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New inequalities similar to Hardy-Hilbert's inequality

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Abstract

In this paper, we establish a new inequality similar to Hardy-Hilbert's inequality. As applications, some particular results and the equivalent form are derived. The integral analogues of the main results are also given.

Key Words: Hardy-Hilbert's inequality; Hölder's inequality; β-function.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$ satisfy $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$,then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \tag{1.1}$$

and an equivalent form is

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p, \tag{1.2}$$

where the constant factor $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. The integral analogues of the inequality is: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ satisfy $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(x)dx < \infty$ then

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{\infty} g^q(x)dx \right)^{1/q}, \tag{1.3}$$

and an equivalent form is

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^{\infty} f^p(x)dx, \tag{1.4}$$

where the constant factor $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. Inequalities (1.1) and (1.3) are called Hardy-Hilbert's inequalities (see [1]) and are important in analysis and its applications (cf. Mitrinovic

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et al. [6]).Recently many generalization and refinements of these inequalities were also obtained, see [10, 4] and the references cited therein.

In the recent year, many new inequalities similar to (1.1) have been established; see [7, 8]. The main objective of this paper is to build a new inequality similar to Hardy-Hilbert's inequality (1.1). The equivalent form which is similar to (1.2) and some particular results are also given. The integral analogues of the main results are also given.

2. Some lemmas

In this section we shall prove lemmas which play crucial roles in proving our main results. We need the formula of the β-function as (cf. Wang et al. [9]):

$$B(p,q) = \int_0^\infty \frac{1}{(1+t)^{p+q}} t^{p-1} dt = B(q,p) \tag{2.1}$$

and the following two inequalities, which are well-known as Hardy's inequality (cf. Hardy et al. [1]).

Lemma 2.1 *If $p > 1$, $a_n \geq 0$ and $A_n = a_1 + a_2 + ... + a_n$, then*

$$\sum_{n=1}^\infty \left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p, \tag{2.2}$$

unless all the $a_n = 0$. The constant is the best possible.

Lemma 2.2 *If $p > 1$, $f \geq 0$ and $F(x) = \int_0^x f(t)dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{2.3}$$

unless $f \equiv 0$. The constant is the best possible.

We need the following results which can be obtained by Euler-Maclaurin summation formula (see [4, 2]).

Let $f : [1,\infty) \to \mathbb{R}$ be non-negative continuously differentiable function such that $\sum_{n=1}^\infty f(n) < \infty$ and $\int_1^\infty f(x)dx < \infty$, then the following equality holds:

$$\sum_{n=1}^\infty f(n) = \int_1^\infty f(x)dx + \frac{1}{2} f(1) + \int_1^\infty \rho_1(x) f'(x)dx, \tag{2.4}$$

where $\rho_1(x) = x - [x] - \frac{1}{2}$. Further, if $f \in C^4[1,\infty)$, $(-1)^k f^{(k)}(x) > 0$ for $k = 1,2,3,4$, and $\lim_{x\to\infty} f^{(k)}(x) = 0$, for $k = 0,1,2,3,4$, then the following inequality holds:

$$-\frac{1}{12} f(1) < \int_1^\infty \rho_1(x) f(x)dx < -\frac{1}{12} f(1) + \frac{1}{720} f''(1) < 0. \tag{2.5}$$

Using the above results Krnic and Pecaric obtained an extension of (1.1) in [4]. We need the following lemma [4, Lemma 2].

Lemma 2.3 *If* $0<\lambda\le 14$ *and* $0<\alpha<\lambda$ *for* $\lambda\le 2$, $0<\alpha\le 2$ *for* $\lambda>2$, *then*

$$\sum_{m=1}^{\infty}\frac{m^{\alpha-1}}{(m+n)^{\lambda}}<n^{\alpha-\lambda}B(\alpha,\lambda-\alpha). \tag{2.6}$$

Now we derive the following lemma.

Lemma 2.4 *If* $0<\alpha<\lambda$ *for* $\lambda\le 2$, $0<\alpha\le 2$ *for* $\lambda>2$, $\beta>0$ *such that* $\alpha+\beta<\lambda$, *and*

$$L:=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\alpha-1}n^{\beta-1}}{(m+n)^{\lambda}}-B(\alpha,\lambda-\alpha)\sum_{n=1}^{\infty}n^{\alpha+\beta-\lambda-1},$$

then

$$-\left(\frac{1}{\alpha}+\frac{\lambda}{2}\right)\sum_{n=1}^{\infty}n^{\beta-\lambda-1}<L<\left(\frac{\lambda-\alpha+7}{12}\right)\sum_{n=1}^{\infty}n^{\beta-\lambda-1} \tag{2.7}$$

Proof. Take $f_n(x)=\frac{x^{\alpha-1}}{(x+n)^{\lambda}}$. Then by (2.4), we have

$$\sum_{m=1}^{\infty}f_n(m)=\int_0^{\infty}f_n(x)dx-\int_0^{1}f_n(x)dx+\frac{1}{2}f_n(1)+\int_1^{\infty}\rho_1(x)f_n'(x)dx. \tag{2.8}$$

From (2.1), we get

$$\int_0^{\infty}f_n(x)dx=n^{\alpha-\lambda}B(\alpha,\lambda-\alpha).$$

Setting $t=\frac{x}{n}$, we obtain

$$0<\int_0^{1}f_n(x)dx=n^{\alpha-\lambda}\int_0^{\frac{1}{n}}\frac{t^{\alpha-1}}{(1+t)^{\lambda}}dt<n^{\alpha-\lambda}\int_0^{\frac{1}{n}}t^{\alpha-1}dt=\frac{1}{\alpha}n^{-\lambda}.$$

Also

$$0<\frac{1}{2}f_n(1)=\frac{1}{2}\frac{1}{(n+1)^{\lambda}}<\frac{1}{2}n^{-\lambda}.$$

Differentiating, we have

$$\begin{aligned}f_n'(x)&=\lambda n\frac{x^{\alpha-2}}{(x+n)^{\lambda+1}}-(\lambda-\alpha+1)\frac{x^{\alpha-2}}{(x+n)^{\lambda}}\\&=g_1(x)-g_2(x)\ \text{ (say)}.\end{aligned}$$

Clearly $g_1(x)$ and $g_2(x)$ satisfy the conditions of (2.5), as $\alpha-2\le 0$. Then

$$\int_1^{\infty}\rho_1(x)g_1(x)dx<0\ \text{ and }\int_1^{\infty}\rho_1(x)g_2(x)dx<0.$$

Hence by (2.5),

$$\int_1^\infty \rho_1(x) f_n'(x)dx = \int_1^\infty \rho_1(x)g_1(x)dx - \int_1^\infty \rho_1(x)g_2(x)dx$$
$$> \int_1^\infty \rho_1(x)g_1(x)dx$$
$$> -\frac{1}{12}g_1(1) = -\frac{1}{12}\frac{\lambda n}{(n+1)^\lambda} > -\frac{\lambda}{12}n^{-\lambda}.$$

Again, by (2.5),

$$\int_1^\infty \rho_1(x) f_n'(x)dx < -\int_1^\infty \rho_1(x)g_2(x)dx$$
$$< -(-\frac{1}{12}g_2(1)) = \frac{1}{12}\frac{\lambda-\alpha+1}{(n+1)^\lambda} < \frac{\lambda-\alpha+1}{12}n^{-\lambda}.$$

Now by (2.8), we get

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{m^{\alpha-1}n^{\beta-1}}{(m+n)^\lambda} = \sum_{n=1}^\infty n^{\beta-1}\left(\sum_{m=1}^\infty f_n(m)\right)$$
$$< B(\alpha,\lambda-\alpha)\sum_{n=1}^\infty n^{\alpha+\beta-\lambda-1} + \left(\frac{1}{2}+\frac{\lambda-\alpha+1}{12}\right)\sum_{n=1}^\infty n^{\beta-\lambda-1},$$

and

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{m^{\alpha-1}n^{\beta-1}}{(m+n)^\lambda} > B(\alpha,\lambda-\alpha)\sum_{n=1}^\infty n^{\alpha+\beta-\lambda-1} - \left(\frac{1}{\alpha}+\frac{\lambda}{12}\right)\sum_{n=1}^\infty n^{\beta-\lambda-1}.$$

Thus (2.7) is valid. This proves the lemma. □

3. Main results

In this section we prove our main result and derive some particular cases.

Theorem 3.1 *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \lambda \leq 4$, $0 < r, s < \lambda$ *if* $\lambda \leq 2$, $0 < r, s \leq 2$ *if* $\lambda > 2$, $r + s = \lambda$, $a_n, b_n \geq 0$, $A_n = \sum_{k=1}^n a_k$, $B_n = \sum_{k=1}^n b_k$. *If* $0 < \sum_{n=1}^\infty a_n^p < \infty$ *and* $0 < \sum_{n=1}^\infty b_n^q < \infty$, *then the following two inequalities hold:*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^\lambda}A_mB_n < pqB(r,s)\left(\sum_{n=1}^\infty a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty b_n^q\right)^{\frac{1}{q}}; \tag{3.1}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^\lambda}A_m\right)^p < (qB(r,s))^p\sum_{n=1}^\infty a_n^p; \tag{3.2}$$

where the constant factors $pqB(r,s)$ *and* $(qB(r,s))^p$ *are the best possible.*

Proof. By Hölder's inequality with weight (cf. Kuang [3]) and Lemma 2.3, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}}A_mB_n$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\left(n^{\frac{s-1}{p}}m^{\frac{r}{p}-1}A_m\right)\left(m^{\frac{r-1}{q}}n^{\frac{s}{q}-1}B_n\right)$$
$$\leq\left\{\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{n^{s-1}}{(m+n)^{\lambda}}\right]m^{r-p}A_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{m^{r-1}}{(m+n)^{\lambda}}\right]n^{s-q}B_n^q\right\}^{\frac{1}{q}}$$
$$< B(r,s)\left\{\sum_{m=1}^{\infty}\left(\frac{A_m}{m}\right)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right\}^{\frac{1}{q}}.$$

Then by Hardy inequality (2.1), (3.1) is valid.

Again by Hölder's inequality and Lemma 2.3, we get

$$\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}A_m=\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\left(n^{\frac{s-1}{p}}m^{\frac{r}{p}-1}A_m\right)\left(m^{\frac{r-1}{q}}n^{\frac{s}{q}}\right)$$
$$\leq\left\{\sum_{m=1}^{\infty}\frac{n^{s-1}}{(m+n)^{\lambda}}m^{r-p}A_m^p\right\}^{\frac{1}{p}}\left\{\sum_{m=1}^{\infty}\frac{m^{r-1}}{(m+n)^{\lambda}}n^{s}\right\}^{\frac{1}{q}}$$
$$<(B(r,s))^{\frac{1}{q}}\left\{\sum_{m=1}^{\infty}\frac{n^{s-1}}{(m+n)^{\lambda}}m^{r-p}A_m^p\right\}^{\frac{1}{p}}.$$

Hence, again applying Lemma 2.3, we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}A_m\right)^p<(B(r,s))^{\frac{p}{q}}\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{n^{s-1}}{(m+n)^{\lambda}}\right]m^{r-p}A_m^p$$
$$<(B(r,s))^p\sum_{m=1}^{\infty}\left(\frac{A_m}{m}\right)^p,$$

then by Hardy inequality (2.1), (3.2) is valid.

For sufficiently small $\varepsilon>0$, take $\tilde{a}_n=n^{-\frac{1+\varepsilon}{p}}$, $\tilde{b}_n=n^{-\frac{1+\varepsilon}{q}}$ for $n\geq 1$. Then

$$\left(\sum_{n=1}^{\infty}\tilde{a}_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\tilde{b}_n^q\right)^{\frac{1}{q}}<1+\frac{1}{\varepsilon}. \tag{3.3}$$

Since $\tilde{A}_1=\tilde{a}_1=1$ and for $n>1$,

$$\tilde{A}_m=\sum_{k=1}^{m}\tilde{a}_k>\sum_{k=1}^{m-1}\int_k^{k+1}x^{-\frac{1+\varepsilon}{p}}dx=\int_1^m x^{-\frac{1+\varepsilon}{p}}dx=\frac{q}{1-\varepsilon(q-1)}\left(m^{\frac{1}{q}-\frac{\varepsilon}{p}}-1\right).$$

Hence

$$\tilde{A}_m > \frac{q}{1-\varepsilon(q-1)}\left(m^{\frac{1}{q}-\frac{\varepsilon}{p}}-1\right), \text{ for } m \geq 1.$$

Similarly

$$\tilde{B}_n > \frac{p}{1-\varepsilon(p-1)}\left(n^{\frac{1}{p}-\frac{\varepsilon}{q}}-1\right), \quad \text{for } n \geq 1.$$

Taking $\phi(\varepsilon) = \frac{pq}{\{1-\varepsilon(p-1)\}\{1-\varepsilon(q-1)\}}$, we have $\lim_{\varepsilon \to 0^+} \phi(\varepsilon) = pq$ and for $m, n \geq 1$,

$$\tilde{A}_m \tilde{B}_n > \phi(\varepsilon)\left(m^{\frac{1}{q}-\frac{\varepsilon}{p}} n^{\frac{1}{p}-\frac{\varepsilon}{q}} - n^{\frac{1}{p}-\frac{\varepsilon}{q}} - m^{\frac{1}{q}-\frac{\varepsilon}{p}}\right).$$

Then

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}} \tilde{A}_m \tilde{B}_n \\
&> \phi(\varepsilon) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{m^{r-\frac{\varepsilon}{p}-1} n^{s-\frac{\varepsilon}{q}-1}}{(m+n)^{\lambda}} - \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{\varepsilon}{q}-1}}{(m+n)^{\lambda}} - \frac{m^{r-\frac{\varepsilon}{p}-1} n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}} \right) \\
&= \phi(\varepsilon)\left(\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\right) \text{ (say).}
\end{aligned}$$

Taking $\alpha = r - \frac{\varepsilon}{p}$, $\beta = s - \frac{\varepsilon}{q}$ in Lemma 2.4, we get

$$\begin{aligned}
\sum\nolimits_1 &:= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{\varepsilon}{p}-1} n^{s-\frac{\varepsilon}{q}-1}}{(m+n)^{\lambda}} \\
&> B\left(r-\frac{\varepsilon}{p}, s+\frac{\varepsilon}{p}\right) \sum_{n=1}^{\infty} n^{-\varepsilon-1} - \left(\frac{p}{pr-\varepsilon}+\frac{\lambda}{12}\right) \sum_{n=1}^{\infty} n^{-r-\frac{\varepsilon}{q}-1} \\
&> B\left(r-\frac{\varepsilon}{p}, s+\frac{\varepsilon}{p}\right) \int_1^{\infty} x^{-\varepsilon-1} dx - \left(\frac{p}{pr-\varepsilon}+\frac{\lambda}{12}\right) \sum_{n=1}^{\infty} n^{-r-\frac{\varepsilon}{q}-1} \\
&= \frac{1}{\varepsilon} B\left(r-\frac{\varepsilon}{p}, s+\frac{\varepsilon}{p}\right) - \left(\frac{p}{pr-\varepsilon}+\frac{\lambda}{12}\right) \sum_{n=1}^{\infty} n^{-r-\frac{\varepsilon}{q}-1},
\end{aligned}$$

where $\sum_{n=1}^{\infty} n^{-r-\frac{\varepsilon}{q}-1} < \infty$.
Again by Lemma 2.4, we obtain

$$\begin{aligned}
\sum\nolimits_2 &:= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{\varepsilon}{q}-1}}{(m+n)^{\lambda}} \\
&< B\left(r-\frac{1}{q}, s+\frac{1}{q}\right) \sum_{n=1}^{\infty} n^{-\frac{1+\varepsilon}{q}-1} + \frac{1}{12}\left(s+\frac{1}{q}+7\right) \sum_{n=1}^{\infty} n^{-r-\frac{\varepsilon}{q}-1} \\
&< \infty.
\end{aligned}$$

Similarly, $\sum_3 < \infty$.

Hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}}\tilde{A}_m\tilde{B}_n > \phi(\varepsilon)\left\{\frac{1}{\varepsilon}B\left(r-\frac{\varepsilon}{p}, s+\frac{\varepsilon}{p}\right)-\bigcirc(1)\right\}. \tag{3.4}$$

If the constant factor $pqB(r,s)$ in (3.1) is not the best possible, then there exists a positive constant K such that $K < pqB(r,s)$ and (3.1) still remains valid if $pqB(r,s)$ is replaced by K. In particular by (3.3) and (3.4), we have

$$\begin{aligned}\phi(\varepsilon)\left\{B\left(r-\frac{\varepsilon}{p}, s+\frac{\varepsilon}{p}\right)-\varepsilon\bigcirc(1)\right\} &< \varepsilon\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}}\tilde{A}_m\tilde{B}_n\\ &< \varepsilon K\left(\sum_{n=1}^{\infty}\tilde{a}_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\tilde{b}_n^q\right)^{\frac{1}{q}}\\ &< K(\varepsilon+1).\end{aligned}$$

Then $pqB(r,s) \leq K$ as $\varepsilon \to 0^+$. This contradiction shows that the constant factor $pqB(r,s)$ in (3.1) is the best possible.

If the constant factor $(qB(r,s))^p$ in (3.2) is not the best possible, then there exists a positive constant $\tilde{K}$ such that $\tilde{K} < qB(r,s)$ and (3.2) still remains valid if $(qB(r,s))^p$ is replaced by $\tilde{K}^p$. Then by Hölder's inequality, (3.2) and Hardy inequality (2.1), we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}}A_mB_n &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}A_m\right)\left(\frac{B_n}{n}\right)\\ &\leq \left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}A_m\right)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right\}^{\frac{1}{q}}\\ &< p\tilde{K}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}},\end{aligned}$$

which gives that the constant factor $pqB(r,s)$ in (3.1) is not the best possible. This contradiction shows that the constant factor $(qB(r,s))^p$ in (3.2) is the best possible. This proves the theorem. □

Now we discuss some particular cases of (3.1) and (3.2). Taking $\lambda = 1,2,3,4$ in Theorem 3.1, we get the following results respectively.

Corollary 3.2 *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $r,s > 0$, $r+s = 1$, $a_n, b_n \geq 0$, $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$. *If* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then the following two inequalities hold.*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{m+n}A_mB_n < pqB(r,s)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{3.5}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{m+n}A_m\right)^p<(qB(r,s))^p\sum_{n=1}^{\infty}a_n^p; \tag{3.6}$$

where the constant factors $pqB(r,s)$ and $(qB(r,s))^p$ are the best possible. In particular

(i) *For $r=\frac{1}{q}$ and $s=\frac{1}{p}$, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{mn(m+n)}<\frac{pq\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.7}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{A_m}{m(m+n)}\right)^p<\left(\frac{q\pi}{\sin\frac{\pi}{p}}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{3.8}$$

(ii) *For $r=s=\frac{1}{2}$ and $p=q=2$, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{mn(m+n)}<4\pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{3.9}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{A_m}{m(m+n)}\right)^2<4\pi^2\sum_{n=1}^{\infty}a_n^2. \tag{3.10}$$

Corollary 3.3 *Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ r,s>0,\ r+s=2,\ a_n,b_n\geq 0,\ A_n=\sum_{k=1}^{n}a_k,\ B_n=\sum_{k=1}^{n}b_k$. If $0<\sum_{n=1}^{\infty}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$, then the following two inequalities hold:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^2}A_mB_n<pqB(r,s)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.11}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^2}A_m\right)^p<(qB(r,s))^p\sum_{n=1}^{\infty}a_n^p, \tag{3.12}$$

where the constant factors $pqB(r,s)$ and $(qB(r,s))^p$ are the best possible. In particular:

(i) *For $r=\frac{2}{q}$ and $s=\frac{2}{p}$, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{m^{\frac{1}{p}}n^{\frac{1}{q}}(m+n)^2}<pqB\left(\frac{2}{p},\frac{2}{q}\right)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.13}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{n^{\frac{1}{p}}A_m}{m^{\frac{1}{p}}(m+n)^2}\right)^p<\left(qB\left(\frac{2}{p},\frac{2}{q}\right)\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{3.14}$$

(ii) *For $r=\frac{1}{2}+\frac{1}{q}$ and $s=\frac{1}{2}+\frac{1}{p}$, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{\sqrt{mn}(m+n)^2}<pqB\left(\frac{1}{2}+\frac{1}{p},\frac{1}{2}+\frac{1}{q}\right)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.15}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\sqrt{n}A_m}{\sqrt{m}(m+n)^2}\right)^p < \left(qB\left(\frac{1}{2}+\frac{1}{p},\frac{1}{2}+\frac{1}{q}\right)\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{3.16}$$

(iii) *For* $r=s=1$ *and* $p=q=2$ *, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{\sqrt{mn}(m+n)^2} < 4\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{3.17}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\sqrt{n}A_m}{\sqrt{m}(m+n)^2}\right)^2 < 4\sum_{n=1}^{\infty}a_n^2. \tag{3.18}$$

Corollary 3.4 *Let* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<r,s\le 2,\ r+s=\lambda,\ a_n,b_n\ge 0,\ A_n=\sum_{k=1}^{n}a_k,\ B_n=\sum_{k=1}^{n}b_k$*. If* $0<\sum_{n=1}^{\infty}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}b_n^q<\infty$*, then the following two inequalities hold.*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^3}A_mB_n < pqB(r,s)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.19}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^3}A_m\right)^p < (qB(r,s))^p\sum_{n=1}^{\infty}a_n^p; \tag{3.20}$$

where the constant factors $pqB(r,s)$ *and* $(qB(r,s))^p$ *are the best possible. In particular*

(i) *For* $r=1+\frac{1}{q}$ *and* $s=1+\frac{1}{p}$ *, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{(m+n)^3} < \frac{\pi}{2\sin\frac{\pi}{p}}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{3.21}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{nA_m}{(m+n)^3}\right)^p < \left(\frac{\pi}{2p\sin\frac{\pi}{p}}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{3.22}$$

(ii) *For* $r=s=\frac{3}{2}$ *and* $p=q=2$ *, we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{(m+n)^3} < \frac{\pi}{2}\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{3.23}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{nA_m}{(m+n)^3}\right)^2 < \frac{\pi^2}{4}\sum_{n=1}^{\infty}a_n^2. \tag{3.24}$$

Taking $\lambda=4,\ r=s=2$ in Theorem 3.1, we get the following result.

Corollary 3.5 *Let* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ a_n, b_n \geq 0,\ A_n = \sum_{k=1}^{n} a_k,\ B_n = \sum_{k=1}^{n} b_k$. *If* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then the following two inequalities holds.*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{\frac{1}{p}} n^{\frac{1}{q}}}{(m+n)^4} A_m B_n < \frac{pq}{6} \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{3.25}$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{m^{\frac{1}{p}} n^{1+\frac{1}{q}}}{(m+n)^4} A_m\right)^p < \left(\frac{q}{6}\right)^p \sum_{n=1}^{\infty} a_n^p; \tag{3.26}$$

where the constant factors $\frac{pq}{6}$ *and* $\left(\frac{q}{6}\right)^p$ *are the best possible. In particular for* $p = q = 2$, *we obtain the two inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sqrt{mn}}{(m+n)^4} A_m B_n < \frac{2}{3} \left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}}; \tag{3.27}$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n^{\frac{3}{2}} \sqrt{m} A_m}{(m+n)^4}\right)^2 < \frac{1}{9} \sum_{n=1}^{\infty} a_n^2, \tag{3.28}$$

where the constant factors $\frac{2}{3}$ *and* $\frac{1}{9}$ *are the best possible.*

4. Integral analogues

In this section we present the integral analogues of the inequalities given in Theorem 3.1, which in fact are similar to the integral analogues of the Hilbert's inequality.

Theorem 4.1 *Let* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda, r, s > 0,\ r + s = \lambda,\ f, g \geq 0$ *and* $F(x) = \int_0^x f(t)dt,\ G(x) = \int_0^x g(t)dt$. *If* $0 < \int_0^{\infty} f^p(x)dx < \infty$ *and* $0 < \int_0^{\infty} g^q(x)dx < \infty$, *then the following two inequalities holds.*

$$\int_0^{\infty}\int_0^{\infty} \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{(x+y)^{\lambda}} F(x)G(y)dxdy < pqB(r,s) \left\{\int_0^{\infty} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^{\infty} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{4.1}$$

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}}}{(x+y)^{\lambda}} F(x)dx\right)^p dy < [qB(r,s)]^p \int_0^{\infty} f^p(x)dx, \tag{4.2}$$

where the constant factors $pqB(r,s)$ *and* $[qB(r,s)]^p$ *are the best possible.*

Proof. For $v > 0$, setting $t = \frac{u}{v}$, by (2.1), we obtain

$$\int_0^{\infty} \frac{u^{\alpha-1}}{(u+v)^{\lambda}} du = v^{\alpha-\lambda} B(\alpha, \lambda - \alpha). \tag{4.3}$$

Then by using Hölder's inequality, (4.3), Hardy inequality (2.3) and proceeding as in the proof of Theorem 3.1, we get (4.1) and (4.2) are valid.

For the best constant factor, let for sufficiently small $\varepsilon > 0$,

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ x^{-\frac{1+\varepsilon}{p}} & \text{if } x \in [1,\infty); \end{cases}$$

$$g_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in (0,1), \\ y^{-\frac{1+\varepsilon}{q}} & \text{if } y \in [1,\infty). \end{cases}$$

Then

$$\left(\int_0^\infty f_\varepsilon^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g_\varepsilon^q(x)dx\right)^{\frac{1}{q}} = \frac{1}{\varepsilon}, \tag{4.4}$$

and

$$F_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ \frac{q}{1-\varepsilon(q-1)}\left(x^{\frac{1}{q}-\frac{\varepsilon}{p}} - 1\right) & \text{if } x \in [1,\infty). \end{cases}$$

$$G_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in (0,1), \\ \frac{p}{1-\varepsilon(p-1)}\left(y^{\frac{1}{p}-\frac{\varepsilon}{q}} - 1\right) & \text{if } x \in [1,\infty). \end{cases}$$

Denote $\phi(\varepsilon) = \frac{pq}{(1-\varepsilon(p-1))(1-\varepsilon(q-1))}$. Then $\phi(\varepsilon) \to pq$, as $\varepsilon \to 0^+$ and for $x, y \geq 1$

$$F_\varepsilon(x)G_\varepsilon(y) > \phi(\varepsilon)\left(x^{\frac{1}{q}-\frac{\varepsilon}{p}}y^{\frac{1}{p}-\frac{\varepsilon}{q}} - y^{\frac{1}{p}-\frac{\varepsilon}{q}} - x^{\frac{1}{q}-\frac{\varepsilon}{p}}\right).$$

Then

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}}{(x+y)^\lambda}F_\varepsilon(x)G_\varepsilon(y)dxdy \\
&> \phi(\varepsilon)\int_1^\infty\int_1^\infty \left(\frac{x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}}{(x+y)^\lambda} - \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{\varepsilon}{q}-1}}{(x+y)^\lambda} - \frac{x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{1}{p}-1}}{(x+y)^\lambda}\right)dxdy \\
&= \phi(\varepsilon)\left(I_1 - I_2 - I_3\right) \text{ (say).}
\end{aligned}$$

Taking $t = \frac{y}{x}$ and using (2.1), we have

$$\begin{aligned}
I_1 :&= \int_1^\infty\int_1^\infty \frac{x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy \\
&= \int_1^\infty x^{-\varepsilon-1}dx\int_0^\infty \frac{t^{s-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt - \int_1^\infty x^{-\varepsilon-1}\left(\int_0^{\frac{1}{x}} \frac{t^{s-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda}dt\right)dx \\
&> \frac{1}{\varepsilon}B\left(r+\frac{\varepsilon}{q},\ s-\frac{\varepsilon}{q}\right) - \int_1^\infty x^{-\varepsilon-1}\left(\int_0^{\frac{1}{x}} t^{s-\frac{\varepsilon}{q}-1}dt\right)dx \\
&= \frac{1}{\varepsilon}B\left(r+\frac{\varepsilon}{q},\ s-\frac{\varepsilon}{q}\right) - \left[\left(s-\frac{\varepsilon}{q}\right)\left(s+\frac{\varepsilon}{p}\right)\right]^{-1}.
\end{aligned}$$

Again taking $t = \frac{y}{x}$ and using (2.1), we obtain

$$\begin{aligned} I_2 :&= \int_1^\infty \int_1^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{\varepsilon}{q}-1}}{(x+y)^\lambda} dxdy \\ &= \int_1^\infty x^{-\frac{\varepsilon+1}{q}-1} dx \int_0^\infty \frac{t^{s-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda} dt - \int_1^\infty x^{-\frac{\varepsilon+1}{q}-1} \left(\int_0^{\frac{1}{x}} \frac{t^{s-\frac{\varepsilon}{q}-1}}{(1+t)^\lambda} dt \right) dx \\ &< \frac{q}{\varepsilon+1} B\left(r+\frac{\varepsilon}{q},\ s-\frac{\varepsilon}{q}\right). \end{aligned}$$

Similarly, we get

$$I_3 := \int_1^\infty \int_1^\infty \frac{x^{r-\frac{\varepsilon}{p}-1} y^{s-\frac{1}{p}-1}}{(x+y)^\lambda} dxdy < \frac{p}{\varepsilon+1} B\left(r-\frac{\varepsilon}{p},\ s+\frac{\varepsilon}{p}\right).$$

Hence

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{(x+y)^\lambda} F_\varepsilon(x) G_\varepsilon(y) dxdy > \phi(\varepsilon) \left\{ \frac{1}{\varepsilon}\, B\left(r+\frac{\varepsilon}{q},\ s-\frac{\varepsilon}{q}\right) - \bigcirc(1) \right\}. \tag{4.5}$$

If the constant factor $pqB(r,s)$ in (4.1) is not the best possible, then there exists a positive constant K such that $K < pqB(r,s)$ and (4.1) still remains valid if $pqB(r,s)$ is replaced by K. In particular by (4.4) and (4.5), we have

$$\begin{aligned} &\phi(\varepsilon) \left\{ B\left(r+\frac{\varepsilon}{q},\ s-\frac{\varepsilon}{q}\right) - \varepsilon \bigcirc (1) \right\} \\ &< \varepsilon \int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{(x+y)^\lambda} F_\varepsilon(x) G_\varepsilon(y) dxdy \\ &< \varepsilon K \left(\int_0^\infty f_\varepsilon^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g_\varepsilon^q(x) dx \right)^{\frac{1}{q}} \\ &= K. \end{aligned}$$

Then $pqB(r,s) \leq K$ as $\varepsilon \to 0^+$. This contradiction shows that the constant factor $pqB(r,s)$ in (4.1) is the best possible.

Proceeding as in the proof for the best constant factor in (3.2), we prove the constant factor $[qB(r,s)]^p$ in (4.2) is the best possible. This proves the theorem. □

Remark 4.2 Taking the different values of the parameters $\lambda,\ r,\ s,$ as following Theorem 3.1, we get the particular inequalities of (4.1) and (4.2).

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