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On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear elliptic systems involving degenerated p-Laplacian operators

H. M. Serag and S. A. Khafagy

Abstract

We study the Maximum Principle and existence of positive weak solutions for the $n\times n$ nonlinear elliptic system

$$\left.\begin{array}{lr} -\Delta_{P,p}u_i = \sum_{j=1}^{n} a_{ij}(x)|u_j|^{p-2}u_j + f_i(x,u_1,u_2,\dots,u_n) & \text{in} \quad \Omega, \\ u_i = 0, \; i=1,2,\dots n & \text{on} \quad \partial\Omega, \end{array}\right\}$$

where the degenerated p-Laplacian defined as $\Delta_{P,p}u = div\ [P(x)|\nabla u|^{p-2}\nabla u]$ with $p>1, p\neq 2$ and $P(x)$ is a weight function. We give some conditions for having the Maximum Principle for this system and then we prove the existence of positive weak solutions for the quasilinear system by using "sub-super solutions method".

Key Words: Maximum principle, existence of positive weak solution, nonlinear elliptic system, degenerated p-Laplacian.

1. Introduction

One of the most useful and best known tools employed in the study of partial differential equations is the Maximum Principle, as it is an useful tool to prove many results such as existence, multiplicity and qualitative properties for their solutions. An excellent overview of the subject up to 1967 can be found in the book by Protter and Weinberger [13]. Several papers have explored Maximum Principle for different systems (linear, semilinear and nonlinear) involving Laplace and p-Laplace operators. The Maximum Principle has also been studied for linear elliptic systems. In particular, de Figueiredo and Mitidieri [4, 5, 6] gave necessary and sufficient conditions for the Maximum Principle. Also, in [8], the authors proved sufficient and necessary conditions for having the Maximum Principle and the existence of positive solutions for cooperative linear elliptic systems involving Laplace operator with constant coefficients. In [9], Fleckinger and Serag presented necessary and sufficient conditions for having the Maximum Principle and for the existence of positive solutions for cooperative semilinear elliptic systems involving Laplace operator with variable coefficients. These results have been extended in [8] to the cooperative nonlinear elliptic system involving the p-Laplacian operators with

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constant coefficients:

$$\left.\begin{array}{lr} -\Delta_p u_i = \sum_{j\neq i}^{n} a_{ij}|u_j|^{p-2}u_j + f_i(x,u_1,u_2,\ldots,u_n) & \text{in } \Omega, \\ u_i = 0,\ i=1,2,\ldots n & \text{on } \partial\Omega. \end{array}\right\} \tag{A}$$

Using an approximation method, the existence of solutions for (A) have been proved in [2].

Bouchekif, Serag and de Thélin [3], proved the validity of the Maximum Principle and the existence of positive solutions for the following nonlinear elliptic system of two equations involving different operators Δ_p, Δ_q defined on bounded domain Ω of $\Re^N$, with constant coefficients a, b, c and d:

$$\left.\begin{array}{lr} -\Delta_p u = a|u|^{p-2}u + b|u|^{\alpha}|v|^{\beta}v + f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = c|u|^{\alpha}|v|^{\beta}u + d|v|^{q-2}v + g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{B}$$

These results have been extended in [14] to a nonlinear system defined on unbounded domain with variable coefficients.

Serag and El-zahrani [15] considered nonlinear elliptic system with p-Laplacian and different variable coefficients:

$$\left.\begin{array}{lr} -\Delta_p u_i = \sum_{j=1}^{n} a_{ij}(x)|u_j|^{p-2}u_j + f_i(x,u_1,u_2,\ldots,u_n) & \text{in } \Omega, \\ u_i = 0,\ i=1,2,\ldots n & \text{on } \partial\Omega. \end{array}\right\} \tag{C}$$

Khafagy and Serag [11] gave a generalization of the p-Laplacian system (B) to the degenerated p-Laplacian system with variable coefficients:

$$\left.\begin{array}{lr} -\Delta_{P,p} u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f(x,u,v) & \text{in } \Omega, \\ -\Delta_{Q,q} v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{D}$$

Here, we consider $n \times n$ nonlinear elliptic system involving degenerated p-Laplacian operators with variable coefficients. We introduce the following nonlinear system:

$$\left.\begin{array}{lr} -\Delta_{P,p} u_i = \sum_{j=1}^{n} a_{ij}(x)|u_j|^{p-2}u_j + f_i(x,u_1,u_2,\ldots,u_n) & \text{in } \Omega, \\ u_i = 0,\ i=1,2,\ldots n & \text{on } \partial\Omega, \end{array}\right\} \tag{S}$$

where Ω is an open bounded subset of $\Re^N$ with a smooth boundary $\partial\Omega$, $\Delta_{P,p}$ denotes the degenerated p-Laplacian defined by $\Delta_{P,p} u = div\ [P(x)|\nabla u|^{p-2}\nabla u]$ with $p > 1,\ p \neq 2$ and $P(x)$ is a weight function, f_i are given functions and the coefficients $a_{ij}(x)$ $(1 \leq i,j \leq n)$ are smooth bounded weight functions. We consider here a generalization of the p-Laplacian system (C) to the degenerated p -Laplacian system (S). We first study the Maximum Principle for system (S) and then we prove the existence of positive weak solutions for this system by using sub-super solutions method.

This paper is organized as follows. In section 2, we introduce some technical results and some notations, which are established in [7], [8]. We give also some assumptions on the coefficients $a_{ij}(x)$ and on the functions f_i to insure the validity of the Maximum Principle and the existence of positive weak solutions for system (S) in a suitable weighted Sobolev space. Section 3 is devoted to the Maximum Principle for the scalar case

$$-\Delta_{P,p} u = a(x)|u|^{p-2}u + f(x) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

and the existence of positive weak solutions for the quasilinear case which extends the scalar case. Finally, in section 4, we consider the system case.

2. Technical results

In this section, we introduce the weighted Sobolev space $W^{1,p}(P,\Omega)$, which is the set of all real valued functions u defined in Ω for which (see [7])

$$\|u\|_{W^{1,p}(P,\Omega)} = \left[\int_\Omega |u|^p + \int_\Omega P(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty, \tag{2.1}$$

where $P(x)$ is a weight function in $\Omega \subset \Re^N$ satisfying the conditions

$$P(x) \in L^1_{\text{Loc}}(\Omega), \ \ (P(x))^{-\frac{1}{p-1}} \in L^1_{\text{Loc}}(\Omega), \ \ (P(x))^{-s} \in L^1(\Omega), \tag{2.2}$$

with

$$s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty). \tag{2.3}$$

Since we are dealing with the Dirichlet problem, we define also the space $W_0^{1,p}(P,\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(P,\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(P,\Omega)} = \left[\int_\Omega P(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty, \tag{2.4}$$

which is equivalent to the norm given by (2.1). Both spaces $W^{1,p}(P,\Omega)$ and $W_0^{1,p}(P,\Omega)$ are well defined reflexive Banach Spaces. Also, besides the conditions given by (2.2) and (2.3) and under the condition

$$p_s = \frac{ps}{s+1} < p < p_s^* = \frac{Np_s}{N-p_s} = \frac{Nps}{N(s+1)-ps}, \tag{2.5}$$

the space $W_0^{1,p}(P,\Omega)$ is compactly imbedded into the space $L^p(\Omega)$, i.e.

$$W_0^{1,p}(P,\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \tag{2.6}$$

which means that

$$\int_\Omega |u|^p \leq c_2 \int_\Omega P(x)|\nabla u|^p, \text{i.e., } \|u\|_{L^p(\Omega)} \leq c\, \|u\|_{W_0^{1,p}(P,\Omega)}, \tag{2.7}$$

for every $u \in W_0^{1,p}(P,\Omega)$ with a constant $c_2 > 0$ independent of u.

Now, let us introduce some technical results concerning the following degenerated homogeneous eigenvalue problem [1], [7]:

$$\left.\begin{array}{lr} -\Delta_{P,P}u = -div[P(x)|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.8}$$

where $P(x)$ is still weight function given by (2.2) and (2.3) and $a(x)$ is a smooth bounded weight functions satisfying

$$a(x) \in L^{\frac{q}{q-p}}(\Omega) \cap L^p(\Omega) \text{ with some } q \text{ satisfies } p < q < p_s^* \text{ and } meas\{x \in \Omega : a(x) > 0\} > 0, \tag{2.9}$$

where p_s^* is given by (2.5).

The hypotheses on P and a will be valid for the whole paper.

Definition 1 *(see [7]) We say that $\lambda \in \Re$ is an eigenvalue of (2.8) if there exists $u \in W_0^{1,p}(P,\Omega)$, $u \neq 0$ such that*

$$\int_\Omega P(x)|\nabla u|^{p-2}\nabla u \nabla \varphi = \lambda \int_\Omega a(x)|u|^{p-2}u\varphi$$

holds for any $\varphi \in W_0^{1,p}(P,\Omega)$. Then u is called an eigenfunction corresponding to the eigenvalue λ.

Lemma 1 *There exists the least (i.e. the first or principal) eigenvalue $\lambda = \lambda_1(p,\Omega) > 0$ and at least one corresponding eigenfunction $u = u_1 \geq 0$ a.e. in Ω of the eigenvalue problem (2.8). Moreover, the first eigenvalue is characterized by*

$$\lambda_1(p,\Omega) = \inf\{\int_\Omega P(x)|\nabla u|^p : \int_\Omega a(x)|u|^p = 1\}, \tag{2.10}$$

and the normalized eigenfunction u_1 associated to the first eigenvalue λ_1 is in $L^\infty(\Omega)$ and unique.

Also, from the characterization of the first eigenvalue given by (2.10), we have

$$\lambda_1(p,\Omega)\int_\Omega a(x)|u|^p \leq \int_\Omega P(x)|\nabla u|^p. \tag{2.11}$$

As in [8, 15], we also introduce the following definition.

Definition 2 *A nonsingular matrix $B = (b_{ij})$ is a M-matrix if $b_{ij} \leq 0$ for $i \neq j$, $b_{ii} > 0$ and $\det B_k > 0$ for $1 \leq k \leq n$, where B_k is the matrix obtained by taking the last $(n-k)$ rows and columns out of the matrix $B = \lambda_1(p,\Omega)I - A$, where $\lambda_1(p,\Omega)$ is the first eigenvalue of the degenerated homogeneous eigenvalue problem given by (2.8) and $A = (a_{ij}) \in M_{n\times n}$, $(1 \leq i,j \leq n)$.*

Lemma 2 *If B_n is a nonsingular M-matrix, then for all $Y \in \Re^N, Y \leq 0$ (resp., $Y \geq 0$), the solution $X \in \Re^N$ of $B_nX = Y$ is non positive (resp. non negative).*

Finally, we write $u = u^+ - u^-$ where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

3. The case of a single equation for the degenerated p-Laplacian

In this section, for a given $p > 1$, we study the scalar case

$$\left.\begin{array}{ll} -\Delta_{P,p}u = -div[P(x)|\nabla u|^{p-2}\nabla u] = a(x)\psi_p(u) + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{T}$$

where $P(x)$ and $a(x)$ are as in section 2, $\psi_p(u) = |u|^{p-2}u$ and $f \in L^{p^*}(\Omega)$ with $\frac{1}{p} + \frac{1}{p^*} = 1$.

Let us denote by Φ_1 the positive eigenfunction associated with $\lambda_1(p,\Omega)$ and normalized by $\|\Phi_1\|_\infty = 1$, then, according to (2.8), Φ_1 satisfies the eigenvalue problem

$$\left.\begin{array}{ll} -\Delta_{P,p}\Phi_1 = \lambda_1(p,\Omega)a(x)\psi_p(\Phi_1) = \lambda_1(p,\Omega)a(x)(\Phi_1)^{p-1}, \quad \Phi_1 > 0 & \text{in } \Omega, \\ \Phi_1 = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{3.1}$$

3.1. Maximum principle

We are concerned with the following form of the Maximum Principle: The hypotheses $f \geq 0$ on Ω implies $u \geq 0$ for any solution u of (T). The validity of the Maximum Principle for the scalar case (T), was proved (see [7]), but we state it in the following theorem for the convenience of the reader.

Theorem 3 *[7] For $f(x) \in L^{p^*}(\Omega)$, the Maximum Principle holds for (T) iff $\lambda_1(p,\Omega) > 1$.*

Proof. Let us prove first that the condition is necessary. Assume that $\lambda_1(p,\Omega) \leq 1$, then the function $f(x) = [1 - \lambda_1(p,\Omega)]a(x)\psi_p(\Phi_1)$ is nonnegative and nevertheless $(-\Phi_1)$ satisfies (T) and hence the Maximum Principle does not hold.

Conversely, assume that $1 < \lambda_1(p,\Omega)$; if u is a solution of (T) for $f \geq 0$, we obtain by multiplying (T) by u^- and integrating over Ω

$$\begin{aligned} \int_\Omega (-\Delta_{P,p}u)u^- = -\int_\Omega P(x)|\nabla u^-|^p &= \int_\Omega a(x)|u|^{p-2}uu^- + \int_\Omega f(x)u^- \\ &= -\int_\Omega a(x)|u^-|^p + \int_\Omega f(x)u^-. \end{aligned}$$

Then, it follows from (2.11) that

$$\lambda_1(p,\Omega)\int_\Omega a(x)|u^-|^p \leq \int_\Omega P(x)|\nabla u^-|^p = \int_\Omega a(x)|u^-|^p - \int_\Omega f(x)u^- \leq \int_\Omega a(x)|u^-|^p.$$

So

$$(\lambda_1(p,\Omega) - 1)\int_\Omega a(x)|u^-|^p \leq 0,$$

and hence $u^- = 0$, where $a(x) \neq 0$ for any x, so that $u \geq 0$.

3.2. Existence of a positive weak solution

We now establish that the same condition $\lambda_1(p,\Omega) > 1$ is also sufficient for the existence of a positive weak solution for the scalar case (T), where $P(x)$ and $a(x)$ are as in section 2 and f is a Caratheodory function, and there exist $M > 0$ and $0 \leq \sigma < 1$ satisfying the condition

$$0 < f(x,u) \leq M(1 + |u|^{\sigma(p-1)}), \quad \text{for any} \quad x \in \Omega \quad \text{and any} \quad u \geq 0. \tag{3.2}$$

Definition 3 *(see [7]) We say that* $u \in W_0^{1,p}(P,\Omega)$ *is a weak solution of (T) if*

$$\int\limits_{\Omega} P(x)|\nabla u|^{p-2}\nabla u \nabla\varphi = \lambda \int\limits_{\Omega} a(x)|u|^{p-2}u\varphi + \int\limits_{\Omega} f(x,u)\varphi$$

holds for any $\varphi \in W_0^{1,p}(P,\Omega)$.

It follows from the continuity and monotonicity of the first eigenvalue $\lambda_1 := \lambda_1(p,\Omega)$ with respect to the domain Ω [8] that there exist a domain $\widetilde{\Omega} \supset \overline{\Omega} = \Omega \cup \partial\Omega$ such that $\widetilde{\lambda}_1 := \lambda_1(p,\widetilde{\Omega})$ satisfies

$$1 < \widetilde{\lambda}_1 < \lambda_1. \tag{3.3}$$

If $\widetilde{\Phi}_1$ is the positive eigenfunction associated with $\widetilde{\lambda}_1$ and normalized by $\left\|\widetilde{\Phi}_1\right\|_\infty = 1$, then, as in (3.1), $\widetilde{\Phi}_1$ satisfies

$$\left.\begin{array}{ll} -\Delta_{P,p}\widetilde{\Phi}_1 = \lambda_1(p,\widetilde{\Omega})a(x)\psi_p(\widetilde{\Phi}_1) = \lambda_1(p,\widetilde{\Omega})a(x)(\widetilde{\Phi}_1)^{p-1}, \ \widetilde{\Phi}_1 > 0 & \text{in } \widetilde{\Omega}, \\ \widetilde{\Phi}_1 = 0 & \text{on } \partial\,\widetilde{\Omega} \end{array}\right\} \tag{3.4}$$

and will be bounded below on $\widetilde{\Omega}$ by a positive number:

$$\beta = \inf\{\widetilde{\Phi}_1(x) : x \in \widetilde{\Omega}\} > 0. \tag{3.5}$$

Let us define the operator T as

$$T : u \longrightarrow (-\Delta_{P,p})^{-1}[a(x)\psi_p(u) + f(x,u)], \tag{3.6}$$

and let us choose δ such that

$$a(x)(\delta\beta)^{p-1}[\lambda_1(p,\widetilde{\Omega}) - 1] \geq M(1 + (\delta)^{\sigma(p-1)}), \tag{3.7}$$

for all $x \in \widetilde{\Omega}$.

It is proved that the operator T is well defined in $L^p(\Omega)$, (see [7]).

Theorem 4 *Assume that (3.2)–(3.7) hold. Then (T) has a positive weak solution if* $\lambda_a(p,\widetilde{\Omega}) > 1$.

Proof. We proceed in three steps.

a) Construction of sub-super solutions for (T):

We claim that $(u_0, u^0) = (0, \delta\widetilde{\Phi}_1)$ is a coupled sub-super solution. For this we prove that

$$-\Delta_{P,p} u^\circ - a(x)\psi_p(u^\circ) - f(x, u^\circ) \geq 0. \tag{3.8}$$

We obtain from (T), (3.4), (3.5), (3.7) and $\left\|\widetilde{\Phi}_1\right\|_\infty = 1$ that

$$\left.\begin{aligned} -\Delta_{P,p} u^\circ - a(x)\psi_p(u^\circ) - f(x, u^\circ) &= -\Delta_{P,p}(\delta\widetilde{\Phi}_1) - a(x)\psi_p(\delta\widetilde{\Phi}_1) - f(x, \delta\widetilde{\Phi}_1) \\ &\geq \lambda_1(p, \widetilde{\Omega})a(x)\psi_p(\delta\widetilde{\Phi}_1) - a(x)\psi_p(\delta\widetilde{\Phi}_1) - M(1 + \left|\delta\widetilde{\Phi}_1\right|^{\sigma(p-1)}) \\ &= [\lambda_1(p, \widetilde{\Omega}) - 1]a(x)\left|\delta\widetilde{\Phi}_1\right|^{p-1} - M(1 + |\delta|^{\sigma(p-1)}) \\ &\geq [\lambda_1(p, \widetilde{\Omega}) - 1]a(x)\left|\delta\beta\right|^{p-1} - M(1 + |\delta|^{\sigma(p-1)}), \end{aligned}\right\}$$

and hence, using (3.7), (3.8) is proved.

b) The operator T defined in (3.6) sends the interval $[0, \delta\widetilde{\Phi}_1]$ into itself, i.e.,

$$T([0, \delta\widetilde{\Phi}_1]) \subseteq [0, \delta\widetilde{\Phi}_1], \tag{3.9}$$

To prove that: let $u \in [0, \delta\widetilde{\Phi}_1]$ and let $v = Tu$; then from (T), (3.4) and (3.5), we obtain

$$\begin{aligned} &\int_\Omega [-\Delta_{P,p} v + \Delta_{P,p}(\delta\widetilde{\Phi}_1)](v - \delta\widetilde{\Phi}_1)^+ \\ = &\int_\Omega [a(x)\psi_p(u) + f(x, u) - \lambda_1(p, \widetilde{\Omega})a(x)\psi_p(\delta\widetilde{\Phi}_1)](v - \delta\widetilde{\Phi}_1)^+ \\ \leq &\int_\Omega [a(x)\psi_p(\delta\widetilde{\Phi}_1) + M(1 + \left|\delta\widetilde{\Phi}_1\right|^{\sigma(p-1)}) - \lambda_1(p, \widetilde{\Omega})a(x)\psi_p(\delta\widetilde{\Phi}_1)](v - \delta\widetilde{\Phi}_1)^+ \\ = &\int_\Omega [-a(x)(\lambda_1(p, \widetilde{\Omega}) - 1)\left|\delta\widetilde{\Phi}_1\right|^{p-1} + M(1 + |\delta|^{\sigma(p-1)})](v - \delta\widetilde{\Phi}_1)^+, \\ \leq &\int_\Omega [-a(x)(\lambda_1(p, \widetilde{\Omega}) - 1)(\delta\beta)^{p-1} + M(1 + |\delta|^{\sigma(p-1)})](v - \widetilde{\Phi}_1)^+. \end{aligned}$$

Using (3.7), we have

$$\int_\Omega [-\Delta_{P,p} v + \Delta_{P,p}(\delta\widetilde{\Phi}_1)](v - \delta\widetilde{\Phi}_1)^+ \leq 0.$$

From the monotonicity of the degenerated p-Laplacian [7], we have $(v - \delta\widetilde{\Phi}_1)^+ = 0$. Hence we deduce that $v \leq \delta\widetilde{\Phi}_1$. Similarly (or by Maximum Principle) $v \geq 0$, and (3.9) is proved.

c) T is compact.

Since the imbedding of $W_0^{1,p}(P,\Omega)$ into $L^p(\Omega)$ is compact, then T is compact.

Hence, it follows from Schauder's Fixed Point Theorem that there is at least a positive weak solution for (T). □

4. The case of a system for the degenerated p-Laplacian

The aim of this section is to extend Theorem (3) and (4) to the following system:

$$\left.\begin{array}{cr} -\Delta_{P,p}u_i = -div[P(x)|\nabla u_i|^{p-2}\nabla u_i] = \sum_{j=1}^{n} a_{ij}(x)|u_j|^{p-2}u_j + f_i(x,u_1,u_2,\ldots,u_n) & \text{in } \Omega, \\ u_i = 0, \ i=1,2,\ldots n & \text{on } \partial\Omega, \end{array}\right\} \tag{S}$$

where $P(x)$ is still weight function given by (2.2) and (2.3) and $a_{ij}(x)$ $(1 \le i,j \le n)$ is a smooth bounded weight functions satisfying– besides (2.9) – also the condition

$$a_{ij}(x) < (a_{ii}(x))^{\frac{1}{p}}\,(a_{jj}(x))^{\frac{1}{p^*}}. \qquad \forall\ 1 \le i,j \le n \tag{4.1}$$

System (S) can be written shortly as

$$\left.\begin{array}{cr} -\Delta_{P,p}U = A\Psi_p(U) + F & \text{in } \Omega, \\ U = 0, & \text{on } \partial\Omega, \end{array}\right\} \tag{I}$$

where U (resp., F) denotes a column matrix with elements u_i (resp. $f_i(x)$), $\Psi_p(U)$ is the column matrix with elements $\psi_p(u_j) := |u_j|^{p-2}u_j$ and $A = (a_{ij}) \in M_{n\times n}$.

We say that system (S) satisfies the Maximum Principle if any nonnegative data: $f_i \ge 0 \ \ \forall\ 1 \le i \le n$, implies that any solution $U := (u_1,u_2,\ldots,u_n)$ for system (S) is non negative: $u_i \ge 0$.

4.1. Maximum principle

For proving the validity of the Maximum Principle, we denote by D the diagonal matrix with diagonal elements $(\lambda_1(p,\Omega)-1)$ and by G the matrix defined by

$$G = (g_{ij}) = \begin{cases} 1 & \text{if } i \ne j, \\ 0 & \text{if } i = j. \end{cases}$$

Theorem 5 *Assume that (4.1) is satisfied. Then, the Maximum Principle holds for system (S) if the matrix $(D-G)$ is a nonsingular M-matrix.*

Proof. Let $U = (u_i) \in (W_0^{1,p}(P,\Omega))^n$ satisfying (S) for $F = (f_i) \ge 0$. Multiplying (S) by u_i^-, and integrating over Ω, we obtain

$$\begin{aligned} \int_\Omega P(x)|\nabla u_i^-|^p &= -\sum_{j\ne i}^{n}\int_\Omega a_{ij}(x)|u_j^+|^{p-2}u_j^+u_i^- + \sum_{j=1}^{n}\int_\Omega a_{ij}(x)|u_j^-|^{p-2}u_j^-u_i^- - \int_\Omega f_iu_i^- \\ &\le \sum_{j=1}^{n}\int_\Omega a_{ij}(x)|u_j^-|^{p-2}u_j^-u_i^- = \int_\Omega a_{ii}(x)|u_i^-|^p + \sum_{j\ne i}^{n}\int_\Omega a_{ij}(x)|u_j^-|^{p-2}u_j^-u_i^-, \end{aligned}$$

Using (2.11), we get

$$(\lambda_1(p,\Omega)-1)\int_\Omega a_{ii}(x)|u_i^-|^p \leq \sum_{j\neq i}^n \int_\Omega a_{ij}(x)|u_j^-|^{p-2}u_j^-u_i^-.$$

From (4.1), we obtain

$$(\lambda_1(p,\Omega)-1)\int_\Omega a_{ii}(x)|u_i^-|^p \leq \sum_{j\neq i}^n \int_\Omega (a_{ii}(x))^{\frac{1}{p}}\,(a_{jj}(x))^{\frac{1}{p^*}}\,|u_j^-|^{p-2}u_j^-u_i^-.$$

Applying Hölder's inequality, we get

$$(\lambda_1(p,\Omega)-1)\int_\Omega a_{ii}(x)|u_i^-|^p \leq \sum_{j\neq i}^n \left(\int_\Omega a_{ii}(x)|u_i^-|^p\right)^{\frac{1}{p}} \left(\int_\Omega a_{jj}(x)(|u_j^-|^{p-1})^{p^*}\right)^{\frac{1}{p^*}}.$$

If

$$\int_\Omega a_{ii}(x)|u_i^-|^p \neq 0,$$

we obtain

$$(\lambda_1(p,\Omega)-1)\left(\int_\Omega a_{ii}(x)|u_i^-|^p\right)^{\frac{1}{p^*}} \leq \sum_{j\neq i}^n \left(\int_\Omega a_{jj}(x)|u_j^-|^p\right)^{\frac{1}{p^*}}.$$

Hence, the column matrix Z, with elements $(\int_\Omega a_{ii}(x)|u_i^-|^p)^{1/p^*}$, satisfies $(D-G)Z \leq 0$. Since $(D-G)$ is a nonsingular M-matrix, then Lemma 2 implies $u_i^- = 0 \quad \forall\ i = 1,2,\ldots,n$, and hence $U=(u_i)\geq 0$. □

4.2. Existence of positive weak solutions

In this subsection, we prove the existence of positive weak solutions for system (S), where $P(x)$ is still weight function given by (2.2) and (2.3), the coefficients $a_{ij}(x)$ are still smooth bounded weight functions given by (2.9) and (4.1) and $f_i(x,U),\ 1\leq i\leq n,$ are Caratheodory functions, and there exist $M>0$ and $0\leq\sigma<1$ satisfying

$$0 < f_i(x,U) \leq M(1+|U|^{\sigma(p-1)}), \quad \text{for any} \quad x\in\Omega \quad \text{and any} \quad U\geq 0. \tag{4.2}$$

To prove the existence theorem, we make use of "sub-super solutions". Following [10], we introduce the following definition.

Definition 4 *Write $A=C+E$, where C is a diagonal matrix and E is with diagonal zero. We say that $(U_\circ,U^\circ)\in (W_0^{1,p}(P,\Omega))^{2n}$ is a sub-super solutions for (S) if*

$$U_\circ \leq U^\circ \quad in \quad \Omega,$$

and for any $U \in (W_0^{1,p}(P,\Omega))^n$*, such that* $U_\circ \le U \le U^\circ$*, we have*

$$\left.\begin{array}{c} -\Delta_{P,p}U^\circ - C\Psi_p(U^\circ) - E\Psi_p(U) - F(x,U\cup U^\circ) \ge 0 \\ \ge -\Delta_{P,p}U_\circ - C\Psi_p(U_\circ) - E\Psi_p(U) - F(x,U\cup U_\circ) \quad in \quad \Omega, \\ U_\circ \le 0 \le U^\circ \quad on \quad \partial\Omega, \end{array}\right\} \tag{4.3}$$

where, for any $U,V \in (W_0^{1,p}(P,\Omega))^n$*, the k-th component of* $F(x,U\cup V)$ *is given by* $(F(x,U\cup V))_k := f_k(x,u_1,\dots,u_{k-1},v_k,u_{k+1},\dots,u_n)$.

We claim now the following theorem.

Theorem 6 *Assume that* $(D-G)$ *is a nonsingular M-matrix; where* D,G *as in subsection 4.1, then for any* $F \in (L^{p^*}(\Omega))^n$*, system (S) admits a positive weak solution.*

Proof. We proceed in three steps

a) Construction of sub-super solutions of (S):

By hypothesis $(D-G)$ is a nonsingular M-matrix. Then, as in the scalar case, we make use of the monotonicity and continuity of the first eigenvalue $\lambda_1(p,\Omega)$ w.r.t. Ω. It follows from [8] that one can choose $\widetilde{\Omega} \supset \overline{\Omega} = \Omega\cup\partial\Omega$ such that $\lambda_1(p,\Omega) > \lambda_1(p,\widetilde{\Omega})$. Hence by use of the same Theorem in [8], $(\widetilde{D}-G)$ is also a nonsingular M-matrix, where $\widetilde{D}$ is a diagonal matrix with elements $\lambda_1(p,\widetilde{\Omega})-1$. As in the previous section, we denote by $\widetilde{\Phi}_1$ the positive eigenfunction associated with $\lambda_1(p,\widetilde{\Omega})$ and normalized by $\|\widetilde{\Phi}_1\|_\infty = 1$ and by β the lower bound of $\widetilde{\Phi}_1$ on $\widetilde{\Omega}$. Choose $0 < Y = \frac{M}{\beta}(1+|\delta|^{\sigma(p-1)})(1,1,\dots,1) \in \Re^N$. The solution $X \in \Re^N$ of

$$(\widetilde{D}-G)X = Y, \tag{4.4}$$

is positive by Lemma 2. Set $U_\circ = 0 \in \Re^n$ and $U^\circ = \Psi_{p^*}(X)\ \widetilde{\Phi}_1$. Combining (4.2) and (4.4), for $U \in [U_\circ, U^\circ]$, we have

$$\left.\begin{array}{c} -\Delta_{P,p}U^\circ - C\Psi_p(U^\circ) - E\Psi_p(U) - F(x,U\cup U^\circ) \ge -\Delta_{P,p}U^\circ - C\Psi_p(U^\circ) - E\Psi_p(U) - \beta Y \\ = [(\lambda_1(p,\widetilde{\Omega})-1)C - CG + CG]\Psi_p(U^\circ) - E\Psi_p(U^\circ) + E[\Psi_p(U^\circ) - \Psi_p(U)] - \beta Y \\ \ge C[(\lambda_1(p,\widetilde{\Omega})-1)I - G]\Psi_p(U^\circ) + (CG-E)\Psi_p(U^\circ) - \beta Y \\ \ge C[(\lambda_1(p,\widetilde{\Omega})-1)I - G]\Psi_p(U^\circ) - \beta Y \\ = C[(\lambda_1(p,\widetilde{\Omega})-1)I - G]X\psi_p(\widetilde{\Phi}_1) - \beta Y \\ = C(\widetilde{D}-G)X\ (\widetilde{\Phi}_1)^{p-1} - \beta Y. \end{array}\right\}$$

Using (3.5) and (4.4), we get

$$-\Delta_{P,p}U^\circ - C\Psi_p(U^\circ) - E\Psi_p(U) - F(x,U\cup U^\circ) \ge 0,$$

hence $(U_\circ, U^\circ)$ satisfies (4.3).

b) Construction of an invariant set.

Let us introduce $K = [0,U^\circ]$ and $\Sigma = (L^p(\Omega))^n$. Next, we define the nonlinear operator $T : K \longrightarrow \Sigma$, where $V = TU$ for any $U \in K$.

Now, we prove that $T(K) \subset K$.

Since $U \in [0, U^\circ]$, it follows from (4.2) and (4.3) that

$$\left.\begin{array}{c} 0 \geq -\Delta_{P,p} V - A\Psi_p(U) - F(x,U) + \Delta_{P,p} U^\circ + C\Psi_p(U^\circ) + E\Psi_p(U) + \beta Y \\ \geq -\Delta_{P,p} V + \Delta_{P,p} U^\circ - C[\Psi_p(U) - \Psi_p(U^\circ] + \beta Y - F(x,U) \\ \geq -\Delta_{P,p} V + \Delta_{P,p} U^\circ. \end{array}\right\} \tag{4.5}$$

Multiplying (4.5) by $(V - U^\circ)^+$ and integrating over Ω, we obtain

$$\int_\Omega P(x)[\Psi_p(\nabla V) - \Psi_p(\nabla U^\circ)][\nabla(V - U^\circ)^+] \leq 0.$$

By the monotonicity of $\Psi_p(\cdot) = |\ \cdot\ |^{p-2}\cdot$ and the positivity of $P(x)$, we have $\nabla(V - U^\circ)^+ = 0$ and hence, $0 \leq V \leq U^\circ$.

c) T is completely continuous

First, we prove that T is compact, let U_j be a bounded sequence in $(L^p(\Omega))^n$, hence $\Psi_p(U_j)$ is bounded in $(L^{p^*}(\Omega))^n$. Multiplying (S) by $V_j := TU_j$, and applying Hölder's inequality, we obtain

$$\int_\Omega P(x)\,|\nabla V_j|^p = \int_\Omega A\Psi_p(U_j)V_j + F(x,U_j)V_j \leq C[\int_\Omega |V_j|^p\,]^{1/p}.$$

Therefore V_j is bounded in $(W_0^{1,p_i}(P_i,\Omega))^n$ and it posses a convergent subsequence in $(L^p(\Omega))^n$.

Now, we proof the continuity of T, let $U_j \longrightarrow U$ in $(L^p(\Omega))^n$. Then, by Dominated Convergence Theorem, we have

$$F(x,U_j) \longrightarrow F(x,U) \quad \text{in} \quad (L^{p^*}(\Omega))^n. \tag{4.6}$$

$$A\Psi_p(U_j) \longrightarrow A\Psi_p(U) \quad \text{in} \quad (L^{p^*}(\Omega))^n. \tag{4.7}$$

Multiplying (4.7) by $(V_j - V)$ and integrating over Ω, we obtain

$$\begin{aligned} \int_\Omega P(x)[\Psi_p(\nabla V_j) - \Psi_p(\nabla V)][\nabla(V_j - V)] &= \int_\Omega A[\Psi_p(U_j) - \Psi_p(U)][V_j - V] \\ &\quad + \int_\Omega [F(x,U_j) - F(x,U)][V_j - V]. \end{aligned} \tag{4.8}$$

It follows from (4.6) and (4.7) that the right hand side of (4.8) tends to zero as j tends to $+\infty$.

It is well known [16] that the following inequality holds:

$$|x-y|^p \leq C\{(|x|^{p-2}\,x - |y|^{p-2}\,y)(x-y)\}^{\frac{\gamma}{2}}(|x|^p + |y|^p)^{1-\frac{\gamma}{2}}, \tag{4.9}$$

for all $x,\ y \in \Re^N$, where $\gamma = p$ if $1 < p \leq 2$ and $\gamma = 2$ if $p > 2$.

Using (4.9), we obtain

$$\int_\Omega P(x)\,|\nabla(V_j - V)|^p \to 0 \quad \text{as} \quad j \to +\infty,$$

and hence T is continuous.

Since K is a convex, bounded and closed subset of $(L^p(\Omega))^n$, we can apply Schauder's Fixed Point Theorem to obtain the existence of a fixed point for T, which gives the existence of, at least, one solution of (S), and this completes the proof. □

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H. M. SERAG, S. A. KHAFAGY
Mathematics Department, Faculty of Science,
Al-Azhar University, Nasr City
(11884), Cairo-EGYPT
e-mail: serraghm@yahoo.com, el_gharieb@hotmail.com.

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