $t \geq 0$, by setting $x_t(s)(\omega) = x(t+s)(\omega)$, $s \in R^-$. Then we say that the process x_t is generalized by the process $x(t)$.

Here $L^p(\Omega, C_h)$ denotes the space of all $\mathcal{F}$-measurable stochastic processes from Ω to C_h with $L^p(\Omega, C_h)$-norm (see Section 2). Let $B(t)$ is a given m-dimensional standard Brownian motion.

In this paper, by using Banach fixed point theorem we shall discuss the existence and the uniqueness of solutions to stochastic functional equations with infinite delay,

$$\begin{cases} d[x(t) - G(x_t)] = f(t, x_t)dt + g(t, x_t)dB(t), & t > 0, \\ x_0 = \phi \in L^p(\Omega, C_h), \end{cases} \tag{1.1}$$

where ϕ is $\mathcal{F}_0$-measurable. Throughout this paper, we shall assume $f : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, R^n)$ and $g : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, L(R^m, R^n)),\ p > 2$.

The contents of this paper are organized as follows. In Section 2, the Banach space $L^p(\Omega, C_h)$ is studied which is fundamental for the subsequent developments. In Section 3, we shall discuss the existence and the uniqueness of solutions to stochastic functional equations with infinite delay. Finally, we shall present in Section 4 some applications about Volterra stochastic integro-differential equation with infinite delay:

$$d[x(t) - \kappa x(t-s)] = \int_{-\infty}^{t} D(t,s)F(x(s))dsdt + \int_{-\infty}^{t} K(t,s)G(x(s))dsdB(t), \quad t > 0.$$

2. Banach space $L^p(\Omega, C_h)$

Suppose $x(t) : [a, b] \to R^n$. Let

$$\|x\|^{[a,b]} = \sup\{|x(s)| : a \le s \le b\}.$$

Assume h is a continuous function from R^- to R with $h(s) > 0$ and $l = \int_{-\infty}^{0} h(s)ds < +\infty$. Let

$$C_h = \{x \in C(R^-, R^n) | \int_{-\infty}^{0} h(s)\|x\|^{[s,0]}ds < +\infty\}.$$

Then C_h is a Banach space with the norm $\|x\|_{C_h} = \int_{-\infty}^{0} h(s)\|x\|^{[s,0]}ds$ (cf. [9], [17], [18]).

Denote by $L^p(\Omega, C_h)$ the space of all $\mathcal{F}$-measurable stochastic processes $\phi : \Omega \to C_h$ such that $\|\phi(\omega)\|_{C_h}$ is of class $L^p (p > 2)$, i.e.

$$L^p(\Omega, C_h) = \{\phi : \Omega \to C_h | (E[\int_{-\infty}^{0} h(s)\|\phi\|^{[s,0]}ds]^p)^{\frac{1}{p}} < +\infty\}.$$

It is easy to see

$$\|\phi\|_{L^p(\Omega, C_h)} := (E[\int_{-\infty}^{0} h(s)\|\phi\|^{[s,0]}ds]^p)^{\frac{1}{p}}$$

is a norm in $L^p(\Omega, C_h)$.

Let

$$L^p(\Omega, C([a,b], R^n)) = \{\phi : \Omega \to C([a,b], R^n) | (E[\|\phi\|^{[a,b]}]^p)^{\frac{1}{p}} < +\infty\}$$

with the norm

$$\|\phi\|_{L^p(\Omega,C([a,b],R^n))} = (E[\|\phi\|^{[a,b]}]^p)^{\frac{1}{p}}.$$

Lemma 2.1 *For any $\epsilon > 0$ and any $k > 0$, there exists $\delta = \delta(\epsilon, k) > 0$ such that for any $\phi_1, \phi_2 \in L^p(\Omega, C_h)$, if $\|\phi_1 - \phi_2\|_{L^p(\Omega,C_h)} \le \delta$, then $\|\phi_1 - \phi_2\|_{L^p(\Omega,C([-k,0],R^n))} \le \epsilon$.*

Proof. If this is not true, suppose there exist $\epsilon_0 > 0$, $k_0 > 0$ such that for any $\delta > 0$, there exist ϕ_1^0, $\phi_2^0 \in L^p(\Omega, C_h)$ satisfying $\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C_h)} \le \delta$, then we have $\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C([-k_0,0],R^n))} \ge \epsilon_0$. Let $l_0 = \int_{-\infty}^{-k_0} h(s)ds > 0$ and select $\delta_0 < \epsilon_0 l_0$, then there exists $\phi_1^0, \phi_2^0 \in L^p(\Omega, C_h)$ satisfying $\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C_h)} \le \delta_0$. On the other hand, $\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C([-k_0,0],R^n))} \ge \epsilon_0$. Thus

$$\begin{aligned}
\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C_h)} &= (E[\int_{-\infty}^{0} h(s)\|\phi_1^0 - \phi_2^0\|^{[s,0]}ds]^p)^{\frac{1}{p}} \\
&= (E[\int_{-\infty}^{-k_0} h(s)\|\phi_1^0 - \phi_2^0\|^{[s,0]}ds + \int_{-k_0}^{0} h(s)\|\phi_1^0 - \phi_2^0\|^{[s,0]}ds]^p)^{\frac{1}{p}} \\
&\ge (E[\int_{-\infty}^{-k_0} h(s)\|\phi_1^0 - \phi_2^0\|^{[s,0]}ds]^p)^{\frac{1}{p}} \\
&\ge (E[\int_{-\infty}^{-k_0} h(s)\|\phi_1^0 - \phi_2^0\|^{[-k_0,0]}ds]^p)^{\frac{1}{p}} \\
&\ge \int_{-\infty}^{-k_0} h(s)ds\|\phi_1^0 - \phi_2^0\|_{L^p(\Omega,C([-k_0,0],R^n))} \\
&\ge \epsilon_0 l_0,
\end{aligned}$$

therefore $\delta_0 \ge \epsilon_0 l_0$, which contradicts $\delta_0 < \epsilon_0 l_0$. This completes the proof. □

Lemma 2.2 *Suppose ϕ_m are sequences in $L^p(\Omega, C(R^-, R^n))$ which are uniformly bounded; then $\lim_{m\to\infty} \|\phi_m - \phi_0\|_{L^p(\Omega,C_h)} = 0$ if and only if for any $k > 0$, $\lim_{m\to\infty} \|\phi_m - \phi_0\|_{L^p(\Omega,C([-k,0],R^n))} = 0$.*

Proof. Necessity can be proved straight away by Lemma 2.1, so we only need to prove the sufficiency. Suppose $\|\phi_m\|_{L^p(\Omega,C(R^-,R^n))} \le H$. Since for any $k > 0$, $\lim_{m\to\infty} \|\phi_m - \phi_0\|_{L^p(\Omega,C([-k,0],R^n))} = 0$, thus $\|\phi_0\|_{L^p(\Omega,C([-k,0],R^n))}$ is bounded, so $\|\phi_0\|_{L^p(\Omega,C(R^-,R^n))}$ is also bounded. Assume $\|\phi_0\|_{L^p(\Omega,C(R^-,R^n))} \le H'$.

By $l = \int_{-\infty}^{0} h(s)ds < +\infty$, then for any $\epsilon > 0$, there exists $k > 0$ such that $\int_{-\infty}^{-k} h(s)ds < \epsilon$, and there exists $N > 0$ such that when $m > N$, we have $\|\phi_m - \phi_0\|_{L^p(\Omega,C([-k,0],R^n))} < \epsilon$. Therefore, if $m > N$, we have

$$
\begin{aligned}
\|\phi_m - \phi_0\|^p_{L^p(\Omega,C_h)} &= E[\int_{-\infty}^0 h(s)\|\phi_m - \phi_0\|^{[s,0]}ds]^p \\
&= E[\int_{-\infty}^{-k} h(s)\|\phi_m - \phi_0\|^{[s,0]}ds + \int_{-k}^0 h(s)\|\phi_m - \phi_0\|^{[s,0]}ds]^p \\
&\leq 2^{p-1}E[\int_{-\infty}^{-k} h(s)\|\phi_m - \phi_0\|^{[s,0]}ds]^p + 2^{p-1}E[\int_{-k}^0 h(s)\|\phi_m - \phi_0\|^{[s,0]}ds]^p \\
&\leq 2^{p-1}[\int_{-\infty}^{-k} h(s)ds]^p E(\|\phi_m\|^{R^-} + \|\phi_0\|^{R^-})^p + 2^{p-1}E[\int_{-k}^0 h(s)\|\phi_m - \phi_0\|^{[-k,0]}ds]^p \\
&\leq 4^{p-1}[\int_{-\infty}^{-k} h(s)ds]^p[\|\phi_m\|^p_{L^p(\Omega,C(R^-,R^n))} + \|\phi_0\|^p_{L^p(\Omega,C(R^-,R^n))}] + 2^{p-1}(\epsilon\ l)^p \\
&\leq 4^{p-1}[(H^p + H'^p)\epsilon^p + \epsilon^p l^p].
\end{aligned}
$$

This completes the proof. □

Lemma 2.3 $L^p(\Omega, C_h)$ *is a Banach space.*

Proof. Suppose $\{\phi_m\}$ are a Cauchy sequence in $(L^p(\Omega, C_h), \|\cdot\|_{L^p(\Omega,C_h)})$, then for any $\epsilon > 0$, there exists a positive integer N, if $m',\ m > N$, then we have

$$
\|\phi_{m'} - \phi_m\|_{L^p(\Omega,C_h)} = (E[\int_{-\infty}^0 h(s)\|\phi_{m'} - \phi_m\|^{[s,0]}ds]^p)^{\frac{1}{p}} < \epsilon
$$

and

$$
\begin{aligned}
|\|\phi_{m'}\|_{L^p(\Omega,C_h)} - \|\phi_m\|_{L^p(\Omega,C_h)}| &= |(E[\int_{-\infty}^0 h(s)\|\phi_{m'}\|^{[s,0]}ds]^p)^{\frac{1}{p}} - (E[\int_{-\infty}^0 h(s)\|\phi_m\|^{[s,0]}ds]^p)^{\frac{1}{p}}| \\
&\leq [E(\int_{-\infty}^0 h(s)\|\phi_{m'}\|^{[s,0]}ds - \int_{-\infty}^0 h(s)\|\phi_m\|^{[s,0]}ds)^p]^{\frac{1}{p}} \\
&= \|\phi_{m'} - \phi_m\|_{L^p(\Omega,C_h)} < \epsilon.
\end{aligned}
$$

Thus $\|\phi_m\|_{L^p(\Omega,C_h)}$ is also a Cauchy sequence, thus it is bounded.

Suppose $\|\phi_m\|_{L^p(\Omega,C_h)} \leq\ M$, we prove that for any $k > 0, \{\|\phi_m\|_{L^p(\Omega,C([-k,0],R^n))}\}$ is bounded.

If this is not true, then there exist $m_i, i = 1, 2, \cdots$ such that $\|\phi_{m_i}\|_{L^p(\Omega,C([-k,0],R^n))} \geq i$. Thus,

$$
\begin{aligned}
\|\phi_{m_i}\|_{L^p(\Omega,C_h)} &= (E[\int_{-\infty}^{0} h(s)\|\phi_{m_i}\|^{[s,0]}ds]^p)^{\frac{1}{p}} \\
&\geq (E[\int_{-\infty}^{-k} h(s)\|\phi_{m_i}\|^{[s,0]}ds]^p)^{\frac{1}{p}} \\
&\geq (E[\int_{-\infty}^{-k} h(s)\|\phi_{m_i}\|^{[-k,0]}ds]^p)^{\frac{1}{p}} \\
&\geq i\int_{-\infty}^{-k} h(s)ds \to +\infty.
\end{aligned}
$$

This contradicts $\|\phi_m\|_{L^p(\Omega,C_h)} \leq M$.

By Lemma 2.1, we have $\|\phi_{m'} - \phi_m\|_{L^p(\Omega,C([-k,0],R^n))} < \epsilon$ if $m, m' > N$ and N is large enough. So there exists a function $\phi \in L^p(\Omega, C([-k,0],R^n))$ such that $\lim\limits_{m\to\infty} \|\phi_m - \phi\|_{L^p(\Omega,C([-k,0],R^n))} = 0$. Since k is arbitrary, ϕ can be extended on R^-, i,e., $\phi \in L^p(\Omega, C(R^-,R^n))$.

For any $k > 0$,

$$(E[\int_{-k}^{0} h(s)\|\phi_{m'} - \phi_m\|^{[s,0]}ds]^p)^{\frac{1}{p}} < \epsilon,$$

let $m' \to \infty$, we have

$$(E[\int_{-k}^{0} h(s)\|\phi_m - \phi\|^{[s,0]}ds]^p)^{\frac{1}{p}} < \epsilon.$$

Thus,

$$\|\phi_m - \phi\|_{L^p(\Omega,C_h)} = (E[\int_{-\infty}^{0} h(s)\|\phi_m - \phi\|^{[s,0]}ds]^p)^{\frac{1}{p}} \leq \epsilon,$$

i.e. $\lim\limits_{m\to\infty} \|\phi_m - \phi\|_{L^p(\Omega,C_h)} = 0$.

Moreover,

$$
\begin{aligned}
\|\phi\|^p_{L^p(\Omega,C_h)} &= E[\int_{-\infty}^{0} h(s)\|\phi\|^{[s,0]}ds]^p \\
&\leq E[\int_{-\infty}^{0} h(s)(\|\phi_m\|^{[s,0]} + \|\phi_m - \phi\|^{[s,0]})ds]^p \\
&\leq 2^{p-1}E[\int_{-\infty}^{0} h(s)\|\phi_m\|^{[s,0]}ds]^p + 2^{p-1}E[\int_{-\infty}^{0} h(s)\|\phi_m - \phi\|^{[s,0]}ds]^p \\
&\leq 2^{p-1}\|\phi_m\|^p_{L^p(\Omega,C_h)} + 2^{p-1}\|\phi_m - \phi\|^p_{L^p(\Omega,C_h)} < \infty.
\end{aligned}
$$

Thus $\phi \in L^p(\Omega, C_h)$, therefore $(L^p(\Omega,C_h), \|\cdot\|_{L^p(\Omega,C_h)})$ is a Banach space. □

Lemmas 2.4 *Suppose* $\phi \in L^p(\Omega, C_h), A > 0,$ $x(t) = \phi(t)$ *on* $(-\infty, 0]$ *and* $x(t) \in L^p(\Omega, C([0, A], R^n))$ *on* $[0, A]$, *then for all* $t \in [0, A], x_t \in L^p(\Omega, C_h)$ *and* x_t *is continuous for* t *in* $L^p(\Omega, C_h)$.

Proof. As a matter of fact, we have

$$\begin{aligned} E[\int_{-\infty}^{0} h(s)\|x_t\|^{[s,0]}ds]^p &= E[\int_{-\infty}^{-t} h(s)\|x_t\|^{[s,0]}ds + \int_{-t}^{0} h(s)\|x_t\|^{[s,0]}ds]^p \\ &\leq 2^{p-1}E[\int_{-\infty}^{-t} h(s)\|x_t\|^{[s,0]}ds]^p + 2^{p-1}E[\int_{-t}^{0} h(s)\|x_t\|^{[s,0]}ds]^p \\ &\leq 2^{p-1}E[\int_{-\infty}^{-t} h(s)\max\{\|x_t\|^{[s,-t]}, \|x_t\|^{[-t,0]}\}ds]^p + 2^{p-1}E[\int_{-t}^{0} h(s)\|x\|^{[0,t]}ds]^p \\ &\leq 4^{p-1}E[\int_{-\infty}^{-t} h(s)\|x\|^{[s+t,0]}ds]^p + 4^{p-1}E[\int_{-\infty}^{-t} h(s)\|x\|^{[0,t]}ds]^p \\ &+ 2^{p-1}E[\int_{-t}^{0} h(s)\|x\|^{[s,0]}ds]^p \\ &\leq 4^{p-1}E[\int_{-\infty}^{0} h(s)\|\phi\|^{[s,0]}ds]^p + (4^{p-1} + 2^{p-1})l^p\|x\|_{L^p(\Omega, C([0,A], R^n))} < \infty, \end{aligned}$$

therefore $x_t \in L^p(\Omega, C_h)$.

Next we prove that x_t is continuous for t. Without loss of generality, we suppose $t_0 \in [0, A],\ t \in [0, A]$. Let $0 \leq t_0 \leq t$, for any $\epsilon > 0$, there exists $M(t_0, \epsilon) > 0$, such that

$$(E[\int_{-\infty}^{-M} h(s)\|x_{t_0}\|^{[s,0]}ds]^p)^{\frac{1}{p}} < \frac{\epsilon}{8}, \quad \int_{-\infty}^{-M} h(s)ds < \frac{\epsilon}{4L},$$

where $\|x\|_{L^p(\Omega, C([0,A], R^n))} \leq L$. Suppose that $|t_0 - t|$ is small enough such that

$$\|x_t - x_{t_0}\|_{L^p(\Omega, C([-M,0], R^n))} < \frac{\epsilon}{2l}.$$

Thus

$$\begin{aligned} \|x_t - x_{t_0}\|^p_{L^p(\Omega, C_h)} &= E[\int_{-\infty}^{0} h(s)\|x_t - x_{t_0}\|^{[s,0]}ds]^p \\ &= E[\int_{-\infty}^{-M} h(s)\|x_t - x_{t_0}\|^{[s,0]}ds + \int_{-M}^{0} h(s)\|x_t - x_{t_0}\|^{[s,0]}ds]^p \\ &\leq 2^{p-1}E[\int_{-\infty}^{-M} h(s)\|x_t\|^{[s,0]}ds + \int_{-\infty}^{-M} h(s)\|x_{t_0}\|^{[s,0]}ds]^p \\ &+ 2^{p-1}E[\int_{-M}^{0} h(s)\|x_t - x_{t_0}\|^{[s,0]}ds]^p \end{aligned}$$

$$
\begin{aligned}
&\leq\ 2^{p-1}E[\int_{-\infty}^{-M} h(s)(\|x_{t_0}\|^{[s,0]}+\|x\|^{[t_0,t]}+\|x_{t_0}\|^{[s,0]})ds]^p\\
&+\ 2^{p-1}E[\int_{-M}^{0} h(s)\|x_t-x_{t_0}\|^{[s,0]}ds]^p\\
&\leq\ 4^{p-1}E[2\int_{-\infty}^{-M} h(s)\|x_{t_0}\|^{[s,0]}ds]^p+4^{p-1}E[\int_{-\infty}^{-M} h(s)\|x\|^{[t_0,t]}ds]^p\\
&+\ 2^{p-1}E[\int_{-M}^{0} h(s)\|x_t-x_{t_0}\|^{[s,0]}ds]^p\\
&\leq\ 4^{p-1}E[2\int_{-\infty}^{-M} h(s)\|x_{t_0}\|^{[s,0]}ds]^p+4^{p-1}E[\int_{-\infty}^{-M} h(s)\|x\|^{[0,A]}ds]^p\\
&+\ 2^{p-1}E[\int_{-M}^{0} h(s)\|x_t-x_{t_0}\|^{[-M,0]}ds]^p\\
&\leq\ 4^{p-1}2^p\frac{\epsilon^p}{8^p}+4^{p-1}(\frac{\epsilon}{4L})^p\|x\|^p_{L^p(\Omega,C([0,A],R^n))}+2^{p-1}l^p\|x_t-x_{t_0}\|^p_{L^p(\Omega,C([-M,0],R^n))}\\
&<\ \frac{\epsilon^p}{4}+\frac{\epsilon^p}{4}+\frac{\epsilon^p}{2}=\epsilon^p.
\end{aligned}
$$

Therefore $\|x_t-x_{t_0}\|_{L^p(\Omega,C_h)}<\epsilon$, thus x_t is continuous for t in $L^p(\Omega,C_h)$. □

3. Existence and uniqueness of solutions

We shall consider the following stochastic integral equation instead of (1.1) by finding a fixed point of the operate Φ:

$$
\begin{cases} x(t)=\phi(0)+G(x_t)-G(x_0)+\int_0^t f(s,x_s)ds+\int_0^t g(s,x_s)dB(s), & t>0,\\ x_0=\phi\in L^p(\Omega,C_h), \end{cases} \tag{3.1}
$$

where ϕ is $\mathcal{F}_0$-measurable.

Throughout this section, we assume that:

$(H_1):$ For arbitrary $\xi,\eta\in L^p(\Omega,C_h)$ and $0\leq t\leq T$, suppose that there exist positive real constants $L_1=L_1(T),\ L_2=L_2(T),\ \kappa=\kappa(T)>0$ such that

$$\|f(t,\xi)-f(t,\eta)\|_{L^p}\leq L_1\|\xi-\eta\|_{L^p(\Omega,C_h)},$$

$$\|g(t,\xi)-g(t,\eta)\|_{L^p}\leq L_2\|\xi-\eta\|_{L^p(\Omega,C_h)},$$

and

$$\|G(\xi)-G(\eta)\|_{L^p}\leq \kappa\|\xi-\eta\|_{L^p(\Omega,C_h)}.$$

$(H_2):$ There exists a $c = c(T) > 0$ such that

$$\|f(t,\xi)\|_{L^p} + \|g(t,\xi)\|_{L^p} + \|G(\xi)\|_{L^p} \le c(1 + \|\xi\|_{L^p(\Omega,C_h)})$$

for $0 \le t \le T$, where $T > 0$ is any fixed time .

Theorem 3.1 *Suppose assumptions* (H_1) *and* (H_2) *hold. Then there exist a unique local uniformly continuous solution to* (1.1) *for any initial value* $(0,\phi)$ *with* $\phi \in L^p(\Omega, C_h)$.

To prove this theorem, assume $T > 0$ is a fixed point to be determined later, and D_T is the subspace of all continuous processes z which belong to the space $C((-\infty,T], L^p(\Omega,C_h))$, $z_t \in L^p(\Omega,C_h)$, with $\|z\|_{D_T} < \infty$, where

$$\|z\|_{D_T} := \sup_{0\le t\le T} \|z_t\|_{L^p(\Omega,C_h)}. \tag{3.2}$$

Now we introduce the following mapping Φ on D_T:

$$\begin{cases} (\Phi z)(t) = \phi(0) + G(z_t) - G(z_0) + \int_0^t f(s,z_s)ds + \int_0^t g(s,z_s)dB(s), & t \ge 0, \\ (\Phi z)(t) = \phi(t) \in L^p(\Omega,C_h), & t \le 0. \end{cases}$$

Lemma 3.1 *Suppose the operator mapping* Φ *and the corresponding domain* D_T *are defined by (3.2), then* $\Phi(D_T) \subset D_T$.

Proof. By the definition of Φ, we have

$$\begin{aligned} \|(\Phi z)_t\|^p_{L^p(\Omega,C_h)} &= E[\int_{-\infty}^0 h(s)\|(\Phi z)_t\|^{[s,0]}ds]^p \\ &= E[\int_{-\infty}^{-t} h(s)\|(\Phi z)_t\|^{[s,0]}ds + \int_{-t}^0 h(s)\|(\Phi z)_t\|^{[s,0]}ds]^p \\ &\le 2^{p-1}E[\int_{-\infty}^{-t} h(s)\|(\Phi z)_t\|^{[s,0]}ds]^p + 2^{p-1}E[\int_{-t}^0 h(s)\|(\Phi z)_t\|^{[s,0]}ds]^p \\ &\le 2^{p-1}E[\int_{-\infty}^{-t} h(s)\max\{\|(\Phi z)_t\|^{[s,-t]}, \|(\Phi z)_t\|^{[-t,0]}\}ds]^p + 2^{p-1}E[\int_{-t}^0 h(s)\|\Phi z\|^{[0,t]}ds]^p \\ &\le 4^{p-1}E[\int_{-\infty}^{-t} h(s)\|\Phi z\|^{[s+t,0]}ds]^p + 4^{p-1}E[\int_{-\infty}^{-t} h(s)\|\Phi z\|^{[0,t]}ds]^p \\ &+ 2^{p-1}E[\int_{-t}^0 h(s)\|\Phi z\|^{[0,t]}ds]^p \\ &\le 4^{p-1}E[\int_{-\infty}^0 h(s)\|\phi\|^{[s,0]}ds]^p + (4^{p-1} + 2^{p-1})l^p\|\Phi z\|^p_{L^p(\Omega,C([0,T],R^n))}. \end{aligned}$$

Almost every sample path of $(\Phi z)(t)$ is uniformly continuous on the interval $[0,T]$, then it is easy to prove that

$$[\sup_{0\le t\le T} |(\Phi z)(t)|]^p = \sup_{0\le t\le T} |(\Phi z)(t)|^p \quad a.s.$$

By the Doob's inequality (cf. Mohammed [14, Theorem 6.1]),

$$E \sup_{0\le t\le T} |(\Phi z)(t)|^p \le (\frac{p}{p-1})^p E|(\Phi z)(T)|^p.$$

So we have

$$\begin{aligned}
\|\Phi z\|^p_{L^p(\Omega,C([0,T],R^n))} &= E[\sup_{0\le t\le T} |(\Phi z)(t)|]^p = E\sup_{0\le t\le T}|(\Phi z)(t)|^p \le (\frac{p}{p-1})^p E|(\Phi z)(T)|^p \\
&= (\frac{p}{p-1})^p E[|\phi(0) + G(z_t) - G(z_0) + \int_0^T f(\tau, z_\tau)d\tau + \int_0^T g(\tau, z_\tau)dB(\tau)|]^p \\
&\le (\frac{p}{p-1})^p 5^{p-1} E|\phi(0)|^p + (\frac{p}{p-1})^p 5^{p-1} c(1+\|\phi\|)^p_{D_T} + (\frac{p}{p-1})^p 5^{p-1} c(1+\|z\|)^p_{D_T} \\
&+ (\frac{p}{p-1})^p 5^{p-1} E|\int_0^T f(\tau, z_\tau)d\tau|^p + (\frac{p}{p-1})^p 5^{p-1} E|\int_0^T g(\tau, z_\tau)dB(\tau)|^p \\
&= (\frac{p}{p-1})^p 5^{p-1} E|\phi(0)|^p + (\frac{p}{p-1})^p 5^{p-1} c^p(1+\|\phi\|)^p_{D_T} \\
&+ (\frac{p}{p-1})^p 5^{p-1} c^p (1+\|z\|)^p_{D_T} + I_1 + I_2,
\end{aligned}$$

$$\begin{aligned}
I_1 &= (\frac{p}{p-1})^p 5^{p-1} E[|\int_0^T f(\tau, z_\tau)d\tau|]^p \\
&\le (\frac{p}{p-1})^p 5^{p-1} T^{\frac{p}{q}} \int_0^T E|f(\tau, z_\tau)|^p d\tau \\
&\le (\frac{p}{p-1})^p 5^{p-1} T^{\frac{p}{q}} \int_0^T c^p(1+\|z_\tau\|_{L^p(\Omega, C_h)})^p d\tau \\
&\le (\frac{p}{p-1})^p 10^{p-1} T^{1+\frac{p}{q}} c^p (1+\|z\|^p_{D_T}),
\end{aligned}$$

$$\begin{aligned}
I_2 &= (\frac{p}{p-1})^p 5^{p-1} E[|\int_0^T g(\tau, z_\tau)dB(\tau)|]^p \\
&\le (\frac{p}{p-1})^p 5^{p-1} [\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p-2}{2}} \int_0^T E|g(\tau, z_\tau)|^p d\tau \\
&\le (\frac{p}{p-1})^p 5^{p-1} [\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p-2}{2}} c^p \int_0^T (1+\|z_\tau\|_{L^p(\Omega, C_h)})^p d\tau \\
&\le (\frac{p}{p-1})^p 10^{p-1} [\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p}{2}} c^p (1+\|z\|^p_{D_T}).
\end{aligned}$$

Therefore, we obtain that $\|\Phi z\|_{D_T} < \infty$. This completes the proof. □

Proof of Theorem 3.1: Let $X, Y \in D_T$, then for any fixed $t \in [0,T]$

$$
\begin{aligned}
&\|(\Phi X)_t - (\Phi Y)_t\|^p_{L^p(\Omega,C_h)} = E[\int_{-\infty}^{0} h(s)\|(\Phi X)_t(\theta) - (\Phi Y)_t(\theta)\|^{[s,0]}ds]^p \\
= \;& E[\int_{-\infty}^{-t} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[s,0]}ds + \int_{-t}^{0} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[s,0]}ds]^p \\
\le \;& E[\int_{-\infty}^{-t} h(s)\max\{\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[s,-t]}, \|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[-t,0]}\}ds \\
+ \;& \int_{-t}^{0} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[s,0]}ds]^p \\
\le \;& E[\int_{-\infty}^{-t} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[s,-t]}ds + \int_{-\infty}^{-t} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[-t,0]}ds \\
+ \;& \int_{-t}^{0} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[-t,0]}ds]^p \\
\le \;& E[\int_{-\infty}^{-t} h(s)\|\phi(t+\theta) - \phi(t+\theta)\|^{[s,-t]}ds + 2\int_{-\infty}^{0} h(s)\|(\Phi X)(t+\theta) - (\Phi Y)(t+\theta)\|^{[-t,0]}ds]^p \\
= \;& E[2\int_{-\infty}^{0} h(s)\|\Phi X - \Phi Y\|^{[0,t]}ds]^p \le l^p 2^p \|\Phi X - \Phi Y\|^p_{L^p(\Omega,C([0,T],R^n))} \\
= \;& l^p 2^p E[\sup_{0\le t\le T} |(\Phi X - \Phi Y)(t)|]^p = l^p 2^p E\sup_{0\le t\le T} |(\Phi X - \Phi Y)(t)|^p \\
\le \;& l^p 3^p (\frac{p}{p-1})^p E|(\Phi X - \Phi Y)(T)|^p \\
\le \;& l^p(\frac{p}{p-1})^p 3^{2p-1} E[|G(X_T) - G(Y_T)|^p + l^p(\frac{p}{p-1})^p 3^{2p-1}\|X_T + Y_T\| E[|\int_0^T [f(\tau,X_\tau) - f(\tau,Y_\tau)]d\tau|]^p \\
+ \;& l^p(\frac{p}{p-1})^p 3^{2p-1} E[|\int_0^T [g(\tau,X_\tau) - g(\tau,Y_\tau)]dB(\tau)|]^p \\
\le \;& l^p(\frac{p}{p-1})^p 3^{2p-1}\kappa^p \|X - Y\|^p_{D_T} + I_3 + I_4,
\end{aligned}
$$

$$
\begin{aligned}
I_3 \;=\;& l^p(\frac{p}{p-1})^p 3^{2p-1} E[|\int_0^T [f(\tau,X_\tau) - f(\tau,Y_\tau)]d\tau|]^p \\
\le \;& l^p(\frac{p}{p-1})^p 3^{2p-1} T^{\frac{p}{q}} \int_0^T E|f(\tau,X_\tau) - f(\tau,Y_\tau)|^p d\tau \\
\le \;& l^p(\frac{p}{p-1})^p 3^{2p-1} T^{\frac{p}{q}} \int_0^T L_1^p \|X_\tau - Y_\tau\|^p_{L^p(\Omega,C_h)} d\tau \\
\le \;& l^p(\frac{p}{p-1})^p 3^{2p-1} T^{1+\frac{p}{q}} L_1^p \|X - Y\|^p_{D_T},
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= l^p(\frac{p}{p-1})^p 3^{2p-1} E[|\int_0^T [g(\tau,X_\tau)-g(\tau,Y_\tau)]dB(\tau)|]^p \\
&\le l^p(\frac{p}{p-1})^p 3^{2p-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p-2}{2}} \int_0^T E|g(\tau,X_\tau)-g(\tau,Y_\tau)|^p d\tau \\
&\le l^p(\frac{p}{p-1})^p 3^{2p-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p-2}{2}} \int_0^T L_2^p \|X_\tau - Y_\tau\|^p_{L^p(\Omega,C_h)} d\tau \\
&\le l^p(\frac{p}{p-1})^p 3^{2p-1}[\frac{p(p-1)}{2}]^{\frac{p}{2}} T^{\frac{p}{2}} L_2^p \|X-Y\|^p_{D_T}.
\end{aligned}
$$

Hence by takeing a suitable $T > 0$ such that T is sufficiently small, we obtain a positive real number $\rho(T) \in (0,1)$ such that

$$\|\Phi X - \Phi Y\|_{D_T} \le \rho(T)\|X-Y\|_{D_T}$$

for any $X, Y \in D_T$. Thus by the well known Banach fixed point theorem we have a unique fixed point $X \in D_T$ which yields

$$
\begin{cases}
X(t) = \phi(0) + G(X_t) - G(X_0) + \int_0^t f(s,X_s)ds + \int_0^t g(s,X_s)dB(s), & t \ge 0, \\
X(t) = \phi(t), \quad t \le 0,
\end{cases}
$$

which proves the existence of a local uniformly continuous solution of (1.1). The uniqueness of solution is proved similarly. Therefore the proof is complete. □

Theorem 3.2 *Let* $f : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, R^n)$, $g : R^+ \times L^p(\Omega, C_h) \to L^p(\Omega, L(R^m, R^n))$ *satisfy the assumption* (H_1). *If there exists a constant* $c > 0$ *such that*

$$\|f(t,\xi)\|_{L^P} + \|g(t,\xi)\|_{L^P} + \|G(\xi)\|_{L^p} \le c(1 + \|\xi\|_{L^p(\Omega,C_h)})$$

for all $\xi \in L^p(\Omega, C_h)$, $t \ge 0$, *then there exist a unique, global uniformly continuous solution* $X(t) : \Omega \to R^n$ *to the equation* (1.1) *for any initial* $(0, \phi)$ *with* $\phi \in L^p(\Omega, C_h)$.

Proof. If f and g satisfy the global Lipschitz condition, then the proof of the theorem can be given similarly as a corollary of Theorem 3.1. If f and g satisfy the local Lipschitz condition, then the proof is given by the standard truncation method [15, Theorem 2.5]. Hence we omit the proof. □

4. Applications

The existence, uniqueness and stability of Volterra integro-differential equation have already been investigated by many authors (cf. [2], [3], [17], [18]). In this section, we shall discuss Volterra neutral stochastic integro-differential equation with infinite delay.

Let $D(t,s), K(t,s) \in C(R^2, R)$. Assume that there exists a positive continuous function $h(s)$ on R^- with

$$|D(t,t+s)|,\ |K(t,t+s)| \le h(s),\ \int_{-\infty}^{0} h(s)ds < +\infty. \tag{4.1}$$

We consider the following Volterra stochastic integro-differential equation:

$$\begin{cases} d[x(t)-\kappa x(t-s)] = \int_{-\infty}^{t} D(t,s)F(x(s))dsdt + \int_{-\infty}^{t} K(t,s)G(x(s))dsdB(t), & t>0, \\ x_0 = \phi \in L^p(\Omega, C_h), \end{cases} \tag{4.2}$$

where $\kappa > 0$, $B(t)$ is a one-dimensional standard Brownian motion, and ϕ is $\mathcal{F}_0$-measurable.

Throughout this section, we assume that

(B) : Both $F: R \to R$ and $G: R \to R$ are continuous functions. They satisfy the Lipschitz condition and the linear growth condition. That is, for arbitrary $x, y \in R$, there exist positive real constants $L_1,\ L_2 > 0$ such that

$$|F(x)-F(y)| \le L_1|x-y|, \quad |G(x)-G(y)| \le L_2|x-y|,$$

and there is moreover a $c > 0$ such that

$$|F(x)| + |G(x)| \le c(1+|x|), \quad x \in R.$$

Theorem 4.1 *Suppose (4.1) and assumption (B) hold. Then there exist a unique, global uniformly continuous solution $X(t): \Omega \to R$ to the equation (4.2) for any initial $(0,\phi)$ with $\phi \in L^p(\Omega, C_h)$.*

Proof. Let

$$f(t,\phi) = \int_{-\infty}^{0} D(t,t+s)F(\phi(s))ds, \quad g(t,\phi) = \int_{-\infty}^{0} K(t,t+s)G(\phi(s))ds, \quad \phi \in L^p(\Omega, C_h),$$

then

$$f(t,x_t) = \int_{-\infty}^{0} D(t,t+s)F(x_t(s))ds = \int_{-\infty}^{t} D(t,s)F(x(s))ds,$$

$$g(t,x_t) = \int_{-\infty}^{0} K(t,t+s)G(x_t(s))ds = \int_{-\infty}^{t} K(t,s)G(x(s))ds.$$

Now we prove that $f(t,\phi)$ and $g(t,\phi)$ satisfy globally Lipschitz condition and the linear growth condition. In fact, for any $\phi,\ \psi \in L^p(\Omega, C_h)$,

$$\begin{aligned} E|f(t,\phi)-f(t,\psi)|^p &= E|\int_{-\infty}^{0} D(t,t+s)(F(\phi(s))-F(\psi(s)))ds|^p \\ &\le E[\int_{-\infty}^{0} h(s)L_1|\phi(s)-\psi(s)|ds]^p \\ &\le L_1^p E[\int_{-\infty}^{0} h(s)\|\phi-\psi\|^{[s,0]}ds]^p = L_1^p\|\phi-\psi\|^p_{L^p(\Omega,C_h)}, \end{aligned}$$

thus $|f(t,\phi)-f(t,\psi)|_{L^p} \leq L_1||\phi-\psi||_{L^p(\Omega,C_h)}$, and

$$\begin{aligned} E|f(t,\phi)|^p &= E|\int_{-\infty}^{0} D(t,t+s)F(\phi(s))ds|^p \\ &\leq c^p E[\int_{-\infty}^{0} h(s)[1+\|\phi\|^{[s,0]}]ds]^p \\ &\leq 2^{p-1}c^p E[\int_{-\infty}^{0} h(s)ds]^p + 2^{p-1}c^p E[\int_{-\infty}^{0} h(s)\|\phi\|^{[s,0]}ds]^p \\ &\leq 2^{p-1}c^p[l^p + \|\phi\|^p_{L^p(\Omega,C_h)}]. \end{aligned}$$

Similarly, g satisfies globally Lipschitz condition and the linear growth condition. Hence, all the conditions in Theorem 3.2 are satisfied. Therefore, there exists a unique uniformly continuous solution to equation (4.2). □

Remark 4.1 *For example, $D(t,s) = K(t,s) = e^{s-t}$, $h(s) = e^s$ satisfy the condition (4.1).*

Acknowledgement

This work was supported by the Natural Science Foundation of Hohai university under grant. No. 2008431011.

References

[1] Arnold, L.: Stochastic Differential Equations: Theory and Applications, Wiley, New York, 1972.

[2] Burton, T. A., Winggins, D. P. and Feng, Youhe : Periodic solutions of functional differential equations with infinite delay, J. London Math. Soc. 40, 81-88 (1989).

[3] Burton, T. A. and Zhang, Shunian : Unified blondeness, periodicity and stability in ordinary and functional differential equations, Ann. Math. Pura. Appl. 145, 129-158 (1986).

[4] Friedman, A.: Stochastic Differential Equations and Their Applications, Vol.2, Academic Press, San Diego, 1976.

[5] Hale, J. K. and Lunel, S. M. V.: Introduction to Functional Differential Equations, Springer-Verlag, Berlin/New York, 1993.

[6] Has'minskii, R. Z.: Stochastic Stability of Differential Equations, Sijthoff and Noorhoff, Rockville, MD, 1981.

[7] Ikeda, N. and Watanabe, S.: Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.

[8] Karatzas, I. and Shreve, S. E.: Brownian Motion and Stochastic Calulus, Springer-Verlag, Berlin New York, 1991.

[9] Li, Y., Zhou, Q.D. and Lu, X. R.: Periodic solutions for functional differential inclusions with infinite delay, Science In China, Series A 37, 1289-1301 (1994).

[10] Liu, K. and Mao, X.: Exponential stability of nonlinear stochastic evolution equations, Stochastic Processes and the Applications 78, 173-193 (1998).

[11] Liu, K. and Xia, X.: On the exponential stability in mean square of neutral stochastic functional differential equations, Systems & Control Letters 37, 207-215 (1999).

[12] Mao, X.: Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.

[13] Mao, X.: Stochastic versions of the Lassalle theorem, J. Differential Equations 153, 175-195 (1999).

[14] Mohammed, S. E. A.: Stochastic Functional Differential Equations, Longman, Harlow/New York, 1986.

[15] Taniguchi, T., Liu, K. and Truman, A.: Existence and uniqueness and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces, J. Differential Equations 18, 172-91 (2002).

[16] Taniguchi, T.: Moment asymptotic behavior and almost sure Lyapunov exponent of stochastic functional differential equations with finite delays via Lyapunov Razubikhin method, Stochastics 58, 191-208 (1996).

[17] Wang, K. and Huang, Q. H.: Norm $|\cdot|_h$ and the periodic solutions of Volterra integro-differential equation, Acta of Northeast Normal University 3, 7-15 (1985).

[18] Wang, K. and Huang, Q. H.: C_h Space, Boundness and periodic solutions of functional differential equations with infinite delay, Science In China, Series A, no. 3, 242-252 (1987).

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Received 02.06.2008