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A note on the Lyapunov exponent in continued fraction expansions

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Abstract

Let $T : [0,1) \to [0,1)$ be the Gauss transformation. For any irrational $x \in [0,1)$, the Lyapunov exponent $\alpha(x)$ of x is defined as

$$\alpha(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|.$$

By Birkoff Average Theorem, one knows that $\alpha(x)$ exists almost surely. However, in this paper, we will see that the non-typical set

$$\{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| \text{ does not exist}\}$$

carries full Hausdorff dimension.

Key Words: Continued fractions, Lévy constant, Hausdorff dimension.

1. Introduction

In the numerical study of a dynamical system, one is often interested in the asymptotic behavior of typical points (with respect to some measure). This study gives important information about the observable properties of the dynamical system, and typical points with respect to some measure give complementary information.

However, the set of non-typical points has rarely been considered in the literature. The first pioneer work in this field is the result given by L. Barreira and J. Schmeling [1, 2], where they showed that, in several situations central in the theory of dynamical system, the set of non-typical points contains complete information about some observable properties. Namely, the set of non-typical points carries full topological entropy and full Hausdorff dimension. L. Barreira and J. Schmeling's efforts are mainly focused on the subshift of finite type, conformal repellers and conformal horseshores.

Then it is natural to ask what happens in the case of infinite symbolic space, or when the underling dynamic system is no longer continuous. As it is known, the example of continued fraction is related to some infinite symbolic space and its underling dynamic system is not continuous.

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It is known that the Gauss transformation $T:[0,1)\to[0,1)$, given as

$$T(0):=0,\quad T(x):=\frac{1}{x}\ (\mathrm{mod}\ 1),\ \text{for } x\in(0,1), \tag{1.1}$$

leads to the continued fraction expansions. For any irrational $x\in[0,1)$, there is a unique infinite continued fraction expansion of the form

$$x=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+T^2(x)}}=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\cfrac{1}{a_3(x)+\ddots}}}, \tag{1.2}$$

where $a_1(x)=\lfloor 1/x\rfloor$, $a_n(x)=a_1(T^{n-1}(x))$, for $n\geq 2$, are called the partial quotients of x.

For any $n\geq 1$, the n-th convergent $\frac{p_n(x)}{q_n(x)}$ of x is obtained by the finite truncation on (1.2) given as

$$\frac{p_n(x)}{q_n(x)}=\cfrac{1}{a_1(x)+\cfrac{1}{\ddots+a_n(x)}},$$

where $p_n(x)$ and $q_n(x)$ can be obtained recursively by the relations

$$p_{-1}=1;\ p_0=0;\ p_k=a_kp_{k-1}+p_{k-2},\quad 1\leq k\leq n.$$

$$q_{-1}=0;\ q_0=1;\ q_k=a_kq_{k-1}+q_{k-2},\quad 1\leq k\leq n. \tag{1.3}$$

It is known that the Gauss transformation is invariant and ergodic with respect to the Gauss measure given by

$$d\mu=\frac{1}{\log 2}\frac{1}{1+x}dx.$$

So, an application of Birkhoff Ergodic Theorem yields that the Lyapunov exponent of $x\in[0,1)$ exists almost surely. However, it is not difficult, if not evidently, that there do exist non-typical points which do not obey the above law. So it would be a natural to consider the size of the exceptional set of such non-typical points. More precisely, we would like to investigate the size of the set

$$\mathcal{D}=\{x\in[0,1):\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log T^j(x)\ \text{does not exists}\}.$$

We show the following theorem.

Theorem 1.1 $\dim_H\mathcal{D}=1$, *where* $\dim_H$ *denotes the Hausdorff dimension.*

The famous result due to P. Lévy [8] asserts that

Theorem 1.2 *[8] For almost all* $x\in[0,1)$, *the constant of* x *exists.*

Now it is known that Lévy's theorem is a direct consequence of the fact that T is an ergodic transformation of measure space $[0,1]$ endowed with the Gauss measure which is invariant under T, see [see 3].

Inspired by L. Barreira and J. Schmeling's work [1, 2], we would like to ask how about the size of the set of non-typical points

$$\mathcal{D} = \{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log T^j(x) \text{ does not exists}\}.$$

2. Preliminary

In this section, we will gather some elementary properties shared by continued fractions.

For any $n \geq 1$ and $(a_1, \cdots, a_n) \in \mathbf{N}^n$, then

$$I(a_1,\cdots,a_n) = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right), & \text{if } n \text{ is odd;} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right], & \text{if } n \text{ is odd} \end{cases}$$

is called an n-th order cylinder, where $p_k,\ q_k\,(1 \leq k \leq n)$ are determined by following recurrence relations

$$p_k = a_k p_{k-1} + p_{k-2},\ q_k = a_k q_{k-1} + q_{k-2},\ 1 \leq k \leq n, \tag{2.4}$$

with the conventions that $p_{-1} = 1, p_0 = 0,\ q_{-1} = 0, q_0 = 1$, and $\frac{p_k}{q_k}$ convergent to ξ. It is well known, (see [7]) that $I(a_1, \cdots, a_n)$ just represents the set of points in $[0,1)$ which have a continued fraction expansions begins with $a_1, \cdots, a_n$. By the recursive relation (2.4), it is easy to see that

Proposition 2.1 *[7, 10] For any $n \geq 1$, $1 \leq k \leq n$ and $(a_1, \cdots, a_n) \in \mathbb{N}^n$, one has*

$$q_n \geq 2^{\frac{n-1}{2}}, \quad \prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n}(a_k+1). \tag{2.5}$$

$$1 \leq \frac{q_n(a_1,\cdots,a_n)}{q_k(a_1,\cdots,a_k)q_{n-k}(a_{k+1},\cdots,a_n)} \leq 2. \tag{2.6}$$

$$\frac{a_k+1}{2} \leq \frac{q_n(a_1,a_2,\cdots,a_n)}{q_{n-1}(a_1,\cdots,a_{k-1},a_{k+1},\cdots,a_n)} \leq a_k+1. \tag{2.7}$$

$$|I(a_1,\cdots,a_n)| = \frac{1}{q_n(q_n+q_{n-1})}, \tag{2.8}$$

where $|I(a_1,\cdots,a_n)|$ denotes the length of $I(a_1,\cdots,a_n)$.

Lemma 2.2 *[5] Let $\{A_n, n \geq\}$ be a sequence of nonempty subset of $\mathbb{N}$. Set*

$$S = \{x \in [0,1) : a_n(x) \in A_n,\ \text{for all } n \geq 1\}.$$

Then for any $N \geq 1$ and $a_k \in A_k$ for $1 \leq k \leq N$, one has

$$\dim_H S = \dim_H S \cap I(a_1, \cdots, a_N).$$

For any $B \geq 2$, let $E(B) = \{x \in [0,1) : a_n(x) \leq B, \text{ for all } n \geq 1\}$. Considering Diophantine problems, Jarnik [6] estimated the Hausdorff dimension $\dim_H E(B)$ of the set $E(B)$ related to the set of points which do not have good rational approximations. More precisely, he proved the following Lemma:

Lemma 2.3 *[6] For any $B \geq 8$,*

$$1 - \frac{4}{B \log 2} < \dim_H E(B) < 1 - \frac{1}{8B \log B}.$$

We end this section with the following lemma:

Lemma 2.4 *[4] Let $E \in \mathbb{R}^n$. If $f : E \to \mathbb{R}^m$ is α-Hölder, i.e., there exists constant $c > 0$ such that for all $x, y \in E$,*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

then $\dim_H f(E) \leq \frac{1}{\alpha} \dim_H E$.

3. Proof of Theorem 1.1

If $x \in [0,1]$ is irrational, then the for any the Lévy constant of x is the number

$$\beta(x) = \lim_{n\to\infty} \frac{\log q_n(x)}{n},$$

from the algorithm (1.2), it easy to see that

$$\prod_{j=0}^{n-1} T^j(x) \leq \frac{1}{q_n(x)} \leq 2 \prod_{j=0}^{n-1} T^j(x).$$

So, it follows

$$\begin{aligned} \mathcal{D} &= \{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log T^j(x) \text{ does not exists}\} \\ &= \{x \in [0,1) : \lim_{n\to\infty} \frac{\log q_n(x)}{n} \text{ does not exists}\}. \end{aligned}$$

The proof of Theorem 1.1 is established in the following steps. First, we construct a subset $\mathcal{D}(B)$ of $\mathcal{D}$. Second, we a define a surjective map f between $\mathcal{D}(B)$ and $E(B)$. Last, we check that the map f is $\frac{1}{1+\epsilon}$-Hölder.

Step I. Fix $0<\delta<\frac{1}{4}$. Choose an integer sequence $\{n_k, k\ge 1\}$ with $(n_1+\cdots+n_{k-1})=o(n_k)$. For any $k\ge 1$, set $\ell_k=\lfloor \delta n_k\rfloor$ and $m_k=n_k-\ell_k$. For any $B\ge 3$, set

$$\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B\big)=\Big\{x\in[0,1): a_{m_k+1}=\cdots=a_{n_k}=B+2, \text{ for all } k\ge 1,$$

$$\text{and } 1\le a_n(x)\le B \ \text{ for all other } n\ge 1\Big\}.$$

We claim that $\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B\big)\subset\mathcal{D}$. It suffices to check that, for any $x\in\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B\big)$,

$$\limsup_{k\to\infty}\left(\frac{\log q_{n_k}(x)}{n_k}-\frac{\log q_{m_k}(x)}{m_k}\right)>0.$$

By Proposition 2.1, it follows that $q_{n_k}(x)\ge q_{m_k}(x)(B+2)^{\ell_k}$ and $q_{m_k}(x)\le (B+1)^{m_k}$. So,

$$\begin{aligned}
&\frac{\log q_{n_k}(x)}{n_k}-\frac{\log q_{m_k}(x)}{m_k}\\
\ge\;& \frac{\log q_{m_k}(x)+\ell_k\log(B+2)}{n_k}-\frac{\log q_{m_k}(x)}{n_k}(1+\frac{\ell_k}{m_k})\\
=\;& \frac{\ell_k\log(B+2)}{n_k}-\frac{\log q_{m_k}(x)}{n_k}\frac{\ell_k}{m_k}\\
\ge\;& \frac{\ell_k}{n_k}\log\frac{B+2}{B+1}.
\end{aligned}$$

Step II. By the choice of n_k, we have

$$\lim_{k\to\infty}\frac{\ell_1+\cdots+\ell_k}{n_k-(\ell_1+\cdots+\ell_k)-1}=\frac{\delta}{1-\delta}.$$

So, for $\epsilon=\frac{4\delta}{1-\delta}\log_2(2B)$, there exists k_0 such that for all $k\ge k_0$,

$$2^{\frac{1}{2}(n_k-(\ell_1+\cdots+\ell_k)-1)\epsilon}\ge (B+3)^{\ell_1+\cdots+\ell_k}. \tag{3.9}$$

Take $x_0\in\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B\big)$, let

$$\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B,x_0\big)=\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B\big)\cap I(a_1(x_0),\cdots,a_{n_{k_0}}(x_0)).$$

Before presenting the map, we give a notation first. For a sequence (or vector) $(a_1,a_2,\cdots)$, we denote by $(a_1,a_2,\cdots)^*$ for the sequence (or vector) by eliminating the terms $a_{m_k}+1,\cdots,a_{n_k}$ in the sequence $(a_1,a_2,\cdots)$ for all $k\ge 1$.

Define $f_\epsilon:\mathcal{D}\big(\{n_k\}_{k=1}^{\infty},\delta,B,x_0\big)\to E(B)\cap I(a_1,\cdots,a_{n_k})^*$,

$$x=[a_1,a_2,\cdots]\to y=[b_1,b_2,\cdots],$$

where $(b_1, b_2, \cdots) = (a_1, a_2, \cdots)^*$.

Step III. We claim that f_ϵ is $\frac{1}{1+\epsilon}$-Hölder. we note that if this is the case, by Lemma 2.3 and Lemma 2.4, it follows that

$$\dim_H \mathcal{D} \geq \frac{1}{1+\epsilon} \dim_H E(B).$$

Letting $\delta \to 0$ and then $B \to \infty$, we will get the desired result.

So, to finish the proof, it suffices to check the claim. For any $x_1, x_2 \in \mathcal{D}\big(\{n_k\}_{k=1}^\infty, \delta, B, x_0\big)$, let $y_1 = f_\epsilon(x_1), y_2 = f_\epsilon(x_2)$. We will estimate the difference between x_1 and x_2 and the difference between y_1 and y_2.

Let n be the smallest integer such that $a_{n+1}(x_1) \neq a_{n+1}(x_2)$ and $n_k \leq n < n_{k+1}$ with some $k \geq k_0$. Since

$$x_i \in I(a_1(x_i), \cdots, a_{n+1}(x_i), a_{n+2}(x_i)) \subset \bigcup_{1 \leq a_{n+2} \leq B+2} I(a_1(x_i), \cdots, a_{n+1}(x_i), a_{n+2})$$

the difference of x_1 and x_2 is larger than the gap between

$$\bigcup_{1 \leq a_{n+2} \leq B+2} I(a_1(x_i), \cdots, a_{n+1}(x_i), a_{n+2}), \ i = 1, 2.$$

Assume that $a_{n+1}(x_1) > a_{n+1}(x_2)$ and n is even (when n is odd, the estimation is same). Then

$$|x_1 - x_2| \geq \left| \frac{p_{n+1}(x_1)}{q_{n+1}(x_1)} - \frac{(B+3)p_{n+1}(x_1) + p_n(x_1)}{(B+3)q_{n+1}(x_1) + q_n(x_1)} \right| \geq \frac{1}{(2B)^4 q_n^2(x_1)}.$$

For the gap between y_1 and y_2, we will distinguish two cases.

(i) $n_k \leq n \leq m_{k+1}$. In this case, by the definition of f_ϵ, it follows that

$$b_j(y_1) = b_j(y_2), \text{ for } 1 \leq j \leq n - (\ell_1 + \cdots + \ell_k).$$

So, we have

$$|y_1 - y_2| \leq \frac{1}{q^2_{n-\ell_1+\cdots+\ell_k}\big(a_1(x_1), \cdots, a_n(x_1)\big)^*}.$$

By Proposition 2.1 and (3.9), we have

$$\begin{aligned} q_n(x_1) &\leq (B+3)^{\ell_1+\cdots+\ell_k} q_{n-\ell_1+\cdots+\ell_k}\big(a_1(x_1), \cdots, a_n(x_1)\big)^* \\ &\leq q^{1+\epsilon}_{n-\ell_1+\cdots+\ell_k}\big(a_1(x_1), \cdots, a_n(x_1)\big)^* \end{aligned}$$

Thus, it follows

$$|f_\epsilon(x_1) - f_\epsilon(x_2)| \leq (2B)^4 |x_1 - x_2|^{\frac{1}{1+\epsilon}}.$$

(ii). $m_{k+1} < n < n_{k+1}$. In this case,

$$b_j(y_1) = b_j(y_2), \text{ for } 1 \leq j \leq m_{k+1} - (\ell_1 + \cdots + \ell_k).$$

In this case, we have

$$|x_1 - x_2| \geq \frac{1}{(2B)^4 q^2_{n_{k+1}}(x_1)}, \quad |y_1 - y_2| \leq \frac{1}{q^2_{m_{k+1}-(\ell_1+\cdots+\ell_k)}(a_1(x_1), \cdots, a_{m_{k+1}}(x_1))}.$$

A similar estimation on $q_{n_{k+1}}$ gives that

$$|f_\epsilon(x_1) - f_\epsilon(x_2)| \leq (2B)^4 |x_1 - x_2|^{\frac{1}{1+\epsilon}}.$$

This completes the proof. □

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