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The equivalence of centro-equiaffine curves

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Abstract

The motivation of this paper is to find formulation of the $SL(n, R)$-equivalence of curves. The types for centro-equiaffine curves and for every type all invariant parametrizations for such curves are introduced. The problem of $SL(n, R)$-equivalence of centro-equiaffine curves is reduced to that of paths. The centro-equiaffine curvatures of path as a generating system of the differential ring of $SL(n, R)$-invariant differential polinomial functions of path are found. Global conditions of $SL(n, R)$-equivalence of curves are given in terms of the types and invariants. It is proved that the invariants are independent.

Key Words: Centro-equiaffine geometry, centro-equiaffine type of a curve, differential invariants of a curve, centro-equiaffine equivalence of curves.

1. Preliminaries

The invariant theory provides a method to find differential invariants of a curve to solve the equivalence problem of curves. In [8] the problem investigated for equiaffine curves and in [13] it is solved for centro-affine curves. The first comprehensive treatment of affine geometry is given in the seminal work of Blaschke [3]. For further developments of the subject, we refer the reader to [14], and the more modern texts [11], [20], the commentaries [16], [17] and survey papers [19], [2], [18]. The fundamental theorem of curves in centro-affine geometry is obtained in [4]. A discussion of centro-affine plane and space curves can be found in [15], [12]. A detailed discussion of plane curves in centro-affine geometry can be obtained in [10]. In [6] equiaffine invariants of 3-dimensional curves and in [5 ,pp.170-172] and [12] equiaffine curvatures of n-dimensional curves are investigated. Complete systems of global equiaffine invariants for plane and space paths are obtained in [1]. The global $SL(n)$-equivalence of path in R^n and C^n is considered in [7] and in [21].

This paper is concerned with the problem of the global equivalence of centro-equiaffine curves. Centro-equiaffine types of a curve is introduced. For every centro-equiaffine type of a curve all possible invariant parametrizations are described. We obtain a generating system of the differential ring of all centro-equiaffine invariant differential polinomials of a path. The conditions of the global centro-equiaffine equivalence of curves are given in terms of the centro-equiaffine type and invariants of a curve. The independence of the invariants is proved.

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2. The centro-equiaffine type of a curve

Let R be the field of real numbers and $I=(a,b)$ be an open interval of R.

Definition 1 *A C^{∞}-map $x:I\to R^n$ will be called an I-path (shortly, a path) in R^n.*

Definition 2 *An I_1-path $x(t)$ and an I_2-path $y(r)$ in R^n will be called D-equivalent if there exists a C^{∞}-diffeomorphism $\varphi:I_2\to I_1$ such that $\varphi^{'}(r)>0$ and $y(r)=x(\varphi(r))$ for all $r\in I_2$. A class of D-equivalent paths in R^n will be called a curve in R^n, ([9], p.9). A path $x\in\alpha$ will be called a parametrization of a curve α.*

Remark 1 *There exist different definitions of a curve ([5], p.2, [7]).*

We denote the group $\{g\in GL(n,R)\mid detg=1\}$ of all $n\times n$ matrices by $SL(n,R)$.

If $x(t)$ is an I-path in R^n then $gx(t)$ is an I-path in R^n for any $g\in SL(n,R)$.

Definition 3 *Two I-paths x and y in R^n will be called $SL(n,R)$-equivalent and written $x\overset{SL(n,R)}{\sim}y$ if there exists $g\in SL(n,R)$ such that $y(t)=gx(t)$.*

Let α be a curve in R^n, that is, $\alpha=\{h_\tau,\tau\in Q\}$, where h_τ is a parametrization of α. Then $g\alpha=\{gh_\tau,\tau\in Q\}$ is a curve in R^n for any $g\in SL(n,R)$.

Definition 4 *Two curves α and β in R^n will be called $SL(n,R)$-equivalent (or $SL(n,R)$-congruent) and written $\alpha\overset{SL(n,R)}{\sim}\beta$ if $\beta=g\alpha$ for some $g\in SL(n,R)$.*

Remark 2 *Our definition is essentially different from the definition ([5], p.21) of a congruence of curves for the group of euclidean motions. By the definition ([5], p.21), two curves with different lengths may be congruent.*

Let x be an I-path in R^n and $x^{'}(t)$ be the derivative of $x(t)$. Put $x^{(0)}=x$, $x^{(n)}=(x^{(n-1)})^{'}$. For $a_k\in R^n$, $k=1,...,n$, the determinant $\det(a_{ij})$ (where a_{ki} are coordinates of a_k) will be denoted by $[a_1a_2...a_n]$. So $\left[x(t)x^{'}(t)...x^{(n-1)}(t)\right]$ is the determinant of the vectors $x(t),x^{'}(t),...,x^{(n-1)}(t)$. For $I=(a,b)$, $q,p\in I$, put

$$l_x(q,p)=\int_q^p\left|\left[x(t)x^{'}(t)...x^{(n-1)}(t)\right]\right|^{\frac{2}{(n-1)n}}dt$$

and $l_x(a,p)=\lim_{q\to a}l_x(q,p)$, $l_x(q,b)=\lim_{p\to b}l_x(q,p)$. There are only four possible cases:

(i) $l_x(a,p)<+\infty$, $l_x(q,b)<+\infty$; (ii) $l_x(a,p)<+\infty$, $l_x(q,b)=+\infty$;

(iii) $l_x(a,p)=+\infty$, $l_x(q,b)<+\infty$; (iv) $l_x(a,p)=+\infty$, $l_x(q,b)=+\infty$.

Suppose that the case (i) or (ii) holds for some q, $p\in I$. Then $l=l_x(a,p)+l_x(q,b)-l_x(q,p)$, where $0\le l\le+\infty$, does not depend on q, p. In this case, we say that x belongs to the centro-equiaffine type of $(0,l)$. The cases (iii) and (iv) do not depend on q, p. In these cases, we say that x belongs to the centro-equiaffine types of $(-\infty,0)$ and $(-\infty,+\infty)$, respectively. There exist paths of all types $(0,l)$ (where $0\le l\le+\infty$), $(-\infty,0)$ and $(-\infty,+\infty)$. The centro-equiaffine type of a path x will be denoted by $L(x)$.

Proposition 1 *(i) If* $x \overset{SL(n,R)}{\sim} y$ *then* $L(x) = L(y)$*;*

(ii) Let α *be a curve and* $x, y \in \alpha$*. Then* $L(x) = L(y)$*.*

Proof. It is obvious. □

The centro-equiaffine type of a path $x \in \alpha$ will be called the centro-equiaffine type of the curve α and denoted by $L(\alpha)$. According to Proposition 1, $L(\alpha)$ is an $SL(n,R)$-invariant of a curve α.

3. Invariant parametrization and reduction theorem

Definition 5 *An* I*-path* $x(t)$ *in* R^n *will be called centro-equiaffine regular (shortly, regular) if* $\left[x(t)x^{'}(t)...x^{(n-1)}(t)\right] \neq 0$ *for all* $t \in I$*. A curve will be called regular if it contains a regular path.*

Now we define an invariant parametrization of a regular curve in R^n.

Let $I = (a,b)$ and $x(t)$ be a regular I-path in R^n. We define the centro-equiaffine arc length function $s_x(t)$ for each centro-equiaffine type as follows. We put $s_x(t) = l_x(a,t)$ for the case $L(x) = (0,l)$, where $0 < l \leq +\infty$, and $s_x(t) = -\ l_x(t,b)$ for the case $L(x) = (-\infty, 0)$. Let $L(x) = (-\infty, +\infty)$. We choose a fixed point in every interval $I = (a,b)$ of R and denote it by a_I. Let $a_I = 0$ for $I = (-\infty, +\infty)$. We set $s_x(t) = l_x(a_I, t)$.

Since $s_x^{'}(t) > 0$ for all $t \in I$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $L(x)$ and $t_x^{'}(s) > 0$ for all $s \in L(x)$.

Proposition 2 *Let* $I = (a,b)$ *and* x *be a regular* I*-path in* R^n*. Then*

(i) $s_{gx}(t) = s_x(t)$ *and* $t_{gx}(s) = t_x(s)$ *for all* $g \in SL(n,R)$*;*

(ii) the equalities $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$ *and* $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s)$ *hold for any* C^∞*-diffeomorphism* $\varphi : J = (c,d) \to I$ *such that* $\varphi^{'}(r) > 0$ *for all* $r \in J$*, where* $s_0 = 0$ *for* $L(x) \neq (-\infty, +\infty)$ *and* $s_0 = l_x(\varphi(a_J), a_I)$ *for* $L(x) = (-\infty, +\infty)$*.*

Proof. The proof of (i) is obvious. We prove (ii). Let $L(x) = (-\infty, +\infty)$. Then we have

$$\begin{aligned} s_{x(\varphi)}(r) &= \int_{a_J}^{r} \left| \left[x(\varphi(r)) \frac{d}{dr}(x(\varphi(r)))...\frac{d^{n-1}}{dr^{n-1}}(x(\varphi(r))) \right] \right|^{\frac{2}{(n-1)n}} dr \\ &= \int_{a_J}^{r} \frac{d\varphi}{dr} \left| \left[x(\varphi(r)) \frac{d}{d\varphi}(x(\varphi(r)))...\frac{d^{n-1}}{d\varphi^{n-1}}(x(\varphi(r))) \right] \right|^{\frac{2}{(n-1)n}} dr \\ &= l_x(\varphi(a_J), \varphi(r)) = l_x(a_I, \varphi(r)) + l_x(\varphi(a_J), a_I). \end{aligned}$$

So $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$, where $s_0 = l_x(\varphi(a_J), a_I)$. This implies that $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s)$. For $L(x) \neq (-\infty, +\infty)$, it is easy to see that $s_0 = 0$. □

Let α be a regular curve and $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 6 *The parametrization $x(t_x(s))$ of a regular curve α will be called an invariant parametrization of α.*

We denote the set of all invariant parametrizations of α by ϕ_α. Every $y \in \phi_\alpha$ is I-path, where $I = L(\alpha)$.

Proposition 3 *Let α be a regular curve, $x \in \alpha$ and x be an I-path, where $I = L(\alpha)$. Then the following conditions are equivalent:*

(i) *x is an invariant parametrization of α;*

(ii) $\left[x(s)x'(s)...x^{(n-1)}(s)\right]^2 = 1$ *for all* $s \in L(\alpha)$;

(iii) $s_x(s) = s$ *for all* $s \in L(\alpha)$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in \phi_\alpha$.Then there exists $y \in \alpha$ such that $x(s) = y\left(t_y(s)\right)$. By Proposition 2, $s_x(s) = s_{y(t_y)}(s) = s_y(t_y(s)) + s_0 = s + s_0,$ where s_0 is as in Proposition 2. Since s_0 does not depend on s, $\frac{ds_x(s)}{ds} = \left|\left[x(s)x'(s)...x^{(n-1)}(s)\right]\right|^{\frac{2}{(n-1)n}} = 1$. Hence $\left[x(s)x'(s)...x^{(n-1)}(s)\right]^2 = 1$ for all $s \in L(\alpha)$.

$(ii) \Rightarrow (iii)$. Let $\left[x(s)x'(s)...x^{(n-1)}(s)\right]^2 = 1$ for all $s \in L(\alpha)$. By the definition of $s_x(t)$, we have $\frac{ds_x(s)}{ds} = \left|\left[x(s)x'(s)...x^{(n-1)}(s)\right]\right|^{\frac{2}{(n-1)n}} = 1$. Therefore $s_x(s) = s + c$ for some $c \in R$. In the case $L(x) \neq (-\infty, +\infty)$, $s_x(s) = s + c$ and $s_x(s) \in L(\alpha)$ for all $s \in L(\alpha)$ implies $c = 0$, that is, $s_x(s) = s$. In the case $L(x) = (-\infty, +\infty)$, $s_x(s) = l_x(a_I, s) = l_x(0, s) = s + c$ implies $0 = l_x(0, 0) = c$, that is, $s_x(s) = s$.

$(iii) \Rightarrow (i)$. The equality $s_x(s) = s$ implies $t_x(s) = s$. Therefore $x(s) = x(t_x(s)) \in \phi_\alpha$. □

Proposition 4 *Let α be a regular curve and $L(\alpha) \neq (-\infty, +\infty)$. Then there exists the unique invariant parametrization of α.*

Proof. Let $x, y \in \alpha$, x be an I_1-path and y be an I_2-path. Then there exists a C^∞-diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. By Proposition 2 and $L(\alpha) \neq (-\infty, +\infty)$, we obtain $y(t_y(s)) = x(\varphi(t_y(s)) = x(\varphi(t_{x(\varphi)}(s))) = x(t_x(s))$. □

Let α be a regular curve and $L(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set ϕ_α is not countable.

Proposition 5 *Let α be a regular curve, $L(\alpha) = (-\infty, +\infty)$ and $x \in \phi_\alpha$. Then* $\phi_\alpha = \{y : y(s) = x\left(s + s'\right), s' \in (-\infty, +\infty)\}$.

Proof. Let $x, y \in \phi_\alpha$. Then there exist $h, k \in \alpha$ such that $x(s) = h(t_h(s))$, $y(s) = k(t_k(s))$, where h be an I_1-path and k be an I_2-path. Since $h, k \in \alpha$ there exists $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $k(r) = h(\varphi(r))$ for all $r \in I_2$. By Proposition 2, $y(s) = k(t_k(s)) = h(\varphi(t_k(s)) = h(\varphi(t_{h(\varphi)}(s))) = h(t_h(s - s_0)) = x(s - s_0)$.

Let $x \in \phi_\alpha$ and $s^{'} \in (-\infty, +\infty)$. We prove $x(\psi) \in \phi_\alpha$, where $\psi(s) = s + s^{'}$. By Proposition 3, $\left[x(s)x^{'}(s)...x^{(n-1)}(s)\right]^2 = 1$ and $s_x(s) = s$. Put $z(s) = x(\psi(s))$. Since ψ is a C^∞-diffeomorphism of $(-\infty, +\infty)$ onto $(-\infty, +\infty)$, then $z = x(\psi) \in \alpha$. Using Proposition 2 and $s_x(s) = s$, we get $s_z(s) = s_{x(\psi)}(s) = s_x(\psi(s)) + s_1 = \left(s + s^{'}\right) + s_1$, where

$$s_1 = \int_{\psi(0)}^{0} \left|\left[x(s)x^{'}(s)...x^{(n-1)}(s)\right]\right|^{\frac{2}{(n-1)n}} ds.$$

This, in view of $\left[x(s)x^{'}(s)...x^{(n-1)}(s)\right]^2 = 1$, implies $s_1 = -\psi(0) = -s^{'}$. Then $s_z(s) = \left(s + s^{'}\right) - s^{'} = s$. By Proposition 3, $z \in \phi_\alpha$. □

Theorem 1 *Let α, β be regular curves and $x \in \phi_\alpha$, $y \in \phi_\beta$. Then,*

(i) for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{SL(n,R)}{\sim} \beta$ if and only if $x(s) \overset{SL(n,R)}{\sim} y(s)$;

(ii) for $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{SL(n,R)}{\sim} \beta$ if and only if $x(s) \overset{SL(n,R)}{\sim} y\left(s + s^{'}\right)$ for some $s^{'} \in (-\infty, +\infty)$.

Proof. (i) Let $\alpha \overset{SL(n,R)}{\sim} \beta$ and $h \in \alpha$. Then there exists $g \in SL(n, R)$ such that $\beta = g\alpha$. This implies $gh \in \beta$. Using Propositions 2 and 4, we get $x(s) = h(t_h(s))$, $y(s) = (gh)(t_{gh}(s))$ and $gx(s) = g(h(t_h(s))) = (gh)(t_h(s)) = (gh)(t_{gh}(s)) = y(s)$. Thus $x \overset{SL(n,R)}{\sim} y$. Conversely, let $x \overset{SL(n,R)}{\sim} y$, that is, there exists $g \in SL(n, R)$ such that $gx = y$. Then $\alpha \overset{SL(n,R)}{\sim} \beta$.

(ii) Let $\alpha \overset{SL(n,R)}{\sim} \beta$. Then there exist I-paths $h \in \alpha$, $k \in \beta$ and $g \in SL(n, R)$ such that $k(t) = gh(t)$. We have $k(t_k(s)) = k(t_{gh}(s)) = k(t_h(s)) = (gh)(t_h(s))$. By Proposition 5, $x(s) = k(t_k(s + s_1))$, $y(s) = h(t_h(s + s_2))$ for some s_1, $s_2 \in (-\infty, +\infty)$. Therefore $x(s - s_1) = gy(s - s_2)$. This implies that $x(s) \overset{SL(n,R)}{\sim} y\left(s + s^{'}\right)$, where $s^{'} = s_1 - s_2$. Conversely, let $x(s) \overset{SL(n,R)}{\sim} y\left(s + s^{'}\right)$ for some $s^{'} \in (-\infty, +\infty)$. Then there exists $g \in SL(n, R)$ such that $y\left(s + s^{'}\right) = gx(s)$. Since $y\left(s + s^{'}\right) \in \beta$, then $\alpha \overset{SL(n,R)}{\sim} \beta$. □

Theorem 1 reduces the problem of the $SL(n, R)$-equivalence of regular curves to that of paths.

4. The generating system

Let $x(t)$ be an I-path in R^n.

Definition 7 *A polynomial $p(x, x^{'}, ..., x^{(k)})$ of x and a finite number of derivatives $x, x^{'}, ..., x^{(k)}$ of x with the coefficients from R will be called a differential polynomial of x. It will be denoted by $p\{x\}$.*

We denote the set of all differential polynomials of x by $R\{x\}$. It is a differential R-algebra. Let G be a subgroup of $SL(n,R)$.

Definition 8 *A differential polynomial $p\{x\}$ will be called G-invariant if $p\{gx\} = p\{x\}$ for all $g \in G$.*

The set of all G-invariant differential polynomials of x will be denoted by $R\{x\}^G$. It is a differential R-subalgebra of $R\{x\}$.

By Proposition 3, an I-path x is an invariant parametrization of a regular curve α if and only if $I = L(\alpha)$ and $\left[x(s)x^{'}(s)...x^{(n-1)}(s)\right]^2 = 1$ for all $s \in L(\alpha)$.

Let I be one of the sets $(0,l)$, $0 < l \leq +\infty$, $(-\infty, 0)$, $(-\infty, +\infty)$. Put $W = \{x : \left[x(s)x^{'}(s)...x^{(n-1)}(s)\right]^2 = 1$ for all s in $I\}$. The restriction of the $SL(n,R)$-invariant differential polynomial $p\{x\}$ to the set W will be denoted by $p\{x\}/_W$. We put $R\{x\}^{SL(n,R)}/_W = \left\{p/_W, p \in R\{x\}^{SL(n,R)}\right\}$. It is a differential R-algebra.

Definition 9 *A subset S of $R\{x\}^{SL(n,R)}/_W$ will be called a generating system of $R\{x\}^{SL(n,R)}/_W$ if the smallest differential R-subalgebra with the unit containing S is $R\{x\}^{SL(n,R)}/_W$.*

Theorem 2 *The system*

$$\left[xx^{'}...x^{(n-1)}\right]/_W,\ \left[xx^{'}...x^{(i-1)}x^{(n)}x^{(i+1)}...x^{(n-1)}\right]/_W,\ i = 1,...n-2,$$

is a generating system of $R\{x\}^{SL(n,R)}/_W$.

Proof. For the proof, we need several lemmas.

By the First Main Theorem for $SL(n)$ ([22], p.45), the system U of $\left[x^{(i_1)}...x^{(i_n)}\right]$, where $0 \leq i_1 < i_2 < ... < i_{n-1} < +\infty$, is a generating system of $R\left\{x^{'}\right\}^{SL(n,R)}$. For the determinant $u = \left[x^{(i_1)}...x^{(i_n)}\right]$, we denote the number of elements of the set $\left\{x^{(i_1)},...,x^{(i_n)}\right\} \setminus \left\{x, x^{'},...,x^{(n-1)}\right\}$ by $\delta(u)$ and we put $\tau(u) = \max(i_1,...,i_n)$. □

Lemma 1 *Let $u = \left[x^{(i_1)}...x^{(i_n)}\right]$ and $\delta(u) \geq 2$. Then u/W is a polynomial of elements $v/_W = \left[x^{(j_1)}...x^{(j_n)}\right]/_W$ such that $\delta(v) < \delta(u)$ and $\tau(v) \leq \tau(u)$.*

Proof. By $\delta(u) \geq 2$, there exists $x^{(k)}$, $1 \leq k \leq n-1$, such that $x^{(k)} \notin \left\{x^{(i_1)},...,x^{(i_n)}\right\}$. We need the following lemma ([22], p.70): □

Lemma 2 *For any vectors $x_0, x_1, ..., x_n, y_2, ...y_n$ in R^n, the following equality holds:*

$$\begin{aligned}&[x_1x_2...x_n] \times [x_0y_2...y_n] - [x_0x_2...x_n] \times [x_1y_2...y_n] - \\ &... - [x_1x_2...x_{n-1}x_0] \times [x_ny_2...y_n] = 0\end{aligned}$$

Proof. In Lemma 3, we put $x_1 = x^{(i_1)}$,..., $x_n = x^{(i_n)}$, $y_3 = x'$,..., $y_{k+1} = x^{(k-1)}$, $y_{k+2} = x^{(k+1)}$,..., $y_n = x^{(n-1)}$. Then

$$\left.\begin{array}{l} \left[x^{(i_1)}...x^{(i_n)}\right] \times \left[x^{(k)}xx^{'}...x^{(k-1)}x^{(k+1)}...x^{(n-1)}\right] - \\ \left[x^{(k)}x^{(i_2)}...x^{(i_n)}\right] \times \left[x^{(i_1)}xx^{'}...x^{(k-1)}x^{(k+1)}...x^{(n-1)}\right] - ... \\ - \left[x^{(i_1)}...x^{(i_{n-1})}x^{(k)}\right] \times \left[x^{(i_n)}xx^{'}...x^{(k-1)}x^{(k+1)}...x^{(n-1)}\right] = 0 \end{array}\right\} \quad (1)$$

Put $v_0 = \left[x^{(k)}xx^{'}...x^{(k-1)}x^{(k+1)}...x^{(n-1)}\right]$, $v_r = \left[x^{(i_r)}xx^{'}...x^{(k-1)}x^{(k+1)}...x^{(n-1)}\right]$, $h_m = \left[x^{(i_1)}...x^{(i_{m-1})}x^{(k)}x^{(i_{m+1})}...x^{(i_n)}\right]$. Then $\delta(v_0) = 0$, $\tau(v_0) \le \tau(u)$, $\delta(v_r) \le 1$, $\tau(h_m) \le \tau(u)$. From equality (1), using $\left[xx^{'}...x^{(n-1)}\right]^2 = 1$, we get $u/_W = v_1h_1v_0/_W + ... + v_nh_nv_0/_W$. By $\delta(u) \ge 2$, the number of multiplications $v_jh_jv_0 \ne 0$ is $\delta(u) + 1 \ge 3$. For h_j such that $v_jh_jv_0 \ne 0$, we have $\delta(h_j) < \delta(u)$. Therefore $u/_W$ is a polynomial of the system $v_0/_W$, $v_j/_W$, $h_j/_W$, with $\delta(v_0) = 0$, $\tau(v_0) \le \tau(u)$, $\delta(v_j) \le 1$, $\tau(v_j) \le \tau(u)$, $\delta(h_j) < \delta(u)$, $\tau(h_j) \le \tau(u)$. So the proof of Lemma 2 is completed. □

Lemma 3 *Let* $u = \left[xx^{'}...x^{(i-1)}x^{(m)}x^{(i+1)}...x^{(n-1)}\right]$ *and* $m > n$. *Then* u *is a differential polynomial of elements* $v = \left[x^{(j_1)}...x^{(j_n)}\right]$ *such that* $\tau(v) < \tau(u)$.

Proof. We have

$$\begin{array}{l} \left[xx^{'}...x^{(i-1)}x^{(m-1)}x^{(i+1)}...x^{(n-1)}\right]^{'} = \left[x^{'}x^{'}...x^{(i-1)}x^{(m-1)}x^{(i+1)}...x^{(n-1)}\right] + ... \\ + \left[xx^{'}...x^{(i-2)}x^{(i)}x^{(m-1)}x^{(i+1)}...x^{(n-1)}\right] + \left[xx^{'}...x^{(i-1)}x^{(m)}x^{(i+1)}...x^{(n-1)}\right] + ... \\ + \left[xx^{'}...x^{(i-1)}x^{(m)}x^{(i+1)}...x^{(n-2)}x^{(n)}\right]. \end{array}$$

In this equality, only the following determinants are nonzero:

$$\begin{array}{rcl} v_1 &=& \left[xx^{'}..x^{(i-1)}x^{(m-1)}x^{(i+1)}..x^{(n-1)}\right], v_2 = \left[xx^{'}..x^{(i-2)}x^{(i)}x^{(m-1)}x^{(i+1)}..x^{(n-1)}\right], \\ v_3 &=& \left[xx^{'}..x^{(i-1)}x^{(m-1)}x^{(i+1)}..x^{(n-2)}x^{(n)}\right], u = \left[xx^{'}..x^{(i-1)}x^{(m)}x^{(i+1)}..x^{(n-1)}\right]. \end{array}$$

So we obtain $u = v_1^{'} - v_2 - v_3$. By $\tau(u) = m$, $\tau(v_1) = \tau(v_2) = \tau(v_3) = m - 1$, the lemma is proved.

Now the proof of Theorem 2 follows from Lemmas 1, 2 and 4 by induction on $\tau(u)$ and $\delta(u)$. □

Theorem 3 *Let* α, β *be regular curves in* R^n *and* $x \in \phi_\alpha$, $y \in \phi_\beta$. *Then,*

(*i*) *for* $L(\alpha) = L(\beta) \ne (-\infty, +\infty)$, $\alpha \overset{SL(n,R)}{\sim} \beta$ *if and only if*

$$\left.\begin{array}{l} sgn[x(s)x^{'}(s)...x^{(n-1)}(s)] = sgn[y(s)y^{'}(s)...y^{(n-1)}(s)], \\ \\ [x(s)x^{'}(s)...x^{(i-1)}(s)\,x^{(n)}(s)\,x^{(i+1)}(s)...x^{(n-1)}(s)] \\ = [y(s)y^{'}(s)...y^{(i-1)}(s)\,y^{(n)}(s)\,y^{(i+1)}(s)...y^{(n-1)}(s)] \end{array}\right\} \quad (2)$$

for all $s \in L(\alpha) = L(\beta)$ *and* $i = 1, ..., n-2$.

(ii) *for* $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{SL(n,R)}{\sim} \beta$ *if and only if there exists* $a \in (-\infty, +\infty)$ *such that*

$$sgn\left[x(s)x'(s)...x^{(n-1)}(s)\right] = sgn\left[y(s+a)y'(s+a)...y^{(n-1)}(s+a)\right]$$

$$\left[x(s)x'(s)...x^{(i-1)}(s)x^{(n)}(s)x^{(i+1)}(s)...x^{(n-1)}(s)\right] =$$

$$\left[y(s+a)y'(s+a)...y^{(i-1)}(s+a)y^{(n)}(s+a)y^{(i+1)}(s+a)...y^{(n-1)}(s+a)\right]$$

for all $s \in (-\infty, +\infty)$ *and* $i = 1, ..., n-2$.

Proof. (i) Let $\alpha \overset{SL(n,R)}{\sim} \beta$. By claim (i) of Theorem 1, $x \overset{SL(n,R)}{\sim} y$. By Proposition 3, $\left|[xx'...x^{(n-1)}]\right| = \left|[yy'...y^{(n-1)}]\right| = 1$. This, in view of $x \overset{SL(n,R)}{\sim} y$, yields (2). Now suppose that (2) holds. By Proposition 3, we have $\left|\left[x(s)x'(s)...x^{(n-1)}(s)\right]\right| = \left|\left[y(s)y'(s)...y^{(n-1)}(s)\right]\right| = 1$ we obtain

$$[x(s)x'(s)...x^{(n-1)}(s)] = [y(s)y'(s)...y^{(n-1)}(s)],$$

$$\begin{aligned} & [x(s)x'(s)...x^{(i-1)}(s)\,x^{(n)}(s)\,x^{(i+1)}(s)...x^{(n-1)}(s)] \\ = \; & [y(s)y'(s)...y^{(i-1)}(s)\,y^{(n)}(s)\,y^{(i+1)}(s)...y^{(n-1)}(s)]. \end{aligned}$$

This, in view of claim (i) of Theorem 1 and Theorems 10.7, 10.8 in [7], implies $\alpha \overset{SL(n,R)}{\sim} \beta$.

The proof of (ii) follows similarly from claim (ii) of Theorem 1. □

Let T be one of the sets $(0, l)$ (where $l \leq +\infty$), $(-\infty, 0)$, $(-\infty, +\infty)$.

Theorem 4 *Let* $h_1(s), ..., h_n(s)$ *be* C^∞*-functions on* T, *where* $|h_n(s)| = 1$ *for all* $s \in T$. *Then there exists an invariant parametrization* y *of a regular curve such that*

$$sgn[y(s)y'(s)...y^{(n-1)}(s)] = h_n(s),$$

$$[y(s)y'(s)...y^{(i-1)}(s)\,y^{(n)}(s)\,y^{(i+1)}(s)...y^{(n-1)}(s)] = h_i(s)$$

for all $s \in T$ *and* $i = 0, ..., n-2$.

Proof. Let $C(s)$ be the matrix $\|c_{ij}(s)\|$, where $c_{j+1j}(s) = 1$ for all $s \in T$, $0 \leq j \leq n-2$; $c_{ij}(s) = 0$ for all $s \in T$, $j \neq n$, $i \neq j+1$, $0 \leq i \leq n-1$; $c_{in}(s) = \frac{h_i(s)}{h_n(s)}$, $i = 0, ..., n-2$, $c_{nn}(s) = \frac{h_n'(s)}{h_n(s)}$. It is known from the theory of differential equations that there exists a solution of the differential equation

$$A_x'(s) = A_x(s)\,C(s) \tag{3}$$

such that $\det A_x(s) \neq 0$ for all $s \in T$, where $A_x(s) = \left\|x(s)x'(s)...x^{(n-1)}(s)\right\|$ is the matrix of column vectors $x(s), x'(s), ..., x^{(n-1)}(s)$. Let $A_x(s)$ be one of such solutions. Put $[x(s)x'(s)...x^{(n-1)}(s)] = \varphi(s)$. By

$\det A_x(s) \neq 0$ for all $s \in T$, we get $\varphi(s) \neq 0$ for all $s \in T$. By $|h_n(s)| = 1$ for all $s \in T$, we have $h_n^{'}(s) = 0$ for all $s \in T$. Then, from (3), we obtain

$$\frac{[x(s)x^{'}(s)...x^{(n-1)}(s)]^{'}}{[x(s)x^{'}(s)...x^{(n-1)}(s)]} = \frac{h_n^{'}(s)}{h_n(s)} = 0.$$

Therefore $\varphi^{'}(s) = 0$. Put $\varphi(s) = \lambda_1$, $\lambda_1 \in R$, $\lambda_1 \neq 0$ and $h_n(s) = \lambda_2$, $\lambda_2 \in R$. By $|h_n(s)| = 1$, we get $|\lambda_2| = 1$. We consider $g \in SL(n,R)$ such that $\det g = \frac{\lambda_2}{\lambda_1}$. So $[gx(gx)^{'}...(gx)^{(n-1)}] = \det g[xx^{'}...x^{(n-1)}] = h_n(s)$. For $y = gx$, we have

$$\frac{\left[yy^{'}...y^{(i-1)}y^{(n)}y^{(i+1)}...y^{(n-1)}\right]}{\left[yy^{'}...y^{(n-1)}\right]} = \frac{\det g\left[xx^{'}...x^{(i-1)}x^{(n)}x^{(i+1)}...x^{(n-1)}\right]}{\det g\left[xx^{'}...x^{(n-1)}\right]} = \frac{h_i(s)}{h_n(s)}$$

$i = 0, ..., n-2$. Hence

$$\begin{aligned} \left[y(s)y^{'}(s)...y^{(n-1)}(s)\right] &= h_n(s), \\ \left[y(s)y^{'}(s)...y^{(i-1)}(s)\,y^{(n)}(s)\,y^{(i+1)}(s)...y^{(n-1)}(s)\right] &= h_i(s) \end{aligned}$$

for all $s \in T$, $i = 0, ..., n-2$. Then by $\left|\left[y(s)y^{'}(s)...y^{(n-1)}(s)\right]\right| = |h_n(s)| = 1$ and Proposition 3, $y \in \phi_\alpha$ for some regular curve α. □

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