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A note on dominant contractions of Jordan algebras

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Abstract

We consider two positive contractions $T, S : L_1(A,\tau) \longrightarrow L_1(A,\tau)$ such that $T \leq S$, here (A,τ) is a semi-finite JBW-algebra. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < 1$, we prove that $\|S^n - T^n\| < 1$ holds for every $n \geq n_0$.

Key Words: Dominant contraction, positive operator, Jordan algebra.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive σ-additive measure μ and let $L_1(X, \mathcal{F}, \mu)$ be the usual associated real L_1-space. A linear operator $T : L_1(X, \mathcal{F}, \mu) \to L_1(X, \mathcal{F}, \mu)$ is called *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$. In [17] the following theorem was proved.

Theorem 1.1 *Let $T, S : L_1(X, \mathcal{F}, \mu) \to L_1(X, \mathcal{F}, \mu)$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$.*

Using this formulated result Zaharapol proved so called "zero-two" law for a positive contraction of L^1-space. Note that the "zero-two" law first appeared in [14].

One can ask: what would happen, for the above contractions, if $\|S - T\| = 1$. In this case we could not apply the formulated theorem. Therefore, there are two options:

(i) one has $\|S^n - T^n\| = 1$ for all $n \in \mathbb{N}$;

(ii) there is an $n_0 \in \mathbb{N}$ such $\|S^{n_0} - T^{n_0}\| < 1$.

So, concerning (ii) we can formulate the following.

Problem 1.2 *Let S, T be as above in Theorem 1.1. If there is an $n_0 \in \mathbb{N}$ such $\|S^{n_0} - T^{n_0}\| < 1$, then can we state $\|S^n - T^n\| < 1$ for every $n \geq n_0$?*

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By denoting $\tilde{S} = S^{n_0}$,$\tilde{T} = T^{n_0}$ as a direct consequence of Theorem 1.1 we get that $\|(\tilde{S})^n - (\tilde{T})^n\| < 1$ for every $n \in \mathbb{N}$ under the statement of the above problem. This means that $\|S^{nn_0} - T^{nn_0}\| < 1$ for every $n \in \mathbb{N}$. But this is not an answer to the question.

The aim of this paper is to give an affirmative answer to the formulated problem for positive L_1-contractions of JBW-algebrasb (In Remark 3.3 we point out that the result can be proved for any partially ordered Banach spaces in which the norm has the additivity property.) Such a result will include as a particular case of the Zaharopl's result. Further, we shall show that indeed that our result is an extension of Theorem 1.1. Namely, we provide an example of two positive contractions for which the condition of Theorem 1.1 is not satisfied, but the statement of the problem holds. Note that Jordan Banach algebras [10],[16] are a non-associative real analogue of von Neumann algebras. The existence of exceptional JBW-algebras does not allow one to use the ideas and methods from von Neumann algebras. Mery papers have been devoted to ergodic type theorems for Jordan algebras were devoted a lot papers (see for example, [1],[2],[9],[13] etc). It worth be worth mentioning that a book [6] that has been written devoted to asymptotic analysis of L_1-contractions on commutative and non-commutative setting. The motivation of these investigations arose in quantum statistical mechanics and quantum field theory (see [5], [15]). We hope that our result will serve to prove the "zero-two" law in a non-associative or non-commutative framework, since nowadays such activities has been reviewed by many authors (see for example, [11]) motivated by various physical reasons.

2. Preliminaries

In this section recall some well known facts concerning Jordan algebras.

Let A be a linear space A over the reals $\mathbb{R}$. A pair $(A, \circ)$, where $\circ$ is a binary operation (e.g. multiplication), is called a *Jordan algebra* if the following conditions are satisfied:

(i) $a \circ (b + c) = a \circ b + a \circ c$; $(b + c) \circ a = b \circ a + c \circ a$ for any $a, b, c \in A$;

(ii) $\lambda(a \circ b) = (\lambda a) \circ b = a \circ (\lambda b)$ for any $\lambda \in \mathbb{R}$, $a, b \in A$;

(iii) $a \circ b = b \circ a$ for any $a, b \in A$;

(iv) $a^2 \circ (b \circ a) = (a^2 \circ b) \circ a$ for any $a, b \in A$.

Let A be a Jordan algebra with unity $\mathbb{1}$ and at the same time be a Banach space over the reals. If a norm on A respects multiplication so that $\|a^2\| = \|a\|^2$ and $\|a^2\| \leq \|a^2 + b^2\|$ for all a, $b \in A$, then A is called a *JB-algebra* (see [3],[4],[10]). Note that in each JB-algebra A the set $A^+ = \{a^2 : a \in A\}$ is regular convex cone and defines in A a partial ordering compatible with the algebraic operations. A JB-algebra A is called a *JBW-algebra* if there exists a Banach space N, which is said to be pre-dual to A, such that A is isometrically isomorphic to the space N^* of continuous linear functionals on N. So, on the JBW-algebra A one can introduce the $\sigma(A, N)$-weak topology. It is known that the pre-dual space N of a JBW-algebra A can be identified with the space of all $\sigma(A, N)$-weak continuous linear functionals A_* on A.

Recall that a *trace* on a JBW-algebra is a map $\tau : A^+ \to [0, \infty]$ such that

(1) $\tau(a + \lambda b) = \tau(a) + \lambda\tau(b)$ for all a, $b \in A^+$ and $\lambda \in \mathbb{R}_+$, provided that $0 \cdot (\infty) = 0$,

(2) $\tau(U_s a) = \tau(a)$ for all $a \in A^+$ and $s \in A$, $s^2 = \mathbb{1}$, where $U_s x = 2s \circ (s \circ x) - s^2 \circ x$.

A trace τ is said to be *faithful* if $\tau(a) > 0$ for all $a \in A^+$, $a \neq 0$; it is normal if, for each increasing net x_α in A^+ that is bounded above, one has $\tau(\sup x_\alpha) = \sup \tau(x_\alpha)$; it is *semi-finite* if there exists a net $\{b_\alpha\} \subset A^+$ increasing to $\mathbb{1}$ such that $\tau(b_\alpha) < \infty$ for all α, and it is *finite* if $\tau(\mathbb{1}) < \infty$.

Throughout the paper we will consider a JBW-algebra A with a faithful semi-finite normal trace τ. Therefore, we omit this condition from the formulation of theorems.

Given $1 \leq p < \infty$, let $A_p = \{x \in A : \tau(|x|^p) < \infty\}$, here $|x|$ denotes the modules of an element x. Define the map $\|\cdot\|_p : A \to [0, \infty)$ by the formula $\|\cdot\|_p = (\tau(|a|^p))^{1/p}$. Then a pair $(A_p, \|\cdot\|_p)$ is a normed space (see [3]). Its completion in the norm $\|\cdot\|_p$ will be denoted by $L_p(A, \tau)$. As usual, we set $L_\infty(A, \tau) = A$ equipped with the norm of A. It is shown [3] that the spaces $L_1(A, \tau)$ and A_* are isometrically isomorphic, therefore they can be indentified. We will use this fact without formal note.

Theorem 2.1 *[3] The space $L_p(A, \tau)$, $p \geq 1$ coincides with the set*

$$L_p = \left\{ x = \int_{-\infty}^{\infty} \lambda de_\lambda : \int_{-\infty}^{\infty} |\lambda|^p d\tau(e_\lambda) < \infty \right\}.$$

Moreover,

$$\|x\|_p = \left(\int_{-\infty}^{\infty} |\lambda|^p d\tau(e_\lambda) \right)^{1/p}.$$

For more information about Jordan algebras we refer a reader to [3],[4],[10].

In the sequel we shall work with mappings of L_1-space. Therefore, recall that a linear bounded operator $T : L_1(A, \tau) \to L_1(A, \tau)$ is *positive* is $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T\| \leq 1$. Here, $\|T\|$ is defined as usual, i.e. $\|T\| = \sup\{\|Tx\|_1 : \|x\|_1 = 1\}$.

3. Main results

In this section we prove the main result of the paper. But before do so we need some auxiliary lemmas.

Lemma 3.1 *Let $T : L_1(A, \tau) \to L_1(A, \tau)$ be a positive operator. Then*

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, x \geq 0} \|Tx\|.$$

Proof. Denote $\alpha = \sup\limits_{\|x\|=1, x\geq 0} \|Tx\|$. It is clear that $\alpha \leq \|T\|$. Let $x \in L_1(A,\tau)$, $\|x\| = 1$, then $x = x^+ - x^-$, $\|x\| = \|x^+\| + \|x^-\|$; we have

$$\begin{aligned} \|Tx\| &= \|Tx^+ - Tx^-\| \\ &= \left\| \|x^+\| T\left(\frac{x^+}{\|x^+\|}\right) - \|x^-\| T\left(\frac{x^-}{\|x^-\|}\right) \right\| \\ &\leq \|x^+\| \left\| T\left(\frac{x^+}{\|x^+\|}\right)\right\| + \|x^-\| \left\| T\left(\frac{x^-}{\|x^-\|}\right)\right\| \\ &\leq \|x^-\|\alpha + \|x^-\|\alpha = \alpha. \end{aligned}$$

Therefore $\|Tx\| \leq \alpha$, hence $\alpha = \|T\|$. □

Lemma 3.2 *Let $T, S : L_1(A,\tau) \to L_1(A,\tau)$ be two positive contractions such that $T \leq S$. Then for every $x \in L_1(A,\tau)$, $x \geq 0$ the equality holds:*

$$\|Sx - Tx\| = \|Sx\| - \|Tx\|.$$

Proof. Let $x \in L_1(A,\tau), x \geq 0$, then we have

$$\begin{aligned} \|(S-T)x\| &= \tau(Sx - Tx) \\ &= \tau(Sx) - \tau(Tx) \\ &= \|Sx\| - \|Tx\|. \end{aligned}$$

□

Now we are ready to formulate the main result.

Theorem 3.3 *Let $T, S : L_1(A,\tau) \longrightarrow L_1(A,\tau)$ be two positive contractions such that $T \leq S$. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < 1$. Then $\|S^n - T^n\| < 1$ for every $n \geq n_0$.*

Proof. Let us assume that $\|S^n - T^n\| = 1$ for some $n > n_0$. Therefore denote

$$m = \min\{n \in \mathbb{N} : \|S^{n_0+n} - T^{n_0+n}\| = 1\}.$$

It is clear that $m \geq 1$. The inequality $T \leq S$ implies that $S^{n_0+m} - T^{n_0+m}$ is a positive operator. Then according to Lemma 3.1 there exists a sequence $\{x_n\} \in L_1(A,\tau)$ such that $x_n \geq 0$, $\|x_n\| = 1, \forall n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(S^{n_0+m} - T^{n_0+m})x_n\| = 1. \quad (3.1)$$

Positivity of $S^{n_0+m} - T^{n_0+m}$ and $x_n \geq 0$ together, with Lemma 3.2, imply that

$$\|(S^{n_0+m} - T^{n_0+m})x_n\| = \|S^{n_0+m}x_n\| - \|T^{n_0+m}x_n\| \quad (3.2)$$

for every $n \in \mathbb{N}$. It then follows from (3.1),(3.2) that

$$\lim_{n\to\infty} \|S^{n_0+m}x_n\| = 1, \tag{3.3}$$

$$\lim_{n\to\infty} \|T^{n_0+m}x_n\| = 0. \tag{3.4}$$

The contractivity of T and S implies that $\|T^{n_0+m-1}x_n\| \le 1$, $\|T^m x_n\| \le 1$ and $\|S^{n_0}T^m x_n\| \le 1$ for every $n \in \mathbb{N}$. Therefore we may choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequences $\{\|T^{n_0+m-1}x_{n_k}\|\}$, $\{\|T^m x_{n_k}\|\}$, $\{\|S^{n_0}T^m x_{n_k}\|\}$ converge. Put $y_k = x_{n_k}, k \in \mathbb{N}$ and

$$\alpha = \lim_{k\to\infty} \|T^{n_0+m-1}y_k\|, \tag{3.5}$$

$$\beta = \lim_{k\to\infty} \|S^{n_0}T^m y_k\|, \tag{3.6}$$

$$\gamma = \lim_{k\to\infty} \|T^m y_k\|. \tag{3.7}$$

From the inequalities $\|S^{n_0+m}y_k\| \le \|S^{n_0+m-1}y_k\|$, $\|S^{n_0+m}y_k\| \le \|S^m y_k\|$ together with (3.3) one gets

$$\lim_{k\to\infty} \|S^{n_0+m-1}y_k\| = 1, \tag{3.8}$$

$$\lim_{k\to\infty} \|S^m y_k\| = 1. \tag{3.9}$$

The inequality $\|S^{n_0+m-1}x_n - T^{n_0+m-1}x_n\| < 1$ with (3.8) implies that $\alpha > 0$. Hence we may choose a subsequence $\{z_k\}$ of $\{y_k\}$ such that $T^{n_0+m-1}z_k \ne 0$, $k \in \mathbb{N}$.

Now from $\|T^{n_0+m-1}z_k\| \le \|T^m z_k\|$ together with (3.5) and (3.7), we find $\alpha \le \gamma$, hence $\gamma > 0$.

Using Lemma 3.2 one gets

$$\begin{aligned} \|S^{n_0}T^m z_k\| &= \|S^{n_0+m}z_k - (S^{n_0+m}z_k - S^{n_0}T^m z_k)\| \\ &= \|S^{n_0+m}z_k\| - \|S^{n_0+m}z_k - S^{n_0}T^m z_k\| \\ &\ge \|S^{n_0+m}z_k\| - \|S^m z_k - T^m z_k\| \\ &= \|S^{n_0+m}z_k\| - \|S^m z_k\| + \|T^m z_k\|. \end{aligned} \tag{3.10}$$

Due to (3.3),(3.9) we have

$$\lim_{k\to\infty} \|S^{n_0+m}z_k\| - \|S^m z_k\| = 0;$$

which with (3.10) implies that

$$\lim_{k\to\infty} \|S^{n_0}T^m z_k\| \ge \lim_{k\to\infty} \|T^m z_k\|,$$

therefore $\beta \ge \gamma$.

On the other hand, by $\|S^{n_0}T^m z_k\| \le \|T^m z_k\|$ one gets $\gamma \ge \beta$, hence $\gamma = \beta$.

Now set

$$u_k = \frac{T^m z_k}{\|T^m z_k\|}, \quad k \in \mathbb{N}.$$

Then using the equality $\gamma = \beta$ and (3.4) one has

$$\lim_{k\to\infty} \|S^{n_0} u_k\| = \lim_{k\to\infty} \frac{\|S^{n_0} T^m z_k\|}{\|T^m z_k\|} = 1,$$

$$\lim_{k\to\infty} \|T^{n_0} u_k\| = \lim_{k\to\infty} \frac{\|T^{n_0+m} z_k\|}{\|T^m z_k\|} = 0.$$

So, owing to Lemma 3.2 and positivity of $S^{n_0} - T^{n_0}$, we get

$$\lim_{k\to\infty} \|(S^{n_0} - T^{n_0}) u_k\| = 1.$$

Since $\|u_k\| = 1, u_k \geq 0, \forall k \in \mathbb{N}$ from Lemma 3.1 one finds $\|S^{n_0} - T^{n_0}\| = 1$, which is a contradiction. This completes the proof. □

As a corollary of the proved theorem we obtain Zaharopol's result in a non-associative setting. Moreover, it recovers one when the algebra is associative.

Corollary 3.4 *Let $T, S : L_1(A,\tau) \to L_1(A,\tau)$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$, then $\|S^n - T^n\| < 1$ for every $n \geq 1$.*

Remark 3.1. It should be noted the following:

(i) Since the dual of $L^1(A,\tau)$ is A, then due to the duality theory Theorem 3.3 holds if we replace L^1-space with JBW-algebra A.

(ii) Unfortunately, Theorem 3.3 is not longer true if one replaces L_1-space by an L_p-space, $1 < p < \infty$. The corresponding example was provided in [17].

(iii) It would be better to note that certain ergodic properties of dominant positive operators has been studied in [8] in a non-commutative setting. In general, to dominant operators were devoted a monograph in [12].

Note that this corollary does not imply the proved theorem. Indeed, let us consider the following example.

Example. Consider $\mathbb{R}^2$ with a norm $\|\mathbf{x}\| = |x_1| + |x_2|$, where $\mathbf{x} = (x_1, x_2)$. An order in $\mathbb{R}^2$ is defined as usual, namely $\mathbf{x} \geq 0$ if and only if $x_1 \geq 0,\ x_2 \geq 0$. Now define mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$, respectively, by

$$S(x_1, x_2) = (Ax_1 + Bx_2, Cx_1 + Dx_2), \tag{3.11}$$

$$T(x_1, x_2) = (\lambda x_2, 0). \tag{3.12}$$

The positivity of S and T implies that $A, B, C, D, \lambda \geq 0$. It is easy to check that $T \leq S$ holds if and only if $\lambda \leq B$.

One can see that

$$\begin{aligned} S^2(x_1,x_2) &= \big((A^2+BC)x_1+(AB+BD)x_2, \\ &\quad (AC+DC)x_1+(D^2+BC)x_2\big), \qquad (3.13) \\ T^2(x_1,x_2) &= (0,0). \qquad (3.14) \end{aligned}$$

By means of Lemma 3.1 let us calculate the norms of operators $S, S-T, S^2-T^2$. Furthermore, we assume that

$$B+D \leq A+C. \tag{3.15}$$

Then using (3.11),(3.12) we have

$$\begin{aligned} \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\geq 0}} \|S\mathbf{x}\| &= \max_{\substack{x_1+x_2=1 \\ x_1,x_2\geq 0}} \{(A+C)x_1+(B+D)x_2\} \\ &= \max_{0\leq x_1\leq 1}\{(A+C-B-D)x_1+B+D\} \\ &= A+C \qquad (3.16) \end{aligned}$$

here we have used (3.15).

Similarly, one has

$$\begin{aligned} \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\geq 0}} \|(S-T)\mathbf{x}\| &= \max_{0\leq x_1\leq 1}\{(A+C-B-D+\lambda)x_1+B+D-\lambda\} \\ &= A+C. \qquad (3.17) \end{aligned}$$

Finally using (3.13), (3.14) together with (3.15) we obtain

$$\begin{aligned} \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\geq 0}} \|(S^2-T^2)\mathbf{x}\| &= \max_{0\leq x_1\leq 1}\Big\{(A+D)(A+C-B-D)x_1 \\ &\quad +D^2+AB+BD+BC\Big\} \\ &= A^2+AC+BC+DC. \qquad (3.18) \end{aligned}$$

Now from (3.16),(3.17) we conclude that the equality $A+C=1$ implies the contractivity of S and $\|S-T\|=1$. The condition (3.15) yields that T is a contraction.

The condition $\|S^2-T^2\|<1$ due to (3.18) yields that

$$A^2+AC+BC+DC<1,$$

which together with $A+C=1$ implies that $C>0$ and $B+D<1$.

Based on the founding conditions, let us provide a more concrete example, i.e. $A=C=1/2$, $B=D=1/3$ and $\lambda=1/4$.

So, we have constructed two positive contractions T and S with $S\geq T$ such that $\|S-T\|=1, \|S^2-T^2\|<1$. This shows that the condition of Corollary 3.4 is not satisfied, but due Theorem 3.3 we have $\|S^n - T^n\| < 1$ for all $n \geq 2$. Therefore the proved Theorem 3.3 is an extension of the Zaharopol's result.

Remark 3.2. Let M be a von Neumann algebra with normal faithful semi-finite trace τ (see [5] for definitions). By M_{sa} we denote the set of all self-adjoint elements of M. Let $\alpha : M \to M$ be a positive linear operator. A linear operator α is said to be *absolute contraction* if $\tau(\alpha(x)) \leq \tau(x)$ for all $x \geq 0, x \in M$ and $\alpha(\mathbb{1}) \leq \mathbb{1}$. Let $L_1(M,\tau)$ be L_1-space associated with M. Then it is known [18] that any absolute contraction can be extended to $L_1(M,\tau)$ such that $\|\alpha(x)\| \leq \|x\|$ for every $x \in L_1(M,\tau)$, $x = x^*$. We also know (see [3], [4]) that the self-adjoint part M_{sa} of M is a JBW-algebra with respect to multiplication $x \circ y = (xy + yx)/2$ and $L_1(M,\tau) = L_1(M_{sa},\tau) + iL_1(M_{sa},\tau)$. Hence, every absolute contraction is L_1-contraction of the JBW-algebra M_{sa}. Therefore, all proved theorems will be valid for any absolute contraction of von Neumann algebras.

Remark 3.3. Note that Theorem 3.3 can be extended to any partially ordered Banach space X in which the norm should satisfy the *additivity condition* on positive part X_+ of X, i.e for any positive elements $x_1, x_2 \in X_+$ one has $\|x_1 - x_2\| = \|x_1\| + \|x_2\|$. The extended theorem's proof will remain the same as the proof of Theorem 3.3. An example of a Banach space which has the additivity condition, besides L_1-spaces, is the dual $A(K)^*$ of the space $A(K)$ of continuous affine functions on a compact convex set K (see [7]).

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