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On purely real surfaces in Kaehler surfaces

Bang-Yen Chen

Abstract

An immersion $\phi\colon M \to \tilde{M}^2$ of a surface M into a Kaehler surface is called *purely real* if the complex structure J on $\tilde{M}^2$ carries the tangent bundle of M into a transversal bundle. In the first part of this article, we prove that the equation of Ricci is a consequence of the equations of Gauss and Codazzi for purely real surfaces in any Kaehler surface. In the second part, we obtain a necessary condition for a purely real surface in a complex space form to be minimal. Several applications of this condition are provided. In the last part, we establish a general optimal inequality for purely real surfaces in complex space forms. We also obtain three classification theorems for purely real surfaces in $\mathbf{C}^2$ which satisfy the equality case of the inequality.

Key word and phrases: Purely real surfaces; integrability condition; equation of Ricci; equation of Gauss-Codazzi; Kaehler surface; Wirtinger angle; optimal inequality.

1. Introduction

Let $\tilde{M}^2$ be a Kaehler surface; that means $\tilde{M}^2$ is endowed with a complex structure J and a Riemannian metric $\tilde{g}$ which is J-Hermitian. Thus, we have

$$\tilde{g}(JX, JY) = \tilde{g}(X, Y), \ \ \forall X, Y \in T_pM^2, \tag{1.1}$$

$$\tilde{\nabla} J = 0 \tag{1.2}$$

for $p \in \tilde{M}^2$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$.

It is well-known that the curvature tensor $\tilde{R}$ of a Kaehler surface $\tilde{M}^2$ satisfies

$$\tilde{R}(X, Y; Z, W) = -\tilde{R}(Y, X; Z, W), \tag{1.3}$$

$$\tilde{R}(X, Y; Z, W) = \tilde{R}(Z, W; X, Y), \tag{1.4}$$

$$\tilde{R}(X, Y; JZ, W) = -\tilde{R}(X, Y; Z, JW), \tag{1.5}$$

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where $\tilde{R}(X,Y;Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W)$.

It is well-known that the three fundamental equations of Gauss, Codazzi and Ricci play fundamental roles in the theory of Riemannian submanifolds (see, for instance, [1, 5, 9]). Since the three equations provide conditions for local isometric embeddability, the three equations also play some important roles in physics; in particular in Kaluza-Klein's theory (see [10, 11]). For surfaces in Riemannian 4-manifolds, the three fundamental equations are independent in general.

In the first part of this article, we prove that the equation of Ricci is a consequence of the equations of Gauss and Codazzi for purely real surfaces in an arbitrary Kaehler surface. In the second part, we obtain a necessary condition in term of Wirtinger angle for a purely real surface in a complex space form to be minimal. Several immediate applications of this condition are provided. In the last part, we establish a general optimal inequality for purely real surfaces in complex space forms. We also obtain three classification theorems for purely real surfaces in $\mathbf{C}^2$ which satisfy the equality case of the inequality.

2. Basic formulas and fundamental equations

Let $(\tilde{M}^2, J, \tilde{g})$ be a Kaehler surface. Assume that M is a surface in $\tilde{M}^2$. Let g be the induced metric on M. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on g and $\tilde{g}$, respectively. And denote by R and $\tilde{R}$ the curvature tensor of M and $\tilde{M}^2$, respectively. Denote by $\langle\ ,\ \rangle$ the inner product associated with $\tilde{g}$ (or with g).

The formulas of Gauss and Weingarten are given respectively by (cf. [1, 5])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection.

The shape operator and the second fundamental form are related by

$$\tilde{g}(h(X,Y),\xi) = g(A_\xi X, Y) \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.4}$$

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.5}$$

$$\tilde{g}(R^D(X,Y)\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \tilde{g}([A_\xi, A_\eta]X, Y), \tag{2.6}$$

where X, Y, Z, W are vectors tangent to M, and $\bar{\nabla} h$ and R^D are defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.7}$$

$$R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}. \tag{2.8}$$

The mean curvature vector $\overrightarrow{H}$ and the squared mean curvature H^2 of the surface are defined respectively by

$$\overrightarrow{H} = \frac{1}{2}\mathrm{trace}\, h, \quad H^2 = \tilde{g}(\overrightarrow{H}, \overrightarrow{H}). \tag{2.9}$$

Let $\tilde{M}^2(4\varepsilon)$ denote a complex space form with constant holomorphic sectional curvature 4ε. The Riemann curvature tensor of $\tilde{M}^2(4\varepsilon)$ satisfies

$$\begin{aligned} \tilde{R}(X,Y;Z,W) =& \varepsilon\big\{ \langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle + \langle JX,W\rangle \langle JY,Z\rangle \\ & - \langle JX,Z\rangle \langle JY,W\rangle + 2\langle X,JY\rangle \langle JZ,W\rangle \big\}. \end{aligned} \tag{2.10}$$

The ellipse of curvature of a surface M in $\tilde{M}^2$ is the subset of the normal plane defined as

$$\{h(v,v) \in T_p^{\perp}M : |v| = 1, v \in T_pM,\, p \in M\}.$$

To see that it is an ellipse, we consider an arbitrary orthogonal tangent frame $\{e_1, e_2\}$. Put $h_{ij} = h(e_i, e_j), i, j = 1, 2$ and look at the formula

$$h(v,v) = \overrightarrow{H} + \frac{h_{11} - h_{22}}{2}\cos 2\theta + h_{12}\sin 2\theta, \quad v = \cos\theta e_1 + \sin\theta e_2. \tag{2.11}$$

As v goes once around the unit tangent circle, $h(v,v)$ goes twice around the ellipse. The ellipse of curvature could degenerate into a line segment or a point.

The center of the ellipse is $\overrightarrow{H}$. The ellipse of curvature is a circle if and only if the following two conditions hold:

$$|h_{11} - h_{22}|^2 = 4|h_{12}|^2, \quad \langle h_{11} - h_{22}, h_{12}\rangle = 0. \tag{2.12}$$

The property that the ellipse of curvature is a circle is a conformal invariant.

3. Basics on purely real surfaces

An immersion $\phi\colon M \to \tilde{M}^2$ of a surface M into a Kaehler surface is called *purely real* if the complex structure J on $\tilde{M}^2$ carries the tangent bundle of M into a transversal bundle (cf. [6], see also [12]). Obviously, every purely real surface admits no complex points.

A point p on a purely real surface M is called a *Lagrangian point* if J carries the tangent space T_pM into its normal space $T_p^{\perp}M$. A purely real surface is called *Lagrangian* if every point is a Lagrangian point.

For each tangent vector X of a purely real surface M, we put

$$JX = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and the normal components of JX.

For an oriented orthonormal frame $\{e_1, e_2\}$, it follows from (3.1) that

$$Pe_1 = (\cos\alpha)e_2, \quad Pe_2 = -(\cos\alpha)e_1 \tag{3.2}$$

for some function α. This function α is called the *Wirtinger angle.* The Wirtinger angle is independent of the choice of e_1, e_2 which preserves the orientation.

A purely real surface is called a *slant surface* if its Wirtinger angle is constant. The Wirtinger angle of a slant surface is called the slant angle (cf. [2]).

For a purely real surface M, if we put

$$e_3 = (\csc\alpha)Fe_1, \quad e_4 = (\csc\alpha)Fe_2, \tag{3.3}$$

then we may derive from (3.1)–(3.3) that

$$Je_1 = \cos\alpha e_2 + \sin\alpha e_3, \qquad Je_2 = -\cos\alpha e_1 + \sin\alpha e_4, \tag{3.4}$$

$$Je_3 = -\sin\alpha e_1 - \cos\alpha e_4, \quad Je_4 = -\sin\alpha e_2 + \cos\alpha e_3, \tag{3.5}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 1, \quad \langle e_3, e_4\rangle = 0. \tag{3.6}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an *adapted orthonormal frame* for M.

For an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we may put

$$\nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2, \tag{3.7}$$

$$D_X e_3 = \Phi(X)e_4, \quad D_X e_4 = -\Phi(X)e_3 \tag{3.8}$$

for some 1-forms ω and Φ. For the second fundamental form h of M, we put

$$h(e_i, e_j) = h^3_{ij}e_3 + h^4_{ij}e_4. \tag{3.9}$$

From (2.3) and (3.9) we have

$$A_{e_3}e_j = h^3_{1j}e_1 + h^3_{2j}e_2, \quad A_{e_4}e_j = h^4_{1j}e_1 + h^4_{2j}e_2. \tag{3.10}$$

We need the following lemma.

Lemma 3.1 *Let M be a purely real surface in a Kaehler surface. Then, with respect to an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we have*

$$e_1\alpha = h^4_{11} - h^3_{12}, \quad e_2\alpha = h^4_{12} - h^3_{22}, \tag{3.11}$$

$$\Phi_1 = \omega_1 - (h^3_{11} + h^4_{12})\cot\alpha, \quad \Phi_2 = \omega_2 - (h^3_{12} + h^4_{22})\cot\alpha, \tag{3.12}$$

where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$ for $j = 1, 2$.

Proof. This is done by straightforward computation using (1.2) and (3.4)–(3.10), as we did in the proof of Lemma 4.1 and Theorem 4.2 in [2, pages 29–31]. □

4. Dependence of Gauss-Codazzi and Ricci equations

In this section we prove the following general result for purely real surfaces.

Theorem 4.1 *The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any purely real surface in any Kaehler surface.*

Proof. Assume that $\phi : M \to \tilde{M}^2$ is a purely real isometric immersion of a surface M into a Kaehler surface $\tilde{M}^2$. We may assume that locally M is equipped with the isothermal metric

$$g = E^2(x,y)(dx^2 + dy^2) \tag{4.1}$$

for some positive function E. The Levi-Civita connection of g satisfies

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{E_x}{E}\frac{\partial}{\partial x} - \frac{E_y}{E}\frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \frac{E_y}{E}\frac{\partial}{\partial x} + \frac{E_x}{E}\frac{\partial}{\partial y}, \\ &\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{E_x}{E}\frac{\partial}{\partial x} + \frac{E_y}{E}\frac{\partial}{\partial y}. \end{aligned} \tag{4.2}$$

If we put

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{E}\frac{\partial}{\partial y}, \tag{4.3}$$

then $\{e_1, e_2\}$ is an orthonormal frame. From (4.2) and (4.3) we derive that

$$\nabla_{e_1} e_1 = -\frac{E_y}{E^2} e_2, \ \nabla_{e_2} e_1 = \frac{E_x}{E^2} e_2, \ \nabla_{e_1} e_2 = \frac{E_y}{E^2} e_1, \ \nabla_{e_2} e_2 = -\frac{E_x}{E^2} e_1. \tag{4.4}$$

It follows from (3.7) and (4.4) that

$$\omega(e_1) = -\frac{E_y}{E^2}, \quad \omega(e_2) = \frac{E_x}{E^2}. \tag{4.5}$$

For simplicity, let us put

$$h(e_1,e_1) = \beta e_3 + \gamma e_4, \ \ h(e_1,e_2) = \delta e_3 + \varphi e_4, \ \ h(e_2,e_2) = \lambda e_3 + \mu e_4. \tag{4.6}$$

In view of (3.6), and (4.6), equation (2.4) of Gauss can be expressed as

$$K = \tilde{K} + \beta\lambda + \gamma\mu - \delta^2 - \varphi^2, \tag{4.7}$$

where $\tilde{K}(p)$ is the sectional curvature of the ambient space $\tilde{M}^2$ with respect to the tangent plane of T_pM of M at $p \in M$. By applying Lemma 3.1 and (4.6), we have

$$e_1\alpha = \gamma - \delta, \ \ e_2\alpha = \varphi - \lambda, \tag{4.8}$$

$$\Phi_1 = -\frac{E_y}{E^2} - (\beta + \varphi)\cot\alpha, \ \ \Phi_2 = \frac{E_x}{E^2} - (\delta + \mu)\cot\alpha. \tag{4.9}$$

By using (3.8) and (4.9) we find

$$\begin{aligned} D_{e_1}e_3 &= -\left(\frac{E_y}{E^2}+(\beta+\varphi)\cot\alpha\right)e_4, \quad D_{e_2}e_3 = \left(\frac{E_x}{E^2}-(\delta+\mu)\cot\alpha\right)e_4, \\ D_{e_1}e_4 &= \left(\frac{E_y}{E^2}+(\beta+\varphi)\cot\alpha\right)e_3, \quad D_{e_2}e_4 = \left(-\frac{E_x}{E^2}+(\delta+\mu)\cot\alpha\right)e_3. \end{aligned} \tag{4.10}$$

So, it follows from (4.4), (4.6) and (4.10) that

$$\begin{aligned} (\bar\nabla_{e_1}h)(e_1,e_2) &= \left(\frac{\delta_x}{E}+\frac{(\lambda+\varphi-\beta)E_y}{E^2}+\varphi(\beta+\varphi)\cot\alpha\right)e_3 \\ &\quad+\left(\frac{\varphi_x}{E}+\frac{(\mu-\gamma-\delta)E_y}{E^2}-\delta(\beta+\varphi)\cot\alpha\right)e_4, \\ (\bar\nabla_{e_2}h)(e_1,e_1) &= \left(\frac{\beta_y}{E}-\frac{(\gamma+2\delta)E_x}{E^2}+\gamma(\delta+\mu)\cot\alpha\right)e_3 \\ &\quad+\left(\frac{\gamma_y}{E}+\frac{(\beta-2\varphi)E_x}{E^2}-\beta(\delta+\mu)\cot\alpha\right)e_4, \\ (\bar\nabla_{e_1}h)(e_2,e_2) &= \left(\frac{\lambda_x}{E}+\frac{(\mu-2\delta)E_y}{E^2}+\mu(\beta+\varphi)\cot\alpha\right)e_3 \\ &\quad+\left(\frac{\mu_x}{E}-\frac{(\lambda+2\varphi)E_y}{E^2}-\lambda(\beta+\varphi)\cot\alpha\right)e_4, \\ (\bar\nabla_{e_2}h)(e_1,e_2) &= \left(\frac{\delta_y}{E}+\frac{(\beta-\lambda-\varphi)E_x}{E^2}+\varphi(\delta+\mu)\cot\alpha\right)e_3 \\ &\quad+\left(\frac{\varphi_y}{E}+\frac{(\gamma+\delta-\mu)E_x}{E^2}-\delta(\delta+\mu)\cot\alpha\right)e_4. \end{aligned} \tag{4.11}$$

On the other hand, from (3.4) we also find

$$\begin{aligned} (\tilde R(e_2,e_1)e_1)^\perp &= \{(\csc\alpha)\tilde R(e_2,e_1;e_1,Je_1)-(\cot\alpha)\tilde K\}e_3 \\ &\quad+(\csc\alpha)\tilde R(e_2,e_1;e_1,Je_2)e_4, \\ (\tilde R(e_1,e_2)e_2)^\perp &= (\csc\alpha)\tilde R(e_1,e_2;e_2,Je_1)e_3 \\ &\quad+\{(\cot\alpha)\tilde K+(\csc\alpha)\tilde R(e_1,e_2;e_2,Je_2)\}e_4. \end{aligned} \tag{4.12}$$

By using (3.6), (4.3), (4.11) and (4.12), we find from the equation of Codazzi that

$$\begin{aligned} \beta_y-\delta_x &= E(\csc\alpha)\tilde R(e_2,e_1;e_1,Je_1)+\{\varphi(\beta+\varphi)-\gamma(\delta+\mu)\}E\cot\alpha \\ &\quad+\frac{1}{E}\{(\gamma+2\delta)E_x+(\lambda+\varphi-\beta)E_y\}-E(\cot\alpha)\tilde K, \\ \gamma_y-\varphi_x &= E(\csc\alpha)\tilde R(e_2,e_1;e_1,Je_2)+(\beta\mu-\delta\varphi)E\cot\alpha \\ &\quad+\frac{1}{E}\{(2\varphi-\beta)E_x+(\mu-\gamma-\delta)E_y\}, \end{aligned} \tag{4.13}$$

$$\lambda_x - \delta_y = E(\csc\alpha)\tilde{R}(e_1, e_2; e_2, Je_1) + (\delta\varphi - \beta\mu)E\cot\alpha$$
$$+ \frac{1}{E}\{(\beta - \varphi - \lambda)E_x + (2\delta - \mu)E_y\},$$
$$\mu_x - \varphi_y = E(\csc\alpha)\tilde{R}(e_1, e_2; e_2, Je_2) + \{\lambda(\beta + \varphi) - \delta(\delta + \mu)\}E\cot\alpha$$
$$+ \frac{1}{E}\{(\gamma + \delta - \mu)E_x + (2\varphi + \lambda)E_y\} + E(\cot\alpha)\tilde{K}.$$

Also, from (3.10), (4.3), (4.6) and (4.8) we have

$$A_{e_3} = \begin{pmatrix} \beta & \delta \\ \delta & \lambda \end{pmatrix}, \; A_{e_4} = \begin{pmatrix} \gamma & \varphi \\ \varphi & \mu \end{pmatrix}, \tag{4.14}$$

$$\alpha_x = (\gamma - \delta)E, \quad \alpha_y = (\varphi - \lambda)E. \tag{4.15}$$

By applying (1.3)–(1.5), (3.4), (4.9), and (4.14) we derive that

$$\tilde{R}(e_1, e_2; e_3, e_4) = \cot\alpha\csc\alpha(\tilde{R}(e_1, e_2; Je_1, e_1) + \tilde{R}(e_1, e_2; Je_2, e_2)) - (1 + 2\cot^2\alpha)\tilde{K}, \tag{4.16}$$

$$\langle [A_{e_3}, A_{e_4}]e_1, e_2\rangle = \varphi(\lambda - \beta) + \delta(\gamma - \mu). \tag{4.17}$$

From (2.8), (4.4) and (4.10) we find

$$\tilde{g}(R^D(e_1, e_2)e_3, e_4) = \frac{(\delta + \mu)\alpha_x - (\beta + \varphi)\alpha_y}{E\sin^2\alpha} - K + \{(\beta + \varphi)E_y - (\delta + \mu)E_x + E(\beta_y + \varphi_y - \delta_x - \mu_x)\}\frac{\cot\alpha}{E^2}. \tag{4.18}$$

By substituting (4.13) and (4.15) into (4.18) we obtain

$$\tilde{g}(R^D(e_1, e_2)e_3, e_4) = (\delta + \mu)(\gamma - \delta) + (\beta + \varphi)(\lambda - \varphi) - K + \csc\alpha\cot\alpha(\tilde{R}(e_1, e_2; Je_1, e_1) + \tilde{R}(e_1, e_2; Je_2, e_2)) - 2(\cot^2\alpha)\tilde{K}. \tag{4.19}$$

Thus, it follows from (2.6), (4.16), (4.17) and (4.19) that the equation of Ricci is

$$K = \tilde{K} + \beta\lambda + \gamma\mu - \delta^2 - \varphi^2, \tag{4.20}$$

which is exactly the equation (4.7) of Gauss. Therefore, the equation of Ricci is a consequence of Gauss and Codazzi. □

Remark 4.1 *If the purely real surface is slant, Theorem 4.1 is due to [8].*

5. Wirtinger angle of purely real surfaces

The following result provides a necessary condition for purely real surfaces in complex space forms to be minimal.

Theorem 5.1 *Let M be a purely real surface in a complex space form $\tilde{M}^2(4\varepsilon)$ of constant holomorphic sectional curvature 4ε. If M is minimal, then the Wirtinger angle α of M satisfies*

$$\Delta\alpha = \left\{||\nabla\alpha||^2 + 6\varepsilon \sin^2\alpha\right\}\cot\alpha, \tag{5.1}$$

where Δ is the Laplace operator of M and $\nabla\alpha$ is the gradient of α.

Proof. Let M be a purely real surface in $\tilde{M}^2(4\varepsilon)$. We follow the notations and definitions given in sections 3 and 4. If M is minimal, we get

$$\lambda = -\beta, \quad \gamma = -\mu. \tag{5.2}$$

So, it follows from (4.15) and (5.2) that

$$\delta = -\frac{\alpha_x}{E} - \mu, \quad \varphi = \frac{\alpha_y}{E} - \beta. \tag{5.3}$$

Moreover, it follows from (2.10) that

$$\tilde{R}(e_2, e_1; e_1, Je_1) = -\tilde{R}(e_1, e_2; e_2, Je_2) = 4\varepsilon\cos\alpha, \tag{5.4}$$

$$\tilde{K} = \varepsilon(1 + 3\cos^2\alpha). \tag{5.5}$$

Substituting (5.2)–(5.5) into the first equation in (4.13) gives

$$\begin{aligned} E^2(\mu_x + \beta_y) &= \alpha_y E_y - \alpha_x E_x - \alpha_{xx} E - 3E(\mu E_x + \beta E_y) \\ &\quad + \alpha_y^2 E\cot\alpha - (\alpha_x\mu + \alpha_y\beta)E^2\cot\alpha + 3\varepsilon E^3\sin\alpha\cos\alpha. \end{aligned} \tag{5.6}$$

Similarly, by substituting (5.2)–(5.5) into the last equation in (4.13) we find

$$\begin{aligned} E^2(\mu_x + \beta_y) &= \alpha_y E_y - \alpha_x E_x + \alpha_{yy} E - 3E(\mu E_x + \beta E_y) \\ &\quad - \alpha_x^2 E\cot\alpha - (\alpha_x\mu + \alpha_y\beta)E^2\cot\alpha - 3\varepsilon E^3\sin\alpha\cos\alpha. \end{aligned} \tag{5.7}$$

Subtracting (5.6) from (5.7) yields

$$\alpha_{xx} + \alpha_{yy} = \left\{\alpha_x^2 + \alpha_y^2 + 6\varepsilon E^2\sin^2\alpha\right\}\cot\alpha. \tag{5.8}$$

Since the Laplace operator Δ of M with respect to the metric (4.1) is given by

$$\Delta = \frac{1}{E^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

equation (5.8) can be simply expressed as (5.1). □

Some easy consequences of Theorem 5.1 are the following.

Corollary 5.1 *[7] Every slant surface in a complex space form $\tilde{M}^2(4\varepsilon)$ with $\varepsilon \neq 0$ is non-minimal unless it is either Lagrangian or complex.*

Proof. Follows immediately from Theorem 5.1. □

Corollary 5.2 *Every compact oriented minimal purely real surface in the complex projective plane $CP^2(4)$ contains some Lagrangian points.*

Proof. Let M be a compact oriented minimal purely real surface in $CP^2(4)$. Then we may choose e_1, e_2 to be an orthonormal frame which gives the orientation on M. So, the α is a global well-defined function on M. Thus, it follows from Theorem 5.1 and Hopf's lemma that $\int_M \cot \alpha dA = 0$, which implies that $\cot \alpha = 0$ holds at some points. Hence, M must admits some Lagrangian points. □

The next two corollaries follows easily from (5.1).

Corollary 5.3 *Let M be a purely real minimal surface in $\mathbf{C}^2$. If the Wirtinger angle α is a harmonic function, then M is slant.*

Corollary 5.4 *Let M be a purely real minimal surface in $CP^2(4)$. If the Wirtinger angle α is a harmonic function, then M is Lagrangian.*

A function f on (M, g) is called *subharmonic* if $\Delta f \geq 0$ holds everywhere on M. The surface M is called *parabolic* if there exists non non-constant negative subharmonic function.

Corollary 5.5 *Let M be an oriented minimal purely real surface in $CP^2(4)$. If M is parabolic, then M contains some Lagrangian points.*

Proof. Let M be an oriented minimal purely real surface in the complex projective plane $CP^2(4)$. Then we have

$$\Delta \alpha = \left\{ ||\nabla \alpha||^2 + 6 \sin^2 \alpha \right\} \cot \alpha. \tag{5.9}$$

If M is parabolic and it admits no Lagrangian points, then $\cot \alpha$ is either a positive function or a negative function on M.

When $\cot \alpha$ is a positive function, α is subharmonic by (5.9). Hence, α must be constant, which is impossible according to Corollary 5.1.

Similarly, when $\cot \alpha$ is a negative function, $-\alpha$ is subharmonic. This is also impossible by the same argument. □

6. A general optimal inequality for purely neal surfaces

We prove the following general optimal inequality for purely real surfaces.

Theorem 6.1 *Let M be a purely real surface in a complex space form $\tilde{M}^2(4\varepsilon)$. Then we have*

$$H^2 \geq 2\{K - ||\nabla\alpha||^2 - (1 + 3\cos^2\alpha)\varepsilon\} + 4\,\langle \nabla\alpha, Jh(e_1, e_2)\rangle \csc\alpha \tag{6.1}$$

with respect to an orthonormal frame $\{e_1, e_2\}$ satisfying $\langle \nabla\alpha, e_2\rangle = 0$, where H^2 and K are the squared mean curvature and the Gauss curvature of M, respectively.

The equality case of (6.1) holds at p if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operator at p take the forms

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}. \tag{6.2}$$

Proof. Assume that M is a purely real surface in $\tilde{M}^2$. Without loss of generality, we may choose an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the gradient of α is parallel to e_1 at p. So, we have $\nabla\alpha = (e_1\alpha)e_1$. As before, let us put

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \; h(e_1, e_2) = \delta e_3 + \varphi e_4, \;\; h(e_2, e_2) = \lambda e_3 + \mu e_4. \tag{6.3}$$

Then, in view of Lemma 3.1 we have

$$A_{e_3} = \begin{pmatrix} \beta & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & \mu \end{pmatrix}. \tag{6.4}$$

Thus, the squared mean curvature H^2 and the Gauss curvature K of M satisfy

$$4H^2 = (\beta + \varphi)^2 + (\delta + \mu + e_1\alpha)^2, \tag{6.5}$$

$$K = \beta\varphi + \delta\mu + \mu e_1\alpha - \delta^2 - \varphi^2 + (1 + 3\cos^2\alpha)\varepsilon. \tag{6.6}$$

Hence, we obtain

$$\begin{aligned} H^2 - 2K + 2||\nabla\alpha||^2 &= \frac{1}{4}\{(\beta - 3\varphi)^2 + (\mu - 3(\delta + e_1\alpha))^2\} - 4\delta e_1\alpha - 2(1 + 3\cos^2\alpha)\varepsilon \\ &\geq -4\delta e_1\alpha - 2(1 + 3\cos^2\alpha)\varepsilon. \end{aligned} \tag{6.7}$$

On the other hand, from $\nabla\alpha = (e_1\alpha)e_1$ and (3.3) we have $F(\nabla\alpha) = (e_1\alpha)\sin\alpha e_3$. Hence, we obtain from (6.3) that

$$\delta e_1\alpha = \langle J(\nabla\alpha), h(e_1, e_2)\rangle \csc\alpha. \tag{6.8}$$

Combining this with (6.7) gives inequality (6.1).

If the equality case of (6.1) holds at a point p, then it follows from (6.7) that $\beta = 3\varphi$ and $\mu = 3\delta + 3e_1\alpha$ hold at p. Hence, we obtain (6.2) from (6.4).

Conversely, if (6.2) holds at a point $p \in M$, then it follows from (6.2) and Lemma 3.1 that we have $e_2\alpha = 0$ at p. Thus, we get $\langle \nabla\alpha, Jh(e_1, e_2)\rangle = -\delta e_1\alpha \sin\alpha$ at p. Now, it is straight-forward to show that (6.2) holds at p implies that the equality case of (6.1) occurs at p. □

The following two corollaries follows immediately from Theorem 6.1.

Corollary 6.1 *[4] If M is a slant surface in a complex space form $\tilde{M}^2(4\varepsilon)$ with slant angle θ, then we have*

$$H^2 \geq 2\{K - (1 + 3\cos^2\theta)\varepsilon\}. \tag{6.9}$$

Corollary 6.2 *Let M be a purely real surface in $\mathbf{C^2}$. Then we have*

$$H^2 \geq 2\{K - ||\nabla\alpha||^2 + 2\,\langle\nabla\alpha, Jh(e_1, e_2)\rangle \csc\alpha\} \tag{6.10}$$

with respect to an orthonormal frame $\{e_1, e_2\}$ satisfying $\langle\nabla\alpha, e_2\rangle = 0$.

The equality case of (6.10) holds if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of M take the forms

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}. \tag{6.11}$$

7. Minimal surfaces satisfying the equality

Example 7.1 Let $\alpha(x)$ be a real-valued function with $\alpha' > 0$ and $0 < \alpha < \pi$ and let b be a nonzero real number. Consider the map:

$$L(x, y) = \big(be^{-\mathrm{i}b^{-1}y} \cot\alpha(x), y\big).$$

Then the induced metric is given by

$$g = b^2\alpha'^2 \csc^4 \alpha dx \otimes dx + \csc^2 \alpha dy \otimes dy.$$

This map L defines a purely real minimal surface with Wirtinger angle α which satisfies the equality case of (6.10). This surface is a helicoid lying in the following real hyperplane of $\mathbf{C}^2$: $\mathcal{H} = \{(z_1, z_2) \in \mathbf{C}^2 : \operatorname{Im} z_2 = 0\}$.

Theorem 7.1 *If M is a purely real minimal surface in $\mathbf{C}^2$ satisfying the equality case of (6.10), then either M is an open part of a totally geodesic slant plane or it is congruent to an open part of a helicoid lying in a real hyperplane of $\mathbf{C}^2$ defined by*

$$L(x, y) = \big(be^{-ib^{-1}y} \cot\alpha(x), y\big) \tag{7.1}$$

with non-constant Wirtinger angle α, where b is a nonzero real number.

Proof. Let M be a purely real minimal surface in $\mathbf{C}^2$ satisfiying the equality case of (6.10). If M is slant, then $\nabla\alpha = 0$ holds. So, Theorem 3 of [4] implies that M is either an open portion of a totally geodesic slant plane or a non-totally geodesic Lagrangian surface. In the later case, M is congruent to an open portion of the Whitney sphere which is non-minimal (cf. [3]).

Next, assume that M is non-slant. Then $U = \{p \in M : \nabla\alpha(p) \neq 0\}$ is a dense open subset of M, since M contains only isolated totally geodesic points. On U we may choose an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ satisfying $\nabla\alpha = (e_1\alpha)e_1$. Then, by Corollary 6.2, the shape operator takes the form (6.11). Hence, we have

$$\begin{aligned} &h(e_1,e_1) = 3\varphi e_3 + (\delta + e_1\alpha)e_4, \\ &h(e_1,e_2) = \delta e_3 + \varphi e_4, \quad h(e_2,e_2) = \varphi e_3 + 3(\delta + e_1\alpha)e_4 \end{aligned} \tag{7.2}$$

for some functions φ and δ.

On the other hand, it follows from the minimality and (7.2) that $\delta = -e_1\alpha$ and $\varphi = 0$. Thus, (7.2) reduces to

$$h(e_1,e_1) = h(e_2,e_2) = 0, \quad h(e_1,e_2) = -e_1\alpha e_3. \tag{7.3}$$

Since Span $\{e_1\}$ and Span $\{e_2\}$ are one-dimensional distributions, there exists a local coordinate system $\{x, y\}$ on U such that $\partial/\partial x$ and $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g on U takes the following form:

$$g = E^2dx^2 + G^2dy^2, \tag{7.4}$$

where E and G are positive functions. The Levi-Civita connection of (7.4) satisfies

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{E_x}{E}\frac{\partial}{\partial x} - \frac{EE_y}{G^2}\frac{\partial}{\partial y}, \\ &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{E_y}{E}\frac{\partial}{\partial x} + \frac{G_x}{G}\frac{\partial}{\partial y}, \\ &\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -\frac{GG_x}{E^2}\frac{\partial}{\partial x} + \frac{G_y}{G}\frac{\partial}{\partial y}. \end{aligned} \tag{7.5}$$

We may put

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{G}\frac{\partial}{\partial y}. \tag{7.6}$$

From $e_2\alpha = 0$, we have $\alpha = \alpha(x)$. Now, it follows from (3.4), (7.3), (7.5), (7.6) and formula (2.1) of Gauss that the immersion satisfies

$$\begin{aligned} &L_{xx} = \frac{E_x}{E}L_x - \frac{EE_y}{G^2}L_y, \quad L_{yy} = -\frac{GG_x}{E^2}L_x + \frac{G_y}{G}L_y, \\ &L_{xy} = \left(\frac{E_y}{E} - \frac{\mathrm{i}\alpha'(x)G}{E\sin\alpha}\right)L_x + \left(\alpha'(x)\cot\alpha + \frac{G_x}{G}\right)L_y. \end{aligned} \tag{7.7}$$

The compatibility conditions of system (7.7) are given by

$$E_y = 0, \quad G_x = -\alpha' G\cot\alpha, \tag{7.8}$$

$$\alpha'' = \alpha'\left(\frac{E_x}{E} - 2\frac{G_x}{G}\right), \tag{7.9}$$

$$G_{xx} = G\alpha'^2 + \frac{E_x}{E}G_x. \tag{7.10}$$

From (7.8) we get $E = E(x)$ and $G = f'(y)\csc\alpha$ for some nonzero real-valued function $f(y)$. Substituting this into (7.9) gives

$$\alpha'' E = 2E\alpha'^2 \cot\alpha + \alpha' E_x. \tag{7.11}$$

After solving this equation for $E(x)$ we obtain $E = b\alpha' \csc^2\alpha$ for some nonzero real number b. Therefore, the metric tensor of the surface is

$$g = b^2\alpha'^2 \csc^4\alpha\, dx^2 + f'^2(y)\csc^2\alpha\, dy^2. \tag{7.12}$$

From (7.7) and (7.12), we obtain

$$\begin{aligned} L_{xx} &= \left(\frac{\alpha''}{\alpha'} - 2\alpha'\cot\alpha\right) L_x, \quad L_{xy} = -\frac{\mathrm{i} f'(y)}{b} L_x, \\ L_{yy} &= \frac{f'^2(y)\sin 2\alpha}{2b\alpha'} L_x + \frac{f''(y)}{f'(y)} L_y. \end{aligned} \tag{7.13}$$

Solving the first equation in (7.13) gives

$$L(x,y) = w(y) + z(y)\cot\alpha(x) \tag{7.14}$$

for some vector functions $z(y), w(y)$. Substituting this into the second equation in (7.13) gives $bz'(y) = -\mathrm{i} f(y) z(y)$, which implies $z = c_1 e^{\mathrm{i} b^{-1} f'(y)}$ for some vector c_1. Combining this with (7.14) yields

$$L(x,y) = w(y) + c_1 e^{\mathrm{i} b^{-1} f(y)} \cot\alpha. \tag{7.15}$$

By substituting (7.15) into the last equation in (7.13) we obtain $f'w'' = f''w'$. Hence, the immersion is congruent to

$$L(x,y) = c_1 e^{\mathrm{i} b^{-1} f(y)} \cot\alpha + c_2 f(y) \tag{7.16}$$

for some vector $c_2 \in \mathbf{C}^2$. Consequently, after choosing suitable initial conditions and reparametrization of y, we obtain (7.1). □

8. Surfaces with circular ellipse of curvature

Example 8.1 Let $w : S^2 \to \mathbf{C}^2$ be the map defined by

$$w(y_0, y_1, y_2) = \frac{1 + iy_0}{1 + y_0^2}(ry_1, ry_2), \quad y_0^2 + y_1^2 + y_2^2 = 1,$$

with $r > 0$. Then w is a Lagrangian immersion of the 2-sphere S^2 into $\mathbf{C}^2$ which is called the *Whitney sphere.*

Theorem 8.1 *Let M be a purely real surface in $\mathbf{C}^2$ satisfying the equality case of* (6.10)*. If M has circular ellipse of curvature, then M is either an open portion of a totally geodesic slant plane or an open portion of the Whitney sphere.*

Proof. Let M be a purely real surface in $\mathbf{C}^2$ with circular ellipse of curvature satisfying the equality case of (6.10). If M is slant, then it follows from Theorem 3 of [4] that M is either an open part of a totally geodesic slant plane or a non-totally geodesic Lagrangian surface. In the later case, M is congruent to an open part of the Whitney sphere (cf. [3]). It is known that there exists an adapted orthonormal from $\{e_1, e_2, e_3, e_4\}$ on the Whitney sphere with $e_3 = Je_1, e_4 = Je_2$ such that

$$h(e_1, e_1) = 3\lambda e_3, \quad h(e_1, e_2) = \lambda e_4, \quad h(e_2, e_2) = \lambda e_3 \tag{8.1}$$

for some function λ. It follows from (2.11) and (8.1) that the Whitney sphere has circular ellipse of curvature.

Next, assume that M is non-slant. Then there exists a non-empty open subset U such that $\nabla\alpha \neq 0$ everywhere on U. Let us work on U to derive a contradiction.

On U we may choose an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ satisfying $\nabla\alpha = (e_1\alpha)e_1$. Then, according to Corollary 6.2, the shape operators of M take the forms (6.11). Hence, the second fundamental form h satisfies

$$\begin{aligned} &h(e_1, e_1) = 3\varphi e_3 + (\delta + e_1\alpha)e_4, \quad h(e_1, e_2) = \delta e_3 + \varphi e_4, \\ &h(e_2, e_2) = \varphi e_3 + 3(\delta + e_1\alpha)e_4 \end{aligned} \tag{8.2}$$

for some functions φ and δ. Since M is assumed to have circular ellipse of curvature, it follows from (2.12) and (8.2) that $\varphi = 0$ and $\delta = -\frac{1}{2}e_1\alpha$. Hence, (8.2) reduces to

$$h(e_1, e_1) = \frac{e_1\alpha}{2}e_4, \quad h(e_1, e_2) = -\frac{e_1\alpha}{2}e_3, \quad h(e_2, e_2) = \frac{3e_1\alpha}{2}e_4. \tag{8.3}$$

As in section 7, there exists a local coordinate system $\{x, y\}$ on U such that $\partial/\partial x$ and $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g on U takes the form of (7.4). We may assume that $e_1 = E^{-1}\partial/\partial x, e_2 = G^{-1}\partial/\partial y$. Then we also have $\alpha = \alpha(x)$ as in section 7.

It follows from (2.1), (3.4), (7.5) and (8.3) that the immersion L satisfies

$$\begin{aligned} L_{xx} &= \left(\frac{\alpha'(x)}{2}\cot\alpha + \frac{E_x}{E}\right)L_x + \left(\frac{\mathrm{i}E}{2G}\alpha'(x)\csc\alpha - \frac{EE_y}{G^2}\right)L_y, \\ L_{xy} &= \left(\frac{E_y}{E} - \frac{\mathrm{i}G\alpha'(x)}{2E}\csc\alpha\right)L_x + \left(\frac{\alpha'(x)}{2}\cot\alpha + \frac{G_x}{G}\right)L_y, \\ L_{yy} &= \left(\frac{3G^2}{2E^2}\alpha'(x)\cot\alpha - \frac{GG_x}{E^2}\right)L_x + \left(\frac{3\,\mathrm{i}G}{2E}\alpha'(x)\csc\alpha + \frac{G_y}{G}\right)L_y, \end{aligned}$$

The compatibility condition of this system are given by

$$E_y = G_y = 0, \tag{8.4}$$

$$\alpha''(x) = \alpha'\left(\frac{E_x}{E} - \frac{G_x}{G} - \alpha' \cot\alpha\right), \tag{8.5}$$

$$G_{xx} - \frac{E_x}{E} G_x = \frac{G}{2}\alpha'^2(x), \tag{8.6}$$

$$\csc^2\alpha = \frac{8G_x(G_x - G\alpha' \cot\alpha)}{G^2\alpha'^2(1 + 3\cos 2\alpha)}. \tag{8.7}$$

From (8.4) we get $E = E(x)$ and $G = G(x)$. So, after solving (8.5) we have $G = c\sqrt{\csc\alpha(x)}$ for some nonzero real number c. Substituting this into (8.6) gives

$$\frac{E_x}{E} = \frac{\alpha''(x)}{\alpha'(x)} - \frac{3}{2}\alpha'(x)\cot\alpha,$$

which implies $E = b\alpha' \csc^{3/2}\alpha$ for some real number $b \neq 0$. Now, by substituting the expression of E and G into (8.7) we obtain $\alpha'(x) = 0$, which is a contradiction. □

9. Surfaces with degenerate second fundamental form

A surface M in $\mathbf{C}^2$ is said to have *full second fundamental form* if its first normal space, $\operatorname{Im} h$, satisfies $\dim(\operatorname{Im} h) = 2$ at each point in M. It is said to have *degenerate second fundamental form* if $\dim(\operatorname{Im} h) < 2$ holds at each point in M.

Example 9.1 Let $\alpha(x)$ be a positive real-valued function with $\alpha' > 0$ defined on open intervals I and b is a nonzero real number. Consider $M = I \times \mathbf{R}$ with metric:

$$g = b^2\alpha'^2(x)\sin^4\alpha(x)dx \otimes dx + \sin^6\alpha(x)dy \otimes dy.$$

The map $\phi : M \to \mathbf{C}^2$ of M into $\mathbf{C}^2$,

$$\phi(x, y) = \frac{b}{12}\left(4e^{3ib^{-1}y}\sin^3\alpha(x), \cos(3\alpha(x)) - 9\cos\alpha(x)\right),$$

defines a purely real isometric immersion of M into $\mathbf{C}^2$ with Wirtinger angle α. It is direct to show that the squared mean curvature H^2, Gauss curvature K, the gradient of α, and the second fundamental form h of ϕ satisfy

$$H^2 = \frac{4}{b^2\sin^4\alpha},\quad K = \frac{3}{b^2\sin^4\alpha},\quad ||\nabla\alpha||^2 = \frac{1}{b^2\sin^4\alpha},\quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0.$$

Hence, this purely real surface satisfies the equality case of (6.10) and it has degenerate second fundamental form. This is a surface of revolution lying in the same real hyperplane $\mathcal{H}$ of $\mathbf{C}^2$ as in Example 7.1.

Theorem 9.1 *Let M be a purely real surface in $\mathbf{C}^2$ satisfying the equality case of* (6.10). *If M has degenerate second fundamental form, then M is congruent to an open portion of one of the following three types of surfaces* :

(1) *A totally geodesic slant plane.*

(2) *A positively curved surface with Wirtinger angle α defined by*

$$L(x,y) = \frac{b}{12}\left(4e^{3ib^{-1}y}\sin^3\alpha(x), \cos(3\alpha(x)) - 9\cos\alpha(x)\right),$$

where $\alpha(x)$ is a non-constant real-valued function and b is a nonzero real number.

(3) *A helicoid lying in a real hyperplane of $\mathbf{C}^2$ with non-constant Wirtinger angle α defined by $L(x,y) = \big(be^{-ib^{-1}y}\cot\alpha(x), y\big)$ with $b > 0$.*

Proof. Let M be a purely real surface in $\mathbf{C}^2$ satisfying the equality case of (6.10). Assume that M has degenerate second fundamental form.

If M is slant, then M is either an open portion of a totally geodesic slant plane or a non-totally geodesic Lagrangian surface. In the first case, we obtain case (1) of the theorem. In the second case, M is congruent to an open portion of the Whitney sphere which has full second fundamental form.

Now, assume that M is non-slant. We may choose an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ satisfying $\nabla\alpha = (e_1\alpha)e_1$. As in section 7, there exists a local coordinate system $\{x, y\}$ on M such that

$$g = E^2dx^2 + G^2dy^2. \tag{9.1}$$

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{G}\frac{\partial}{\partial t}, \quad \alpha = \alpha(x). \tag{9.2}$$

Moreover, since M satisfies the equality case of (6.10), the shape operator takes the form (cf. Corollary 6.2):

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}. \tag{9.3}$$

Because M has degenerate second fundamental form, it follows from (9.3) that we have either (a) $\delta = \varphi = 0$ or (b) $\delta = -e_1\alpha$ and $\varphi = 0$.

Case (a): $\delta = \varphi = 0$. In this case, the second fundamental form satisfies

$$h(e_1, e_1) = (e_1\alpha)e_4, \ \ h(e_1, e_2) = 0, \ \ h(e_2, e_2) = 3(e_1\alpha)e_4. \tag{9.4}$$

It follows from (3.4), (9.1), (9.2) and (9.4) that

$$\begin{aligned} L_{xx} &= \left(\alpha'(x)\cot\alpha + \frac{E_x}{E}\right)L_x + \left(\mathrm{i}\alpha'(x)(\csc\alpha)\frac{E}{G} - \frac{EE_y}{G^2}\right)L_y, \\ L_{xy} &= \frac{E_y}{E}L_x + \frac{G_x}{G}L_y, \\ L_{yy} &= \left(3\alpha'(x)(\cot\alpha)\frac{G^2}{E^2} - \frac{GG_x}{E^2}\right)L_x + \left(3\,\mathrm{i}\alpha'(x)(\csc\alpha)\frac{G}{E} + \frac{G_y}{G}\right)L_y, \end{aligned} \tag{9.5}$$

The compatibility from (9.5) are given by

$$E_y = 0, \quad \alpha'(x) = \frac{G_x}{3G} \tan\alpha, \tag{9.6}$$

$$\tan^2\alpha = \frac{3G(E_x G_x - E G_{xx})}{E G_x^2}. \tag{9.7}$$

Since $\alpha'(x) \neq 0$, (9.6) implies that $E = E(x), G_x \neq 0$ and $G = f'(y)\sin^3\alpha(x)$ for some function $f(y)$. By substituting these into (9.7) we get

$$\alpha''(x)E = E_x\alpha'(x) - 2\alpha'^2(x)E\cot\alpha, \tag{9.8}$$

which implies that $E(x) = b\alpha'(x)\sin^2\alpha$ for some nonzero real number b. Hence, the metric tensor is given by

$$g = b^2\alpha'^2(x)\sin^4\alpha(x)dx^2 + f'^2(y)\sin^6\alpha(x)dy^2. \tag{9.9}$$

It follows from (9.5) and (9.9) that

$$\begin{aligned} L_{xx} &= \left(3\alpha'\cot\alpha + \frac{\alpha''(x)}{\alpha'(x)}\right)L_x + \frac{\mathrm{i}b\alpha'^2\csc^2\alpha}{f'(y)}L_y, \\ L_{xy} &= 3\alpha'\cot\alpha L_y, \quad L_{yy} = \left(\frac{f''(y)}{f'(y)} + \frac{3\mathrm{i}f'(y)}{b}\right)L_y. \end{aligned} \tag{9.10}$$

Solving the second equation in (9.10) gives

$$L(x,y) = w(x) + z(y)\sin^3\alpha(x) \tag{9.11}$$

for some $w(x), z(y)$. Substituting (9.11) into the last equation in (9.10) yields

$$z''(y) = \left(\frac{f''(y)}{f'(y)} + \frac{3\mathrm{i}f'(y)}{b}\right)z'(y). \tag{9.12}$$

Solving this equation gives $z(y) = c_0 + c_1 e^{3\mathrm{i}b^{-1}f(y)}$ for some vectors $c_0, c_1 \in \mathbf{C}^2$. Hence, the immersion is congruent to

$$L(x,y) = w(x) + c_1 e^{3\mathrm{i}b^{-1}f(y)}\sin^3\alpha(x). \tag{9.13}$$

After substituting this into the first equation in (9.10) we obtain

$$w''(x) = \left(3\alpha'(x)\cot\alpha + \frac{\alpha''(x)}{\alpha'(x)}\right)w'(x). \tag{9.14}$$

Now, by solving this equation we get

$$w(x) = c_3 + c_2(\cos 3\alpha - 9\cos\alpha), \quad c_2, c_3 \in \mathbf{C}^2. \tag{9.15}$$

Combining this with (9.13) shows that the immersion is congruent to

$$L(x,y) = c_1 e^{3\mathrm{i}b^{-1}f(y)} + c_2(\cos 3\alpha - 9\cos\alpha). \tag{9.16}$$

So, after choosing initial conditions and reparametrization of y, we obtain case (2).

Case (b): $\delta = -e_1\alpha$ *and* $\varphi = 0$. In this case, we have

$$h(e_1, e_1) = h(e_2, e_2) = 0, \quad h(e_1, e_2) = -e_1\alpha e_3, \quad e_1\alpha \neq 0. \tag{9.17}$$

Thus, the surface is minimal. Consequently, it follows from Theorem 7.1 that the surface is congruent to the one given in case (3) of the theorem. □

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Bang-Yen CHEN
Department of Mathematics,
Michigan State University,
East Lansing, MI 48824–1027, USA
e-mail: bychen@math.msu.edu

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