Turk J Math
34 (2010) , 293 – 303.

doi:10.3906/mat-0805-2

Finite subquandles of sphere

Nülifer Özdemir and Hüseyin Azcan

Abstract

In this work finite subquandles of sphere are classified by using classification of subgroups of orthogonal group $O(3)$. For any subquandle Q of sphere there is a subgroup G_Q of $O(3)$ associated with Q. It is shown that if Q is a finite (infinite) subquandle, then G_Q is a finite (infinite) subgroup. Finite subquandles of sphere are obtained from actions of finite subgroups of $SO(3)$ on sphere. It is proved that the finite subquandles Q_1 and Q_2 of sphere whose all elements are not on the same great circle are isomorphic if and only if the subgroups G_{Q_1} and G_{Q_2} of $O(3)$ are isomorphic by which finite subquandles of sphere are classified.

Key Words: Quandle, orthogonal group.

1. Introduction

In this section some basic definitions about quandles are given. More details may be found in [1],[2],[3]. A quandle is a set with a binary operation which can be defined formally as follows.

Definition 1 *A quandle is a non-empty set X with a binary operation satisfying the following three axioms:*

- $a * a = a$, *for all* $a \in X$.
- *There is a unique* $a \in X$ *such that* $a * b = c$ *for* $b, c \in X$.
- *The formula* $(a * b) * c = (a * c) * (b * c)$ *holds for* $a, b, c \in X$.

As an example, considers the Coxeter quandle defined as follows. Let $\langle , \rangle$ be a symmetric bilinear form on $\mathbb{R}^n$ and $S = \{v \in \mathbb{R}^n | \langle v, v\rangle \neq 0\}$, then S has a quandle structure by the binary operation

$$u * v := \frac{2\langle u, v\rangle}{\langle v, v\rangle} v - u,$$

where $u, v \in S$. Note that, $u * v$ is the image of u under the reflection in v.

As in all algebraic structures one can define the concept of subquandle and subquandle generated by a subset:

2000 *AMS Mathematics Subject Classification:* 57M27, 57M25, 57M99.

Definition 2 *A nonempty subset H of a quandle Q is said to be a subquandle of Q if, under the binary operation on Q, H itself forms a quandle.*

Definition 3 *Let Q be a quandle and W be a subset of Q. The intersection of all the subquandles of Q which contain W is called the subquandle of Q generated by W.*

In this work we consider $S^{n-1} = \left\{x = (x_1, x_2, \ldots, x_n) \,|x_1^2 + x_2^2 + \ldots + x_n^2 = 1\right\}$ as a quandle with respect to the binary operation

$$x * y = 2\langle x, y\rangle\, y - x,$$

where $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in S^{n-1}$.

The main theorem of this paper is as follows.

Theorem 4 *Let Q be a finite subquandle of sphere. Then Q is isomorphic to one of the following:*

- *Dihedral quandle: A finite subquandle whose all elements lie on the same great circle.*
- *Biprism quandle: A subquandle of S^2 which has $2n + 2$ points and is genereted by vertices of a spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}$.*
- *Tetrahedral quandle: A subquandle of S^2 which has 12 points and is generated by vertices of a spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$.*
- *Octahedral quandle:A subquandle of S^2 which has 18 points and is genereted by vertices of a spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$.*
- *Icosahedral quandle: A subquandle of S^2 which has 30 points and is generated by vertices of a spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}$.*

2. Subgroups of O(n) associated with subquandles of S^{n-1}

Let Q be a subquandle of S^{n-1}. Then we have a map

$$\begin{array}{ccccccc} Q & \xrightarrow{\Psi} & O(n) & & & & \\ y & \longmapsto & \Psi(y) = \sigma_y : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n & \\ & & & x & \longmapsto & \sigma_y(x), & \end{array}$$

where $\sigma_y(x) = x - 2\langle x, y\rangle\, y$. Note that $\sigma_y(y) = -y$ and $\sigma_{-y}(x) = x - 2\langle x, -y\rangle(-y) = \sigma_y(x)$ for all $x \in \mathbb{R}^n$ and $y \in Q$.

We associate $G_Q := \langle \sigma_y : y \in Q\rangle \leq O(n)$ to Q. If the subquandle Q contains only one element, then $G_Q = \{\sigma_y, I\}$, and the group G_Q is isomorphic to the group $\mathbb{Z}_2$. If the subquandle Q consists of two elements y_1 and y_2, then these points must be antipodal. Hence $\sigma_{y_1} = \sigma_{y_2}$ and we get again $G_Q \cong \mathbb{Z}_2$.

If the equality $\sigma_{y_1}(x) = \sigma_{y_2}(x)$ holds for all $x \in \mathbb{R}^n$, then $y_1 = \pm\ y_2$. We put $y_1,\ y_2$ instead of x in the equation $\sigma_{y_1}(x) = \sigma_{y_2}(x)$, we get following equations

$$y_1 = \langle y_1, y_2\rangle\, y_2 \text{ and } y_2 = \langle y_1, y_2\rangle\, y_1.$$

From these equations we obtain $y_1 = \langle y_1, y_2\rangle^2 y_1$. Since $y_1,\ y_2 \in Q \subset S^{n-1},\ y_1 = \pm\ y_2$ is obtained. If y_1 and y_2 are different and not antipodal points of any subquandle of S^{n-1}, then $\sigma_{y_1} \neq \sigma_{y_2}$. Hence we can say that if any subquandle of S^{n-1} is infinite, so is G_Q. For the finite situation we give the following proposition.

Proposition 5 *If Q is a finite subquandle of S^{n-1}, then G_Q is a finite subgroup of $O(n)$.*

Proof. Let $Q = \{y_1, y_2, \ldots, y_m\}$ be a finite subquandle of S^{n-1} where $m > 2$. Define the map σ'_{y_i} by

$$\begin{array}{rccl} \sigma'_{y_i}: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & \sigma'_{y_i}(x) = -\sigma_{y_i}(x) \end{array}$$

for each $y_i \in Q$ and consider the group $G'_Q = \left\langle \sigma'_{y_i} : y_i \in Q \right\rangle$. Since the quandle binary operation on S^{n-1} is defined by

$$x * y = 2\langle x, y\rangle\, y - x$$

for all $x, y \in S^{n-1}$, we can write $\sigma'_{y_i}(x) = x * y_i$. From the definition of quandle there exist a unique element x in Q such that $x * y_i = x'$ for all $x' \in Q$, hence the restriction of σ'_{y_i} to Q is one-to-one and onto. So we think of $\sigma'_{y_i}|_Q$ to be an element of symmetric group S_m. Hence we can take the group G generated by the restriction of σ'_{y_i} to Q as a subgroup of symmetric group S_m. Since S_m is a finite group, the subgroup G is finite.

Now we define following map:

$$\begin{array}{rccl} \Psi: & G'_Q & \longrightarrow & G \\ & \sigma'_{y_i} & \longmapsto & \sigma'_{y_i}|_Q. \end{array}$$

This map is a group isomorphism. From this isomorphism the subgroup G'_Q is finite.

We may clarify the relation between G_Q and G'_Q as follows. Let us define the map $\varphi : G_Q \longrightarrow G'_Q$, $\varphi(\sigma_{y_i}) = \det(\sigma_{y_i})\sigma_{y_i}$ on the generators and extend it on G_Q such that φ is a group homomorphism. The map φ is onto and the kernel of φ is $\{g \in G'_Q : \det g.g = I\}$. Note that $\det(\det g.g) = (\det g)^{n+1} = 1$. If n is even then $\det g = 1$. Hence the kernel of φ is trivial and $G_Q \cong G'_Q$. If n is odd then $\det g = \pm 1$. If G_Q contains the element $-I$ kernel of φ is $\{I, -I\}$. Hence the map φ is two-to-one and $G_Q \cong G'_Q \times \mathbb{Z}_2$. If the group G_Q does not contain the element $-I$ kernel of φ is trivial and $G_Q \cong G'_Q$. Therefore the group G_Q is finite subgroup of the orthogonal group $O(n)$. □

Another example of a quandle is the conjugation quandle: Let G be a group, then the conjugation operation in G, i.e. $g * h = h^{-1}gh$, turns G into a quandle and denote by *cong*(G). Hence the quandle group $G_Q = \langle \sigma_y : y \in Q\rangle$ where Q is a subquandle of S^{n-1} is a quandle with respect to the binary operation

$$\sigma_{y_1} * \sigma_{y_2} = \sigma_{y_2}\sigma_{y_1}\sigma_{y_2} \quad (\text{since } \sigma_{y_2} = \sigma_{y_2}^{-1}).$$

Then we obtain following proposition.

Proposition 6 *If a and b are points of S^{n-1}, then $\sigma_{a*b} = \sigma_b \sigma_a \sigma_b$, where $a * b = 2\langle a, b\rangle b - a$.*

Proof. For each x in $\mathbb{R}^n$,

$$\begin{aligned}
\sigma_{a*b}(x) &= x - 2\langle a*b, x\rangle a*b \\
&= x - 2\langle 2\langle a,b\rangle b - a, x\rangle (2\langle a,b\rangle b - a) \\
&= x - 8\langle a,b\rangle^2 \langle x,b\rangle b + 4\langle a,b\rangle \langle x,b\rangle a + 4\langle a,b\rangle\langle a,x\rangle b - 2\langle a,x\rangle a
\end{aligned}$$

and

$$\begin{aligned}
\sigma_b\sigma_a\sigma_b(x) &= \sigma_b(\sigma_a(\sigma_b(x))) \\
&= \sigma_b(\sigma_a(x - 2\langle b,x\rangle b)) \\
&= \sigma_b(x - 2\langle b,x\rangle b - 2\langle x - 2\langle b,x\rangle b, a\rangle a) \\
&= x - 8\langle a,b\rangle^2 \langle x,b\rangle b + 4\langle a,b\rangle \langle x,b\rangle a + 4\langle a,b\rangle\langle a,x\rangle b - 2\langle a,x\rangle a
\end{aligned}$$

where $a, b \in S^{n-1}$. Thus, $\sigma_{a*b} = \sigma_b\sigma_a\sigma_b$. □

The map $\Psi : Q \longrightarrow cong(G_Q)$, $\Psi(x) = \sigma_x$ is a quandle homomorphism since

$$\Psi(x_1 * x_2) = \sigma_{x_1 * x_2} = \sigma_{x_2}\sigma_{x_1}\sigma_{x_2} = \sigma_{x_1} * \sigma_{x_2} = \Psi(x_1) * \Psi(x_2).$$

Let G_{Q_1} and G_{Q_2} be finite quandle groups. If the map $\varphi : G_{Q_1} \to G_{Q_2}$ is a group homomorphism, then φ induces a quandle homomorphism between conjugation quandles by the following equation:

$$\varphi(\sigma_{x_1} * \sigma_{x_2}) = \varphi(\sigma_{x_2}\sigma_{x_1}\sigma_{x_2}) = \varphi(\sigma_{x_1})\varphi(\sigma_{x_1})\varphi(\sigma_{x_2}) = \varphi(\sigma_{x_1}) * \varphi(\sigma_{x_2}).$$

Conversely, if φ is a quandle homomorphism, then the map φ may not be a group homomorphism.

Proposition 7 *If Q_1 and Q_2 are isomorphic finite (infinite) subquandles of sphere, then the quandle groups G_{Q_1} and G_{Q_2} are also isomorphic.*

Proof. Let $Q_1 = \{x_1, x_2, ..., x_n, ...\}$, $Q_2 = \{y_1, y_2, ..., y_n, ...\}$ and $f : Q_1 \to Q_2$, $f(x_i) = y_i$ be a quandle isomorphism. Then

$$G_{Q_1} = \langle \sigma_{x_1}, \sigma_{x_2}, ..., \sigma_{x_n} | \sigma_{x_j}\sigma_{x_i}\sigma_{x_j} = \sigma_{x_k} \text{ for suitable } x_i, x_j, x_k\rangle$$

and

$$G_{Q_2} = \langle \sigma_{y_1}, \sigma_{y_2}, ..., \sigma_{y_n} | \sigma_{y_l}\sigma_{y_m}\sigma_{y_l} = \sigma_{y_n} \text{ for suitable } y_l, y_m, y_n\rangle.$$

We can define the map $\Psi(\sigma_{x_i}) = \sigma_{f(x_i)}$ on generators and extend it on G_{Q_1} such that φ is a group homomorphism, where $\sigma_{x_i} \in G_{Q_1}$. Note that this map is preserves the relations on G_{Q_1}:

$$\begin{aligned}
\Psi(\sigma_{x_k}) = \Psi(\sigma_{x_j}\sigma_{x_i}\sigma_{x_j}) &= \Psi(\sigma_{x_i * x_j}) \\
&= \sigma_{f(x_i * x_j)} \\
&= \sigma_{f(x_i) * f(x_j)} \\
&= \sigma_{f(x_j)}\sigma_{f(x_i)}\sigma_{f(x_j)} = \Psi(\sigma_{x_j})\Psi(\sigma_{x_i})\Psi(\sigma_{x_j}).
\end{aligned}$$

Hence the map Ψ is a group isomorphism. □

Remark Converse of this proposition is not true. For example, the subquandle of circle which has three elements, and the subquandle of circle which has six elements, have the same quandle group namely D_6. But converse of this proposition is also true for the subquandles of sphere whose all elements are not on the same great circle as we will see in the course of the paper.

If the quandle groups which are obtained from finite subquandles, these lying on same great circles and not containing antipodal points, are isomorphic then we can easily check that these quandles are also isomorphic: Let Q_1 and Q_2 be finite subquandles of sphere which are not contain antipodal points and $\varphi : G_{Q_1} \longrightarrow G_{Q_2}$ be a group isomorphism. Then the maps

$$\begin{array}{ccc} Q_1 & \longrightarrow & Q_2 \\ \downarrow_{\Psi_{Q_1}} & & \downarrow_{\Psi_{Q_2}} \\ G_{Q_1} & \xrightarrow{\varphi} & G_{Q_2} \end{array}$$

$\Psi_{Q_1} : Q_1 \longrightarrow G_{Q_1}$, $\Psi_{Q_1}(x) = \sigma_x$ and $\Psi_{Q_2} : Q_2 \longrightarrow G_{Q_2}$, $\Psi_{Q_2}(y) = \sigma_y$ are one-to-one. We define the map f

$$\begin{array}{rccl} f: & Q_1 & \longrightarrow & Q_2 \\ & x & \longmapsto & f(x) = \left(\Psi_{Q_2}^{-1}\varphi\Psi_{Q_1}\right)(x). \end{array}$$

Since

$$\begin{aligned} f(x*y) &= \left(\Psi_{Q_2}^{-1}\varphi\Psi_{Q_1}\right)(x*y) = \Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_{x*y}\right)\right) \\ &= \Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_y\sigma_x\sigma_y\right)\right) = \Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_y\right)\varphi\left(\sigma_x\right)\varphi\left(\sigma_y\right)\right) \\ &= \Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_x\right)*\varphi\left(\sigma_y\right)\right) = \Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_x\right)\right)*\Psi_{Q_2}^{-1}\left(\varphi\left(\sigma_y\right)\right) \end{aligned}$$

where $x, y \in Q_1$, the map f is a quandle homomorphism. If $f(x) = f(y)$, then $\Psi_{Q_2}^{-1}(\varphi(\sigma_x)) = \Psi_{Q_2}^{-1}(\varphi(\sigma_y))$. Since Ψ_{Q_2} and φ are one-to-one functions we obtain that $\sigma_x = \sigma_y$. Since subquandle Q_1 does not contain antipodal points, we get $x = y$. Therefore the map f is a quandle isomorphism.

3. Finite Subquandles of Sphere

We already know that the set of vertices of a regular n-gon forms a quandle which we call as the dihedral quandle. Now we show that any finite subquandle of S^1 is isomorphic with a dihedral quandle. Let $Q = \{x_1, x_2, \cdots, x_n\}$ be a subquandle of S^1 and $\theta = \min\{d(x_i, x_j) \,|x_i, x_j \in Q\}$ where $d(x_i, x_j)$ denotes the spherical distance between x_i and x_j. Assume that $\theta = d(x_k, x_l)$ then x_k and x_l obviously generate Q . If $D_n = \{y_1, y_2, \cdots, y_n\}$ where $y_j = \cos j\theta + i \sin j\theta$ denotes the dihedral quandle then define $\varphi : Q \to D_n$, by $\varphi(x_k) = y_1$, $\varphi(x_l) = y_2$ and extend φ such that it is a homomorphism. Clearly such φ is an isomorphism as well therefore any finite subquandle of S^1 is a dihedral quandle. As it can be noticed when n is an odd number, if $x_i \in Q$ then $-x_i \notin Q$ and when n an even number, if $x_i \in Q$ then $-x_i \in Q$ as well. Hence if a subquandle of S^1 includes one antipodal pair then it includes all antipodal pairs. We also note that quandle groups obtained from finite subquandles of S^1 are dihedral groups. If $Q \subset S^1$ has n elements then we have $G_Q \cong D_n$ when n is odd and $G_Q \cong D_{n/2}$ when n is an even integer.

It is well known that any finite subgroup of orthogonal group $O(3)$ which does not lie completely in $SO(3)$ is isomorphic with one of the following groups:

$$\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_n, S_4, D_n \times \mathbb{Z}_2, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2, A_5 \times \mathbb{Z}_2.$$

Since the quandle group G_Q is generated by reflections and finite subgoup of orthogonal group $O(3)$, G_Q can be isomorphic to one of the above groups. But a subgroup of $O(3)$ which is isomorphic to $A_4 \times \mathbb{Z}_2$ can not be obtained as a quandle group: The group A_4 has only three elements a, b, c such that $a^2 = b^2 = c^2 = 1$, $ab = c$ and the subgroup $\langle a, b, c\rangle$ is isomorphic to the Klein-four group V_4. The subgroup of $O(3)$ which is isomorphic to $A_4 \times \mathbb{Z}_2$ has six element of degree 2. Then the subgroup which is generated by these six element of degree 2 is isomorphic to the group $V_4 \times \mathbb{Z}_2$. Hence a quandle group can not be isomorphic to the group $A_4 \times \mathbb{Z}_2$.

We present with following proposition a property about the finite subquandle of the quandle $\left(S^{n-1}, *\right)$.

Proposition 8 *Let Q be a finite subquandle of S^{n-1}. If $-a$ is in Q for some $a \in Q$, then Q contains all antipodal points.*

Proof. Let b be an element of Q. Consider the great circle passing trough the points $a, -a$ and b. Let θ be the distance between a and b. Since the quandle Q is finite, θ is a rational factor of 2π. Let $\theta = 2\pi\frac{p}{q}$ where $p \le q$ and $(p,q) = 1$. From this situation the distance between b and $-a$ is $\pi - 2\pi\frac{p}{q}$. The elements which are generated by $a, -a$ and b are in Q. Since $2\pi\frac{p}{q}$, the distance between $-a$ and $-b$, is $2p$ factor of $\frac{\pi}{q}$, the point $-b$ is in Q. □

From following Proposition one can easily check that a finite subquandle of sphere all elements of whose are not on the same great circle contains all antipodal points because of the structure of groups S_4, $D_{2n} \times \mathbb{Z}_2$, $S_4 \times \mathbb{Z}_2$, $A_5 \times \mathbb{Z}_2$.

Proposition 9 *Let Q_1 and Q_2 be two finite subquandles of sphere whose points do not lie on the same great circles. And G_{Q_1} and G_{Q_2} be two quandle groups which are obtained from this two quandles. If the group G_{Q_1} is isomorphic to the group G_{Q_2}, then the quandle Q_1 is isomorphic to the quandle Q_2.*

Proof. Since the quandles Q_1 and Q_2 are subquandles of sphere such that there isn't any great circle of S^2 containing all points of subquandles Q_1 and Q_2, the quandle groups G_{Q_1} and G_{Q_2} are isomorphic to one of the groups $S_4, D_{2n} \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2, A_5 \times \mathbb{Z}_2$.

Any subgroup of $O(3)$ which is isomorphic to one of the groups $A_5 \times \mathbb{Z}_2$, S_4, $S_4 \times \mathbb{Z}_2, D_{2n} \times \mathbb{Z}_2$ is generetad by three suitably chosen reflections.

If $G_{Q_1} \cong G_{Q_2} \cong S_4$: Let

$$G_{Q_1} = \left\langle \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_3} : (\sigma_{x_1}\sigma_{x_2})^2 = I,\ (\sigma_{x_1}\sigma_{x_3})^3 = I,\ (\sigma_{x_2}\sigma_{x_3})^3 = I \right\rangle,$$

where $\sigma_{x_1}, \sigma_{x_2}$ and σ_{x_3} are suitable reflections. The subquandle Q_1 is generated by the points x_1, x_2 and x_3 such that there isn't any great circle which contains these points. If these three points are on the same great circle, then all points of quandle lie on this great circle. But in this situation quandle group obtained from this quandle is isomorphic to dihedral group. This is a contradiction since $G_{Q_1} \cong S_4$.

Since $(\sigma_{x_1}\sigma_{x_2})^2 = I$, the map $\sigma_{x_1}\sigma_{x_2}$ is a rotation by π radians. Then the distance between x_1 and x_2 is $\frac{\pi}{2}$. From the equations $x_1 * x_2 = -x_1$ and $x_2 * x_1 = -x_2$, the points $-x_1$ and $-x_2$ are elements of the quandle Q_1. Since $(\sigma_{x_1}\sigma_{x_3})^3 = I$, the map $\sigma_{x_1}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ radians.

- If $\sigma_{x_1}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$ radians, then distance between x_1 and x_3 is $\frac{\pi}{3}$. Since $(\sigma_{x_2}\sigma_{x_3})^3 = I$, the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ radians.

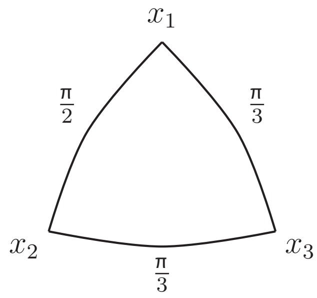

Figure 1

  - If the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$ radians, then distance between x_2 and x_3 is $\frac{\pi}{3}$. These three points determine a triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$ (see Figure 1). The subquandle which is generated by the vertices of the this triangle have 12 elements.
  - If the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{4\pi}{3}$ radians, then distance between x_2 and x_3 is $\frac{2\pi}{3}$. Since antipodal point $-x_2$ is in subquandle Q_1 and we consider the great circle of S^2 containing $-x_2, x_2, x_3$, the point x_2' which is shown Figure 2 is an element of Q_1. From the equation $x_2 * x_2' = x_3$ we get $\sigma_{x_2 * x_2'} = \sigma_{x_3}$. Since distance between x_2 and x_2' is $\frac{\pi}{3}$, we get

$$\begin{aligned} S_4 \cong G_{Q_1} &= \langle \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_2'} : (\sigma_{x_1}\sigma_{x_2})^2 = I, \\ &\quad (\sigma_{x_1}\sigma_{x_2'})^3 = I, (\sigma_{x_2}\sigma_{x_2'})^3 = I \rangle \end{aligned}$$

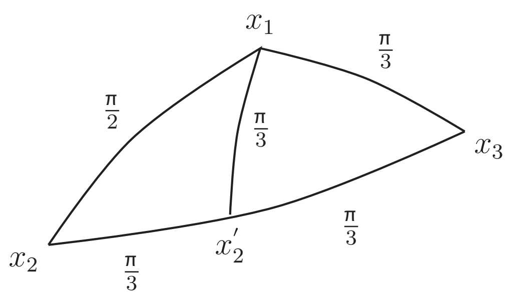

Figure 2

Again we obtain a new spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$.

- If the map $\sigma_{x_1}\sigma_{x_3}$ is a rotation by $\frac{4\pi}{3}$ radians, then distance between x_1 and x_3 is $\frac{2\pi}{3}$. Since $(\sigma_{x_2}\sigma_{x_3})^3 = I$, the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

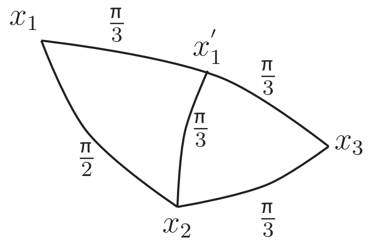

Figure 3

  - If the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{2\pi}{3}$, distance between x_2 and x_3 is $\frac{\pi}{3}$. Since $-x_1 \in Q_1$ and we consider the great circle of S^2 containing $-x_1, x_1, x_3$, the point x_1' which is shown Figure 3 is in Q_1. Also distance between x_2 and x_1' is $\frac{\pi}{3}$. Since $x_1 * x_1' = x_3$, we obtain

$$\begin{aligned} S_4 \cong G_{Q_1} &= \langle \sigma_{x_1}, \sigma_{x_1'}, \sigma_{x_2}; (\sigma_{x_1}\sigma_{x_2})^2 = I, \\ & \left(\sigma_{x_1'}\sigma_{x_2}\right)^3 = I, \left(\sigma_{x_1}\sigma_{x_1'}\right)^3 = I\rangle. \end{aligned}$$

  - If the map $\sigma_{x_2}\sigma_{x_3}$ is a rotation by $\frac{4\pi}{3}$, then distance between x_2 and x_3 is $\frac{2\pi}{3}$. Since $-x_2 \in Q_1$, the point x_2', which is shown in Figure 4, is in Q_1. And distance between x_2 and x_2' is $\frac{\pi}{3}$. Hence

$$\begin{aligned} S_4 \cong G_{Q_1} &= \langle \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_2'} (\sigma_{x_1}\sigma_{x_2})^2 = I, \\ & \left(\sigma_{x_1}\sigma_{x_2'}\right)^3 = I, \ \left(\sigma_{x_2}\sigma_{x_2'}\right)^3 = I\rangle. \end{aligned}$$

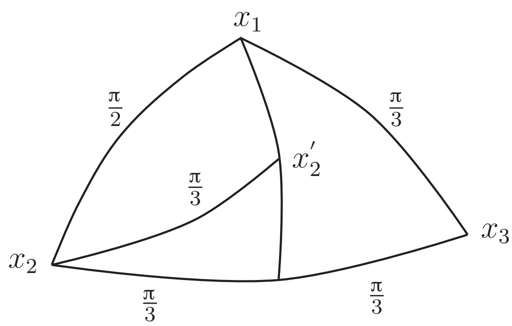

Figure 4

Let y_1, y_2 and y_3 be vertices of the spherical triangle whose side lengths are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$ (Figure 5) and

$$S_4 \cong G_{Q_2} = \langle \sigma_{y_1}, \sigma_{y_2}, \sigma_{y_3} | (\sigma_{y_1}\sigma_{y_2})^2 = I, \ (\sigma_{y_1}\sigma_{y_3})^3 = I, \ (\sigma_{y_2}\sigma_{y_3})^3 = I\rangle.$$

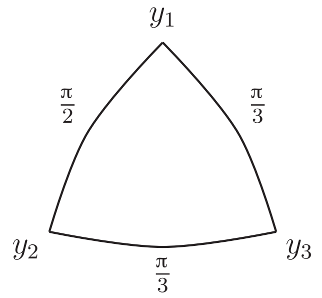

Figure 5

We obtain a subquandle of S^2 which has 12 elements and generated by the vertices y_1, y_2 and y_3. Then there exists a map $f \in O(3)$ such that $f(x_1) = y_1$, $f(x_2) = y_2$, and $f(x_3) = y_3$. Since f can be written as a composition of at most 4 reflections, f can be equal σ_a, $\sigma_a\sigma_b$, $\sigma_a\sigma_b\sigma_c$ or $\sigma_a\sigma_b\sigma_c\sigma_d$ (See [5] page 23). If $\sigma_a(x_i) = y_i$ and $\sigma_a(x_j) = y_j$, $x_i, x_j \in Q_1, y_i, y_j \in Q_2$, then

$$\sigma_a(x_i * x_j) = -((x_i * x_j) * a) = (-x_i * a) * (-x_i * a) = \sigma_a(x_i) * \sigma_a(x_j) = y_i * y_j.$$

Hence σ_a is a quandle homomorphism. In a similar way we can show that the maps $\sigma_a\sigma_b$, $\sigma_a\sigma_b\sigma_c$ and $\sigma_a\sigma_b\sigma_c\sigma_d$ are also quandle homomorphisms. The quandles Q_1 and Q_2 are finite, as a result f is a quandle isomorphism. Therefore we have shown that if $S_4 \cong G_{Q_1} \cong G_{Q_2}$, then $Q_1 \cong Q_2$.

The following cases can be proved similarly:

$$\begin{aligned} G_{Q_1} &\cong G_{Q_2} \cong S_4 \times \mathbb{Z}_2 \\ G_{Q_1} &\cong G_{Q_2} \cong A_5 \times \mathbb{Z}_2 \\ G_{Q_1} &\cong G_{Q_2} \cong D_{2n} \times \mathbb{Z}_2 \end{aligned}$$

□

3.1. The List of the Finite Subquandle of Sphere

Let G be a subgroup of $SO(3)$.

If G is isomorphic to the group $\mathbb{Z}_n$, all elements of G leave same two points fixed. Elements of orbit are only antipodal two points. These antipodal points are a subquandle of sphere.

If G is isomorphic to dihedral group D_{2n}, there are 3 orbits. One of these orbits contains only antipodal 2 points. Another orbits consist of m points. These m points are on the same circle. Total number of the elements of 3 orbits is $2n+2$. These $2n+2$ points form a subquandle of sphere. These subquandle is also generated three points which are vertices of a spherical triangle whose sides lengths are $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}$.

If G is isomorphic to the alternating group A_4, there are 3 orbits. The points of two orbits are the vertices of a regular tetrahedron. The points of the other orbit are the vertices of a regular octahedron. While the set of the vertices of a regular tetrahedron is not a subquandle, the set of the vertices of a regular octahedron is a subquandle of sphere. These subquandle is also generated by three points which are vertices of a spherical triangle whose lengths of the sides $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$.

If G is isomorphic to the alternating group S_4, there are 3 orbits. The points of first orbit are the vertices of a regular octahedron. The elements of second orbit are the vertices of a cube. The number of the elements of the last orbit is 12 and the set of these 12 points are a subquandle of sphere. Observe that these subquandle is also generated by three points which are vertices of a spherical triangle with side lengths $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$. But the set of the vertices of a cube and 12 elements of the last orbit is type of a subquandle of sphere. These subqunadle which has 18 points is generated by three points which are vertices of a spherical triangle whose lengths of the sides $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$.

If G is isomorphic to the alternating group A_5, again there are 3 orbits. The 12 points of first orbit are the vertices of a regular icosahedron. The 20 elements of second orbit are the vertices of a regular dodecahedron. The vertices of a regular icosahedron and the vertices of a regular dodecahedron are not type of a quandle. The number of the elements of the last orbit is 30 and the set of these 30 points are a subquandle of sphere. Observe that these subquandle is also generated by three points which are vertices of a spherical triangle whose lengths of sides are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}$.

Remark We have classified the finite subquandles of S^2. We can generalize this classification to higher dimensions as follows. This generalization will be done for the quandles whose points do not lie on the same great circle. Let Q_1 and Q_2 be finite subquandles of S^n which are generated by the point sets $\{x_1, x_2, \cdots x_n\}$ and $\{y_1, y_2, \cdots y_n\}$ of S^n respectively. And let

$$G_{Q_1} = \langle \sigma_{x_1}, \sigma_{x_2}, \cdots, \sigma_{x_n} \rangle \text{ and } G_{Q_1} = \langle \sigma_{y_1}, \sigma_{y_2}, \cdots, \sigma_{y_n} \rangle$$

be two quandle groups which are obtained from this quandles. Suppose that the group G_{Q_1} is isomorphic to the group G_{Q_2}. Now we will construct a quandle isomorphism Φ from Q_1 to Q_2. For an element $x_i \in Q_1$, fix $\Phi(x_i) = y_i$, where y_i is any element of Q_2. Take any element $x_j \in Q_1$ different from x_i. Let the distance between x_i and x_j be θ_{ij}, where θ_{ij} is a rational factor of 2π. If θ_{ij} were an irrational factor of 2π, Q_1 would be an infinite subquandle. Just like the S^2 case, there exists a point $y'_j \in Q_2$ such that $d(y_i, y_j) = \theta_{ij}$. Hence we can define $\Phi(x_j) = y'_j$. Now take $x_k \in Q_1$ different from both x_i and x_j. As explicitly shown in S^2 case, if $d(x_i, x_k) = \theta_{ik}$ and $d(x_j, x_k) = \theta_{jk}$, there exists a point $y'_k \in Q_2$ such that $d(y_i, y'_k) = \theta_{ik}$, $d(y'_j, y_k) = \theta_{jk}$. Thus we define $\Phi(x_k) = y'_k$. Let $x_l \in Q_1$ be a point different from x_i, x_j, x_k and $d(x_i, x_l) = \theta_{il}$, $d(x_j, x_l) = \theta_{jl}$, $d(x_k, x_l) = \theta_{kl}$. If the point $x_l \in Q_1$ is on the same great circle with any two of x_i, x_j, x_k, then these four points are on the same S^2. Thus, as explicitly shown in S^2, there exists a point $y'_l \in Q_2$ such that y'_l is on the same sphere with y_i, y'_j, y'_k and we define $\Phi(x_l) = y'_l$. If the point $x_l \in Q_1$ is not on the same great circle with any two of x_i, x_j, x_k, then $\Phi(x_l)$ is obtained as follows: Now we must find a point $y'_l \in Q_2$ such that $d(y_i, y'_l) = \theta_{il}$, $d(y'_j, y'_l) = \theta_{jl}$, $d(y'_k, y'_l) = \theta_{kl}$. As in the S^2, we can find a point $y'_l \in Q_2$ such that $d(y_i, y'_l) = \theta_{il}$, $d(y'_k, y'_l) = \theta_{kl}$. But we must show that the point $y'_l \in Q_2$ can be chosen to satisfied $d(y'_j, y'_l) = \theta_{jl}$. Now we consider two spheres having centers y_i and y'_k, and radii θ_{il} and θ_{kl} respectively. Intersection of these two spheres is a great circle denoted by C. Hence the distance between the points of C and y_i is θ_{il} and the distance between the points of C and the point y'_k is θ_{kl}. And there exists a point y'_l on C such that the distance between y'_l and y'_j is θ_{jl}: Again as in S^2, there exists a point $y' \in Q_2$ such that $d(y_i, y') = \theta_{il}$ and $d(y'_j, y') = \theta_{jl}$. Also the point $y' \in Q_2$ is on the C. If we take the sphere having centre y'_j and radius θ_{jl}, then this sphere intersects with C. It can be shown that y' and y'_l can be taken as the same intersection point. This is the point y_l' we are looking for. As a result $\Phi(x_l) =$ is defined as y_l'. Continuing like this we get the isomorphism Φ. Hence $Q_1 \cong \langle y_1', y_2', \cdots, y_i, \cdots, y_n' \rangle \cong Q_2$ is obtained. Note that the group $\langle \sigma_{y_1}, \sigma_{y_2}, \cdots, \sigma_{y_n} \rangle \cong \langle \sigma_{y_1'}, \sigma_{y_2'}, \cdots, \sigma_{y_i} \cdots, \sigma_{y_n'} \rangle$.

For $n \geq 3$, finite subgroups of $O(n)$ that is generated by reflections is given in [6]. Giving explicit finite subquandles of S^n by using the finite reflction subgroups of $O(n+1)$ may be the subject of another work.

References

[1] Brieskorn, E.: Automorphic sets and braids and singularities, Contemporary Mathematics, **78**, 45-115 (1988).

[2] Joyce, D.: A classifying invariant of knots, the knot quandle, Journal of Pure and Applied Algebra, **23**, 37-65 (1982).

[3] Roger, F. and Rourke, C.: Racks and links in codimension two, Journal of Knot Theory and Its Ramifications, **1**, 343-406 (1992).

[4] Armstrong, M.A.: Groups and symmetry, Springer-Verlag (1988).

[5] Beardon, A.F.: The geometry of discrete groups, Springer-Verlag (1983).

[6] Grove, L.C., Benson, C.T.: Finite reflection groups, Springer-Verlag (1985).

Nülifer ÖZDEMİR and Hüseyin AZCAN
Department of Mathematics,
Science Faculty, Anadolu University,
26470, Eskişehir, TURKEY
e-mail: nozdemir@anadolu.edu.tr
e-mail: hazcan@anadolu.edu.tr

Received 01.05.2008