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Characterizations of slant helices in Euclidean 3-space

L. Kula, N. Ekmekci, Y. Yaylı and K. İlarslan

Abstract

In this paper we investigate the relations between a general helix and a slant helix. Moreover, we obtain some differential equations which they are characterizations for a space curve to be a slant helix. Also, we obtain the slant helix equations and its Frenet aparatus.

Key Words: Slant helix, genaral helix, spherical helix, tangent indicatrix, principal normal indicatrix and binormal indicatrix.

1. Introduction

In differential geometry, a curve of constant slope or general helix in Euclidean 3-space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 (see [11, 13] for details) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of $\varkappa$ and τ are non-zero constant it is, of course, a general helix. We call it a circular helix. Its known that straight line and circle are degenerate-helix examples ($\varkappa = 0$, if the curve is straight line and $\tau = 0$, if the curve is a circle).

The study of these curves in $\mathbb{R}^3$ as spherical curves is given by Monterde in [12]. The Lancret theorem was revisited and solved by Barros (in [2]) in 3-dimensional real space forms by using killing vector fields as along curves. Also in the same space-forms, a characterization of helices and Cornu spirals is given by Arroyo, Barros and Garay in [1].

On the studies of general helices in Lorentzian space forms, Lorentz-Minkowski spaces, semi-Riemannian manifolds, we refer to the papers [3, 4, 5, 6, 7, 9].

In [8], A slant helix in Euclidean space $\mathbb{R}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that γ is a slant helix in $\mathbb{R}^3$ if and only if the geodesic curvature of the principal normal of a space curve γ is a constant function.

In [10], Kula and Yayli have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix.

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In this paper we consider the relationship between the curves slant helices and general helices in $\mathbb{R}^3$. We obtain the differential equations which are characterizations of a slant helix. Also, we give some slant helix examples in Euclidean 3-space

2. Preliminaries

We now recall some basic concepts on classical differential geometry of space curves and the definitions of general helix, slant helix in Euclidean 3-space. A curve $\gamma : I \subset R \to \mathbb{R}^3$, with unit speed, is a general helix if there is some constant vector u, so that $t.u = \cos\theta$ is constant along the curve, where $t(s) = \gamma^{'}(s)$ is a unit tangent vector of γ at s. We define the curvature of γ by $\varkappa(s) = \left\|\gamma^{''}(s)\right\|$. If $\varkappa(s) \neq 0$, then the unit principal normal vector $n(s)$ of the curve γ at s is given by $\gamma^{''}(s) = \varkappa(s)\, n(s)$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at s. For the derivatives of the Frenet frame, the Frenet-Serret formulae hold:

$$\begin{aligned} t^{'}(s) &= \varkappa(s)\, n(s) \\ n^{'}(s) &= -\varkappa(s)\, t(s) + \tau(s)\, b(s) \\ b^{'}(s) &= -\tau(s)\, n(s), \end{aligned} \tag{2.1}$$

where $\tau(s)$ is the torsion of the curve γ at s. It his known that curve γ is a general helix if and only if $\left(\frac{\tau}{\varkappa}\right)(s)$ =constant. If both of $\varkappa(s) \neq 0$ and $\tau(s)$ are constant, we call as a circular helix.

Definition 2.1. *Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors t , n and b. The unit tangent vectors along the curve α generate a curve (t) on the sphere of radius 1 about the origin. The curve (t) is called the spherical indicatrix of t or more commonly, (t) is called tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $(t) = t(s)$ will be a representation of (t). Similarly one considers the principal normal indicatrix $(n) = n(s)$ and binormal indicatrix $(b) = b(s)$ [13].*

Definition 2.2. *A curve γ with $\varkappa(s) \neq 0$ is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix (n) of γ*

$$\sigma_n(s) = \left(\frac{\varkappa^2}{(\varkappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)^{'}\right)(s) \tag{2.2}$$

is a constant function [8].

In this paper, by D we denote the covariant differentiation of $\mathbb{R}^3$.

Remark 2.1 If the Frenet frame of the tangent indicatrix (t) of a space curve γ is $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathcal{T}}\mathcal{T} &= \varkappa_t\mathcal{N} \\ D_{\mathcal{T}}\mathcal{N} &= -\varkappa_t\mathcal{T} + \tau_t\mathcal{B} \\ D_{\mathcal{T}}\mathcal{B} &= -\tau_t\mathcal{N}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathcal{T} &= n \\ \mathcal{N} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}(-\varkappa t + \tau b) \\ \mathcal{B} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}(\tau t + \varkappa b) \end{aligned} \tag{2.4}$$

and $\varkappa_t = \frac{\sqrt{\varkappa^2+\tau^2}}{\varkappa}$ is the curvature of (t), $\tau_t = \frac{\varkappa\tau' - \varkappa'\tau}{\varkappa(\varkappa^2+\tau^2)}$ is the torsion of (t).

Remark 2.2. If the Frenet frame of the principal normal indicatrix (n) of a space curve γ is $\{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathsf{T}}\mathsf{T} &= \varkappa_n\mathsf{N} \\ D_{\mathsf{T}}\mathsf{N} &= -\varkappa_n\mathsf{T} + \tau_n\mathsf{B} \\ D_{\mathsf{T}}\mathsf{B} &= -\tau_n\mathsf{N}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \mathsf{T} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}(-\varkappa t + \tau b) \\ \mathsf{N} &= \frac{1}{\sqrt{(\varkappa^2+\tau^2)\left(\varkappa\tau' - \varkappa'\tau\right)^2 + (\varkappa^2+\tau^2)^4}}[(\varkappa\tau' - \varkappa'\tau)(\tau t + \varkappa b) - (\varkappa^2+\tau^2)^2 n] \\ \mathsf{B} &= \frac{1}{\sqrt{\left(\varkappa\tau' - \varkappa'\tau\right)^2 + (\varkappa^2+\tau^2)^3}}[(\varkappa^2+\tau^2)(\tau t + \varkappa b) + (\varkappa\tau' - \varkappa'\tau)n], \end{aligned} \tag{2.6}$$

the curvature of (n) is

$$\varkappa_n = \frac{\sqrt{\left(\varkappa^2+\tau^2\right)^3 + \left(\varkappa\tau' - \varkappa'\tau\right)^2}}{\left(\varkappa^2+\tau^2\right)^{3/2}},$$

and the torsion of (n) is

$$\tau_n = \frac{\left[\left(\varkappa\tau'' - \varkappa''\tau\right)\left(\varkappa^2+\tau^2\right) - 3\left(\varkappa\tau' - \varkappa'\tau\right)\left(\varkappa\varkappa' - \tau'\tau\right)\right]}{\left(\varkappa^2+\tau^2\right)^3 + \left(\varkappa\tau' - \varkappa'\tau\right)^2}.$$

Remark 2.3. If the Frenet frame of the binormal indicatrix (b) of a space curve γ is $\{\mathbb{T}, \mathbb{N}, \mathbb{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathbb{T}}\mathbb{T} &= \varkappa_b\mathbb{N} \\ D_{\mathbb{T}}\mathbb{N} &= -\varkappa_b\mathbb{T} + \tau_b\mathbb{B} \\ D_{\mathbb{T}}\mathbb{B} &= -\tau_b\mathbb{N}, \end{aligned} \tag{2.7}$$

where

$$\begin{array}{l} \mathbb{T} = -n \\ \mathbb{N} = \frac{1}{\sqrt{\varkappa^2+\tau^2}} \left(\varkappa t - \tau b\right) \\ \mathbb{B} = \frac{1}{\sqrt{\varkappa^2+\tau^2}} \left(\tau t + \varkappa b\right) \end{array} \tag{2.8}$$

and $\varkappa_b = \frac{\sqrt{\varkappa^2+\tau^2}}{\tau}$ is the curvature of (b), $\tau_b = \frac{-\left(\varkappa\tau^{'} - \varkappa^{'}\tau\right)}{\tau(\varkappa^2+\tau^2)}$ is the torsion of (b).

3. Characterizations of slant helices

In this section, we give some characterizations for a unıt speed curve γ in $\mathbb{R}^3$ to be a slant helix by using its *tangent indicatrix* (t), *principal normal indicatrix* (n) and *binormal indicatrix* (b), respectively.

Theorem 3.1. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. γ is a slant helix if and only if the principal normal vector field N of the principal normal indicatrix (n) satisfies the equation*

$$D_{\mathsf{T}}^2\mathsf{N}+\varkappa_n^2\mathsf{N}= 0, \tag{3.1}$$

where $\varkappa_n$ is curvature of the principal normal indicatrix (n) of the curve γ.

Proof. Suppose that γ is a slant helix. From remark 2.2. the curvature of (n) is

$$\varkappa_n = \sqrt{1+\sigma_n^2\left(s\right)} \tag{3.2}$$

and the torsion of (n) is

$$\tau_n = \frac{\left(\varkappa^2+\tau^2\right)^{5/2}}{\left(\varkappa\tau^{'} - \varkappa^{'}\tau\right)^2 + \left(\varkappa^2+\tau^2\right)^3}\sigma_N^{'}\left(s\right). \tag{3.3}$$

Since, $\sigma_n\left(s\right)$ is a constant function, we get

$$\varkappa_n = \text{non-zero constant, and } \ \tau_n = 0.$$

Hence the principal normal indicatrix of γ is a circle. From frame equations (2.5), we obtain that

$$D_{\mathsf{T}}^2\mathsf{N} + \varkappa_n^2\mathsf{N} = 0.$$

Conversely, let us assume that (3.1) holds. We show that the curve γ is a slant helix. From frame equations (2.5)

$$D_{\mathsf{T}}^2\mathsf{N} + \varkappa_n^2\mathsf{N} = -\varkappa_n^{'}\mathsf{T} - \tau_n^2\mathsf{N} + \tau_n^{'}\mathsf{B} = 0. \tag{3.4}$$

Then we see that

$$\varkappa_n \text{ is a constant and } \tau_n = 0,$$

which means that γ is a slant helix. □

In the next six theorems, we obtain the differential equations of a slant helix according to the *tangent vector field* $\mathcal{T}$, *principal normal vector field* $\mathcal{N}$ and *binormal vector field* $\mathcal{B}$ *of the principal normal indicatrix* (t) of the curve.

Theorem 3.2. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$ The curve γ is a slant helix if and only if the tangent vector field $\mathcal{T}$ of the tangent indicatrix (t) of the curve γ satisfies the following equation:*

$$D_{\mathcal{T}}^3\mathcal{T} - 3\frac{\varkappa_t'}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{T} - \left\{\frac{\varkappa_t''}{\varkappa_t} - 3\left(\frac{\varkappa_t'}{\varkappa_t}\right)^2 - \lambda_1\varkappa_t^2\right\}D_{\mathcal{T}}\mathcal{T} = 0, \tag{3.5}$$

where $\lambda_1 \in \mathbb{R}^+$ ($\lambda_1 = 1 + \frac{1}{c_1^2}$ and $c_1 \in \mathbb{R}_0$) and $\varkappa_t$, is curvatures of the tangent indicatrix (t) of the curve γ.

Proof. Suppose that γ is a slant helix. Thus the tangent indicatrix (t) of γ is a general helix. From (2.3), we have $D_{\mathcal{T}}\mathcal{T} = \varkappa_t\mathcal{N}$. By differentiating $D_{\mathcal{T}}\mathcal{T} = \varkappa_t\mathcal{N}$, we get

$$D_{\mathcal{T}}^3\mathcal{T} = -2\varkappa_t\varkappa_t'\mathcal{T} - \varkappa_t^2 D_{\mathcal{T}}\mathcal{T} + \varkappa_t''\mathcal{N} + \varkappa_t' D_{\mathcal{T}}\mathcal{N} + 2\varkappa_t'\tau_t\mathcal{B} + \varkappa_t\tau_t D_{\mathcal{T}}\mathcal{B}. \tag{3.6}$$

By using the frame equations in (2.3), we get (3.5).

Conversely let us assume that (3.5) holds. From (2.3), we have

$$\mathcal{B} = \frac{1}{\tau_t}D_{\mathcal{T}}\mathcal{N} + \frac{\varkappa_t}{\tau_t}\mathcal{T}. \tag{3.7}$$

Differentiating the last equality, we have

$$\begin{aligned} D_{\mathcal{T}}\mathcal{B} = {} & \tfrac{1}{\varkappa_t\tau_t}\left\{D_{\mathcal{T}}^3\mathcal{T} - 3\tfrac{\varkappa_t'}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{T} - \left[\tfrac{\varkappa_t''}{\varkappa_t} - 3\left(\tfrac{\varkappa_t'}{\varkappa_t}\right)^2 - \varkappa_t^2 - \tau_t^2\right]D_{\mathcal{T}}\mathcal{T}\right\} \\ & + \tfrac{1}{\varkappa_t^2}\left(\tfrac{\varkappa_t}{\tau_t}\right)' D_{\mathcal{T}}^2\mathcal{T} - \left(\tfrac{\tau_t}{\varkappa_t} + \tfrac{\varkappa_t'}{\varkappa_t^3}\left(\tfrac{\varkappa_t}{\tau_t}\right)'\right)D_{\mathcal{T}}\mathcal{T} + \left(\tfrac{\varkappa_t}{\tau_t}\right)'\mathcal{T}. \end{aligned} \tag{3.8}$$

Using equations (2.3) and (3.5), we get

$$\left(\frac{\varkappa_t}{\tau_t}\right)' = 0 \text{ and } \frac{\varkappa_t}{\tau_t} = \sqrt{\frac{1}{\lambda_1 - 1}} = c_1(\text{non-zero constant}).$$

Thus, from (2.2), we obtain $\sigma_n = \frac{\tau_t}{\varkappa_t}$ =constant which means that γ is a slant helix. □

By using the properties of general helix, we restate the theorem 3.2 according to the τ_t torsion of the tangent indicatrix (t) of the curve γ as follows.

Theorem 3.3. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ is a slant helix if and only if the tangent vector field $\mathcal{T}$ of the tangent indicatrix (t) of the curve γ satisfies the equation*

$$D_{\mathcal{T}}^3\mathcal{T}-3\frac{\tau_t^{'}}{\tau_t}D_{\mathcal{T}}^2\mathcal{T}-\left\{\frac{\tau_t^{''}}{\tau_t}-3\left(\frac{\tau_t^{'}}{\tau_t}\right)^2-\mu_1\tau_t^2\right\}D_{\mathcal{T}}\mathcal{T}=0, \tag{3.9}$$

where $\mu_1 \in \mathbb{R}^+$ ($\mu_1=1+c_1^2$ and $c_1\in\mathbb{R}_0$) and τ_t, is torsion of the tangent indicatrix (t) of the curve γ.

Theorem 3.4. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ is a slant helix if and only if the principal normal vector field $\mathcal{N}$ of the tangent indicatrix (t) of the curve γ satisfies the equation*

$$D_{\mathcal{T}}^2\mathcal{N}-\frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}\mathcal{N}+\lambda_1\varkappa_t^2\mathcal{N}=0, \tag{3.10}$$

where $\lambda_1 \in \mathbb{R}^+$ ($\lambda_1=1+\frac{1}{c_1^2}$ and $c_1\in\mathbb{R}_0$) and $\varkappa_t$, is curvatures of the tangent indicatrix (t) of the curve γ.

Proof. Suppose that γ is a slant helix. Thus the tangent indicatrix (t) of γ is a general helix. By differentiating $D_{\mathcal{T}}\mathcal{N}=-\varkappa_t\mathcal{T}+\tau_t\mathcal{B}$, we get

$$D_{\mathcal{T}}^2\mathcal{N}=-\varkappa_t^{'}\mathcal{T}+\tau_t^{'}\mathcal{B}-\left(\varkappa_t^2+\tau_t^2\right)\mathcal{N}. \tag{3.11}$$

By using the frame equations in (2.3), equation (3.11) is reduced to (3.10).

Conversely, suppose that (3.10) holds. From (2.3), we have

$$\mathcal{T}=-\frac{1}{\varkappa_t}D_{\mathcal{T}}\mathcal{N}+\frac{\tau_t}{\varkappa_t}\mathcal{B}. \tag{3.12}$$

By differentiating equation (3.12), we get

$$D_{\mathcal{T}}\mathcal{T}=-\frac{1}{\varkappa_t}\left[D_{\mathcal{T}}^2\mathcal{N}-\frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}\mathcal{N}+\left(\varkappa_t^2+\tau_t^2\right)\mathcal{N}\right]+\varkappa_t\mathcal{N}+\left(\frac{\tau_t}{\varkappa_t}\right)^{'}\mathcal{B}. \tag{3.13}$$

Using equations (2.3) and (3.9), we get

$$\left(\frac{\tau_t}{\varkappa_t}\right)^{'}=0 \text{ and } \frac{\varkappa_t}{\tau_t}=\sqrt{\frac{1}{\lambda-1}}=c(\text{non-zero constant}).$$

Thus, from (2.2), we obtain $\sigma_n=\frac{\tau_t}{\varkappa_t}=$constant, which means that γ is a slant helix. This completes the proof. □

By using the properties of general helix, we restate the theorem 3.4 according to the τ_t torsion of the tangent indicatrix (t) of the curve γ as follows.

Theorem 3.5. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ is a slant helix if and only if the principal normal vector field $\mathcal{N}$ of the tangent indicatrix (t) of the curve γ satisfies the equation*

$$D_{\mathcal{T}}^2\mathcal{N}-\frac{\tau_t^{'}}{\tau_t}D_{\mathcal{T}}\mathcal{N}+\mu_1\tau_t^2\mathcal{N}=0, \tag{3.14}$$

where $\mu_1 \in \mathbb{R}^+$ ($\mu_1 = 1 + c_1^2$ and $c_1 \in \mathbb{R}_0$) and τ_t, is torsion of the tangent indicatrix (t) of the curve γ.

We omit the proofs of the following theorems, since they are analogous to the proofs of the above theorems.

Theorem 3.6. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ is a slant helix if and only if the binormal vector field $\mathcal{B}$ of the principal normal indicatrix (t) of the curve γ satisfies the equation*

$$D_{\mathcal{T}}^3\mathcal{B}-3\frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{B}-\left\{\frac{\varkappa_t^{''}}{\varkappa_t}-3\left(\frac{\varkappa_t^{'}}{\varkappa_t}\right)^2-\lambda_1\varkappa_t^2\right\}D_{\mathcal{T}}\mathcal{B}=0, \tag{3.15}$$

where $\lambda_1 \in \mathbb{R}^+$ ($\lambda_1 = 1+\frac{1}{c_1^2}$ and $c_1 \in \mathbb{R}_0$) and $\varkappa_t$, is curvatures of the tangent indicatrix (t) of the curve γ.

Theorem 3.7. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$ The curve γ is a slant helix if and only if the binormal vector field $\mathcal{B}$ of the principal normal indicatrix (t) of the curve γ satisfies the equation*

$$D_{\mathcal{T}}^3\mathcal{B}-3\frac{\tau_t^{'}}{\tau_t}D_{\mathcal{T}}^2\mathcal{B}-\left\{\frac{\tau_t^{''}}{\tau_t}-3\left(\frac{\tau_t^{'}}{\tau_t}\right)^2-\mu_1\tau_t^2\right\}D_{\mathcal{T}}\mathcal{B}=0, \tag{3.16}$$

where $\mu_1 \in \mathbb{R}^+$ ($\mu_1 = 1+c_1^2$ and $c_1 \in \mathbb{R}_0$) and τ_t, is torsion of the tangent indicatrix (t) of the curve γ.

In the next six theorems, we obtain the differential equations of a slant helix according to the *tangent vector field* $\mathbb{T}$, *principal normal vector field* $\mathbb{N}$ and *binormal vector field* $\mathbb{B}$ *of the binormal indicatrix* (b) of the curve.

Theorem 3.8. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ is a slant helix if and only if the tangent vector field $\mathbb{T}$ of the binormal indicatrix (b) of the curve γ satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{T}-3\frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}^2\mathbb{T}-\left\{\frac{\varkappa_b^{''}}{\varkappa_b}-3\left(\frac{\varkappa_b^{'}}{\varkappa_b}\right)^2-\lambda_2\varkappa_b^2\right\}D_{\mathbb{T}}\mathbb{T}=0, \tag{3.17}$$

where $\lambda_2 \in \mathbb{R}^+$ *(* $\lambda_2 = 1 + \frac{1}{c_2^2}$ *and* $c_2 \in \mathbb{R}_0$ *) and* $\varkappa_b$, *is curvatures of the binormal indicatrix* (b) *of the curve* γ.

Proof. Suppose that γ is a slant helix. Hence the binormal indicatrix (b) of γ is a general helix. By differentiating $D_{\mathbb{T}}\mathbb{T} = \varkappa_b \mathbb{N}$, we get

$$D_{\mathbb{T}}^3\mathbb{T} = -2\varkappa_b\varkappa_b^{'}\mathbb{T} - \varkappa_b^2 D_{\mathbb{T}}\mathbb{T} + \varkappa_b^{''}\mathbb{N} + \varkappa_b^{'} D_{\mathbb{T}}\mathbb{N} + 2\varkappa_b^{'}\tau_b\mathbb{B} + \varkappa_b\tau_b D_{\mathbb{T}}\mathbb{B}. \tag{3.18}$$

By using the frame equations in (2.7), we get (3.17).

Conversely let us assume that (3.17) holds. From (2.7), we have

$$\mathbb{B} = \frac{1}{\tau_b} D_{\mathbb{T}}\mathbb{N} + \frac{\varkappa_b}{\tau_b}\mathbb{T}. \tag{3.19}$$

By differentiating equation (3.19),

$$\begin{aligned} D_{\mathbb{T}}\mathbb{B} = \tfrac{1}{\varkappa_b\tau_b}\left\{ D_{\mathbb{T}}^3\mathbb{T} - \tfrac{3\tau_b^{'}}{\tau_b} D_{\mathbb{T}}^2\mathbb{T} - \left[\tfrac{\varkappa_b^{''}}{\varkappa_b} - \tfrac{3\left(\tau_b^{'}\right)^2}{\tau_b^2} - \varkappa_b^2 - \tau_b^2 \right] D_{\mathbb{T}}\mathbb{T} \right\} \\ + \tfrac{1}{\varkappa_b^2}\left(\tfrac{\varkappa_b}{\tau_b}\right)^{'} D_{\mathbb{T}}^2\mathbb{T} - \left(\tfrac{\tau_b}{\varkappa_b} + \tfrac{\varkappa_b^{'}}{\varkappa_b^3}\left(\tfrac{\varkappa_b}{\tau_b}\right)^{'} \right) D_{\mathbb{T}}\mathbb{T} + \left(\tfrac{\varkappa_b}{\tau_b}\right)^{'}\mathbb{T}. \end{aligned} \tag{3.20}$$

Using equations (2.7) and (3.17), we get

$$\left(\frac{\varkappa_b}{\tau_b}\right)^{'} = 0 \text{ and } \frac{\varkappa_b}{\tau_b} = \sqrt{\frac{1}{\lambda_2 - 1}} = c_2(\text{non-zero constant})$$

Since $\sigma_n = -\frac{\tau_b}{\varkappa_b}$, γ is a slant helix. Thus the proof of theorem 3.8 is completed. □

Theorem 3.9. *Let* γ *be a unit speed curve with Frenet vectors* t, n, b *and with non-zero curvatures* $\varkappa$ *and* τ *in* $\mathbb{R}^3$ *The curve* γ *is a slant helix if and only if the tangent vector field* $\mathbb{T}$ *of the binormal indicatrix* (b) *of the curve* γ *satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{T} - 3\frac{\tau_b^{'}}{\tau_b} D_{\mathbb{T}}^2\mathbb{T} - \left\{ \frac{\tau_b^{''}}{\tau_b} - 3\left(\frac{\tau_b^{'}}{\tau_b}\right)^2 - \mu_2\tau_b^2 \right\} D_{\mathbb{T}}\mathbb{T} = 0, \tag{3.21}$$

where $\mu_2 \in \mathbb{R}^+$ *(* $\mu_2 = 1 + c_2^2$ *and* $c_2 \in \mathbb{R}_0$ *) and* τ_b, *is torsion of binormal indicatrix* (b) *of the curve* γ.

Theorem 3.10. *Let* γ *be a unit speed curve with Frenet vectors* t, n, b *and with non-zero curvatures* $\varkappa$ *and* τ *in* $\mathbb{R}^3$ *The curve* γ *is a slant helix if and only if the principal normal vector field* $\mathbb{N}$ *of the binormal indicatrix* (b) *satisfies the equation*

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\tau_b^{'}}{\tau_b} D_{\mathbb{T}}\mathbb{N} + \mu_2\tau_b^2\mathbb{N} = 0, \tag{3.22}$$

where $\mu_2 \in \mathbb{R}^+$ *(* $\mu_2 = 1 + c_2^2$ *and* $c_2 \in \mathbb{R}_0$ *) and* τ_b, *is torsion of binormal indicatrix* (b) *of the curve* γ.

Proof. Suppose that γ is a slant helix. Therefore the tangent indicatrix (b) of γ is a general helix. By differentiating $D_{\mathbb{T}}\mathbb{N} = -\varkappa_b\mathbb{T} + \tau_b\mathbb{B}$, we get

$$D_{\mathbb{T}}^2\mathbb{N} = -\varkappa_b^{'}\mathbb{T} + \tau_b^{'}\mathbb{B} - \left(\varkappa_b^2 + \tau_b^2\right)\mathbb{N}. \tag{3.23}$$

By using the frame equations in (1.7), equation (2.21) is reduced to (2.20).

Conversely suppose that (2.20) holds. From (1.7), we have

$$\mathbb{T} = -\frac{1}{\varkappa_b}D_{\mathbb{T}}\mathbb{N} + \frac{\tau_b}{\varkappa_b}\mathbb{B}. \tag{3.24}$$

Differentiating the last equality, we have

$$D_{\mathbb{T}}\mathbb{T} = -\frac{1}{\varkappa_b}\left[D_{\mathbb{T}}^2\mathbb{N} - \frac{\tau_b^{'}}{\tau_b}D_{\mathbb{T}}\mathbb{N} + \left(\varkappa_b^2 + \tau_b^2\right)\mathbb{N}\right] + \varkappa_b\mathbb{N} + \left(\frac{\tau_b}{\varkappa_b}\right)^{'}\mathbb{B}.$$

Using equations (1.7) and (2.20), we get

$$\left(\frac{\tau_b}{\varkappa_b}\right)^{'} = 0 \text{ and } \frac{\tau_b}{\varkappa_b} = \sqrt{\mu_2 - 1} = c_2(\text{non-zero constant})$$

Since $\sigma_n = -\frac{\tau_b}{\varkappa_b}$, γ is a slant helix. This completes the proof of the theorem. □

Theorem 3.11. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$ The curve γ is a slant helix if and only if the principal normal vector field $\mathbb{N}$ of the binormal indicatrix (b) of the curve γ satisfies the equation*

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}\mathbb{N} + \lambda_2\varkappa_b^2\mathbb{N} = 0, \tag{3.25}$$

where $\lambda_2 \in \mathbb{R}^+$ ($\lambda_2 = 1 + \frac{1}{c_2^2}$ and $c_2 \in \mathbb{R}_0$) and $\varkappa_b$, is curvatures of the binormal indicatrix (b) of the curve γ.

With the similar proof, we have the following theorems.

Theorem 3.12. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ* is a slant helix if and only if the *the binormal vector field $\mathbb{B}$ of the binormal indicatrix (b) of the curve γ satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{B} - 3\frac{\tau_b^{'}}{\tau_b}D_{\mathbb{T}}^2\mathbb{B} - \left\{\frac{\tau_b^{''}}{\tau_b} - 3\left(\frac{\tau_b^{'}}{\tau_b}\right)^2 - \mu_2\tau_b^2\right\}D_{\mathbb{T}}\mathbb{B} = 0, \tag{3.26}$$

where $\mu_2 \in \mathbb{R}^+$ ($\mu_2 = 1 + c_2^2$ and $c_2 \in \mathbb{R}_0$) and τ_b, is torsion of binormal indicatrix (b) of the curve γ.

Theorem 3.13. *Let γ be a unit speed curve with Frenet vectors t, n, b and with non-zero curvatures $\varkappa$ and τ in $\mathbb{R}^3$. The curve γ* is a slant helix if and only if the *the binormal vector field $\mathbb{B}$ of the binormal indicatrix (b) satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{B} - 3\frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}^2\mathbb{B} - \left\{\frac{\varkappa_b^{''}}{\varkappa_b} - 3\left(\frac{\varkappa_b^{'}}{\varkappa_b}\right)^2 - \lambda_2\varkappa_b^2\right\}D_{\mathbb{T}}\mathbb{B} = 0, \tag{3.27}$$

where $\lambda_2 \in \mathbb{R}^+$ ($\lambda_2 = 1 + \frac{1}{c_2^2}$ and $c_2 \in \mathbb{R}_0$) and $\varkappa_b$, is curvatures of the binormal indicatrix (b) of the curve γ.

Example.

In [10], Kula and Yayli showed that the tangent indicatrix of a slant helix in Euclidean 3-space is a spherical general helix. The general equation of spherical helix obtained by Monterde in [11] as follows:

$$\begin{aligned} \beta_c(s) = \quad & (\cos s\cos(\omega s) + \tfrac{1}{\omega}\sin s\sin(\omega s), \\ & -\cos s\sin(\omega s) + \tfrac{1}{\omega}\sin s\cos(\omega s), \tfrac{1}{c\,\omega}\sin s), \end{aligned} \tag{3.28}$$

where $\omega = \frac{\sqrt{1+c^2}}{c}$ and $c \in \mathbb{R}_0$. Now we can easily obtained the general equation of a slant helix in Euclidean 3-space. Let α be a unit speed slant helix, then we have $\frac{d\alpha}{ds} = T = \beta_c(s)$. Thus by one integration we can easily obtained the family of slant helix according to the non-zero constant c as follows. If we denote the family of slant helix by α_c, then

$$\begin{aligned} \alpha_c(s) = \quad & (\tfrac{w+1}{2w(1-w)}\sin[(1-w)s] + \tfrac{w-1}{2w(1+w)}\sin[(1+w)s], \\ & \tfrac{w+1}{2w(1-w)}\cos[(1-w)s] + \tfrac{w-1}{2w(1+w)}\cos[(1+w)s], \tfrac{1}{wc}\cos s), \end{aligned}$$

where $\omega = \frac{\sqrt{1+c^2}}{c}$ and $c \in \mathbb{R}_0$. Also it is easily show that the slant helix fully lies in the hyperboloid of one sheet with equation

$$\frac{x^2}{4c^4} + \frac{y^2}{4c^4} - \frac{z^2}{4c^2} = 1.$$

Now we give an example of slant helices in Euclidean 3-space and draw pictures of tangent indicatricies, normal indicatricies of the family of slant helix for $c = \pm\frac{1}{4}, \pm 1, \pm 4, \pm 6$.

(i) For $c = \pm\frac{1}{4}, \pm 1, \pm 4, \pm 6$, normal indicatricies of the family of slant helix lie on the unit sphere which is rendered in Figure 1.

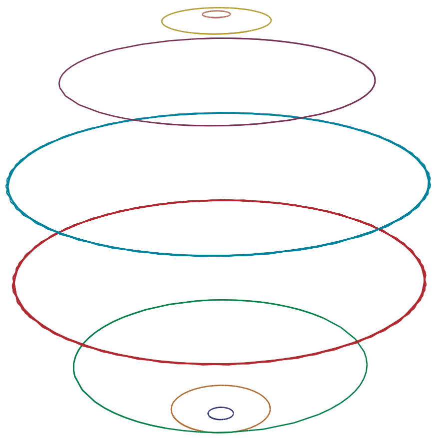


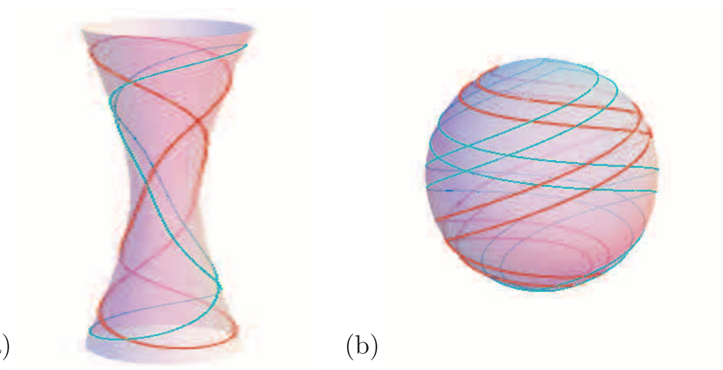
Figure 1. Normal indicatricies of the family of slant helix for $c = \pm\frac{1}{4}, \pm 1, \pm 4, \pm 6$.

(ii) For $c = \pm\frac{1}{4}$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 2.

(iii) For $c = \pm 1$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 3.

(iv) For $c = \pm 4$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 4.

(v) For $c = \pm 6$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 5.



(a) (b)

Figure 2. The slant helices for $c = \pm\frac{1}{4}$ (a) and tangent indicatricies of the slant helices for $c = \pm\frac{1}{4}$ (b)

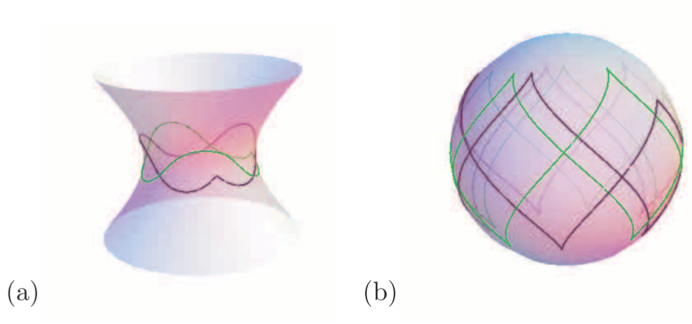


Figure 3. The slant helices for $c = \pm 1$ (a) and tangent indicatricies of the slant helices for $c = \pm 1$ (b)

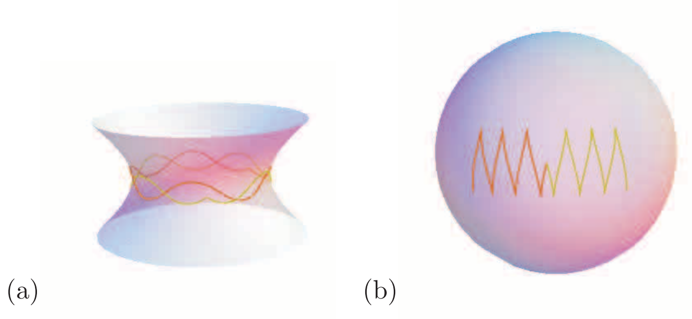


Figure 4. The slant helices for $c = \pm 4$ (a) and tangent indicatricies of the slant helices for $c = \pm 4$ (b)

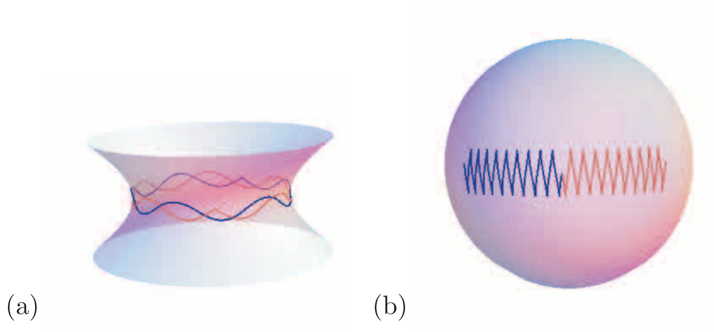


Figure 5. The slant helices for $c = \pm 6$ (a) and tangent indicatricies of the slant helices for $c = \pm 6$ (b)

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L. KULA
Ahi Evran University, Faculty of Sciences and Arts
Department of Mathematics, Kirsehir-TURKEY
e-mail: kula@science.ankara.edu.tr

N. EKMEKCİ, Y. YAYLI
Ankara University, Faculty of Science
Department of Mathematics,
06100, Tandogan, Ankara-TURKEY
e-mail: ekmekci@science.ankara.edu.tr
e-mail: yayli@science.ankara.edu.tr

K. İLARSLAN
Kırıkkale University, Faculty of Sciences
and Arts Department of Mathematics,
71450, Yahsihan, Kirikkale-TURKEY
e-mail: kilarslan@yahoo.com