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Trace formulae for Schrödinger systems on graphs

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Abstract

For Schrödinger systems on metric graphs with δ'-type conditions at the central vertex, firstly, we obtain precise description for the square root of the large eigenvalue up to the $o(1/n)$-term. Secondly, the regularized trace formulae for Schrödinger systems are calculated with some techniques in classical analysis. Finally, these formulae are used to obtain a result of inverse problem in the spirit of Ambarzumyan.

Key Words: Schrödinger systems, metric graph, δ'-type conditions, trace formula, Ambarzumyan-type theorem

1. Introduction

In a finite-dimensional space, an operator has a finite trace. But in an infinite-dimensional space, ordinary differential operators do not necessarily have finite trace (the sum of all eigenvalues). But Gelfand and Levitan [15] observed that the sum $\sum_n(\lambda_n - \mu_n)$ often makes sense, where $\{\lambda_n\}$ and $\{\mu_n\}$ are the eigenvalues of the "perturbed problem" and "unperturbed problem", respectively. The sum $\sum_n(\lambda_n - \mu_n)$ is called a regularized trace. Gelfand and Levitan first obtained an identity of trace for the Schrödinger operator [15]. We describe briefly here the result. Let $\lambda_j, j = 0, 1, \cdots$, be eigenvalues of the eigenvalue problem

$$-y''(x) + q(x)y(x) = \lambda y(x),\ \ y'(0) = y'(\pi) = 0.$$

Then there is the following identity of trace:

$$\sum_{n=0}^{\infty}[\lambda_n - n^2 - \frac{1}{\pi}\int_0^{\pi} q(x)dx] = \frac{1}{4}[q(\pi) + q(0)] - \frac{1}{2\pi}\int_0^{\pi} q(x)dx.$$

The trace identity of a differential operator deeply reveals spectral structure of the differential operator and has important applications in the numerical calculation of eigenvalues, inverse problem, theory of solitons, theory of integrable system [22, 41]. However, the calculation of every eigenvalue for the differential operator is very difficult. The most important application of the trace formulae is in solving inverse problems [41], i.e., given some spectral-related data, how to reconstruct the unknown potential function.

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A Quantum graph is the differential (self-adjoint) operator on a metric graph, i.e., the domain of the operator is a function space, each element in the space satisfying certain boundary conditions at the vertices. Differential operator on a metric graph (quantum graph) is a rather new and rapidly-developing area of modern mathematical physics. Such operators can be used to describe the motion of quantum particles confined to certain low dimensional structures. Spectral and scattering properties of Schrödinger operator in such structures attract a considerable attention during past years.

Recently, the spectral problems of quantum graphs have become a rapidly-developing field of mathematics and mathematical physics, and spectral properties of quantum graphs and different inverse problems have been studied in both forward [25, 26, 27, 32, 34, 39] and inverse [3, 7, 28, 33, 36, 37, 42, 45, 46], etc. Some results on trace formula and the inverse scattering problems for Laplacians on metric graphs have been studied [6, 16, 29, 40, 43], etc.

2. Main results

In this paper, we consider the following boundary value problems for Schrödinger systems on star-shaped metric graphs consisting of d segments of equal length:

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \cdots, d; \ d \geq 2, \ d \in \mathbf{N}, \tag{2.1}$$

which are subject to the boundary conditions

$$y_j(0) = 0, \ j = 1, 2, \cdots, d \tag{2.2}$$

or

$$y_j'(0) = 0, \ j = 1, 2, \cdots, d, \tag{2.3}$$

at the pendant vertices 0, and

$$y_1'(\lambda, \pi) = y_2'(\lambda, \pi) = \cdots = y_d'(\lambda, \pi), \tag{2.4}$$

$$y_1(\lambda, \pi) + y_2(\lambda, \pi) + \cdots + y_d(\lambda, \pi) = 0, \tag{2.5}$$

at the central vertex π. In equation (2.1), $q_j \in C[0, \pi]$, $j = 1, 2, \cdots, d$, are real-valued functions. (2.4) and (2.5) are called a δ'-type conditions.

For convenience, we denote by A_1, A_2 the operator acting in Hilbert space $L_d^2[0, \pi] = \bigoplus_{i=1}^d L^2[0, \pi]$ for the problem (2.1), (2.2), (2.4) and (2.5) or (2.1), (2.3), (2.4) and (2.5), respectively.

It is easy to verify that operators A_1 and A_2 are both self-adjoint, and each operator's spectrum, which consists of eigenvalues with the unique accumulation point $+\infty$, is real and lower bounded, and can be determined by the variational principle. Counting multiplicities of the eigenvalues, we can arrange those eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ in an ascending order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots .$$

The asymptotic expressions of eigenvalues and trace formulae for the operators A_1 and A_2 are established with residue techniques and asymptotic analysis method. In particular, the formulae presented here can be helpful in solving inverse problems. We end this paper with results in the spirit of Ambarzumyan.

In the case $q_j = 0, j = 1, 2, \cdots, d$, in (2.1), we can calculate the eigenvalues of operators A_1 and A_2 (for the detail, see the proofs of Theorems 2.1 and 2.2 in section 3). Denote by $\mu^D_{n,j}, j = 1, 2, \cdots, d, n = 1, 2, \cdots$, the spectrum of self-adjoint operator A_1, then

$$\mu^D_{n,d} = n^2 \tag{2.6}$$

and

$$\mu^D_{n,j} = (n - \frac{1}{2})^2,\ j = 1, 2 \cdots, d-1,\ n = 1, 2, \cdots. \tag{2.7}$$

Each of the eigenvalues n^2 is simple, and $(n - \frac{1}{2})^2$ is of multiplicity $d-1$.

Denote by $\mu^N_{n,j}, j = 1, 2, \cdots, d, n = 0, 1, 2, \cdots$, the spectrum of self-adjoint operator A_2, then

$$\mu^N_{n,d} = \left(n - \frac{1}{2}\right)^2,\ n = 1, 2, \cdots \tag{2.8}$$

and

$$\mu^N_{n,j} = n^2, j = 1, 2 \cdots, d-1, n = 0, 1, 2, \cdots. \tag{2.9}$$

Each of the eigenvalues $\left(n - \frac{1}{2}\right)^2$, $n = 1, 2, \cdots$, is simple, and each of the eigenvalues n^2, $n = 0, 1, 2, \cdots$, is of multiplicity $d-1$.

Suppose that $q_j(x) \in C^1[0, \pi]$, $j = 1, 2, \cdots, d$, let $\{\lambda^D_{n,j}, j = 1, 2, \cdots, d\}_{n=1}^{\infty}$ be the sequence of the eigenvalues of the operator A_1 and $\{\lambda^N_{n,j}, j = 1, 2, \cdots, d\}_{n=0}^{\infty}$ be the sequence of eigenvalues of the operator A_2, and denote

$$\bar{q_j} = \frac{1}{2\pi}\int_0^{\pi} q_j(x)dx. \tag{2.10}$$

The main results of this paper is as follows.

Theorem 2.1 *For sufficiently large n, the eigenvalues of the operator A_1 possess the asymptotic expression*

$$\sqrt{\lambda^D_{n,d}} = n + \frac{1}{nd}\sum_{j=1}^{d} \bar{q_j} + o\left(\frac{1}{n}\right), \tag{2.11}$$

and

$$\sqrt{\lambda^D_{n,j}} = n - \frac{1}{2} + \frac{c_{j,0}}{n - \frac{1}{2}} + o\left(\frac{1}{n}\right),\ j = 1, 2, \cdots, d-1, \tag{2.12}$$

where $c_{j,0}$, $1 \le j \le d-1$, are the solutions of the equation for c

$$\sum_{j=1}^{d} \prod_{j \neq l \in \{1,2,\cdots,d\}} (c - \bar{q_j}) = 0. \tag{2.13}$$

Theorem 2.2 *For sufficiently large n, the eigenvalues of the operator A_2 possess the asymptotic expression*

$$\sqrt{\lambda_{n,d}^N} = (n - \frac{1}{2}) + \frac{1}{(n-\frac{1}{2})d}\sum_{j=1}^{d} \bar{q}_j + o\left(\frac{1}{n}\right), \tag{2.14}$$

and

$$\sqrt{\lambda_{n,j}^N} = n + \frac{c_{j,0}}{n} + o\left(\frac{1}{n}\right), \ j = 1, 2, \cdots, d-1, \tag{2.15}$$

where $c_{j,0}$, $1 \le j \le d-1$, are solutions of the equation (2.13).

Theorem 2.3 *The trace formula for the operator A_1 reads as*

$$\begin{aligned} &\textstyle\sum_{n=1}^{\infty}[\sum_{j=1}^{d}(\lambda_{n,j}^D - \mu_{n,j}^D) - 2\sum_{j=1}^{d}\bar{q}_j] \\ &\quad = \textstyle\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi) - q_j(0)] - \frac{1}{2d}\sum_{j=1}^{d} q_j(\pi) + \frac{1}{d}\sum_{j=1}^{d}\bar{q}_j. \end{aligned} \tag{2.16}$$

Theorem 2.4 *The trace formula for the operator A_2 reads as*

$$\begin{aligned} &\textstyle\sum_{j=1}^{d-1}\lambda_{0,j}^N + \sum_{n=1}^{\infty}[\sum_{j=1}^{d}(\lambda_{n,j}^N - \mu_{n,j}^N) - 2\sum_{j=1}^{d}\bar{q}_j] \\ &\quad = \textstyle\frac{1}{4}\sum_{j=1}^{d}[q_j(\pi) + q_j(0)] - \frac{1}{2d}\sum_{j=1}^{d} q_j(\pi) + \frac{d-1}{d}\sum_{j=1}^{d}\bar{q}_j. \end{aligned} \tag{2.17}$$

Denote the set of eigenvalues of the operator A_i, $i = 1, 2$, by $\sigma(A_i)$, respectively.

Theorem 2.5 *Let the real-valued functions $q_j \in C[0, \pi]$, $j = 1, 2, \cdots, d$, and $m_k, k = 1, 2, \cdots$, be a strictly ascending infinite sequence of positive integers.*

(a) If either $\{(m_k - \frac{1}{2})^2 : k = 1, 2, \cdots\} \subset \sigma(A_1)$ and the multiplicity of each eigenvalue $(m_k - \frac{1}{2})^2$ is $d-1$ or $\{m_k^2 : k = 1, 2, \cdots\} \subset \sigma(A_1)$ holds, then $\sum_{j=1}^{d}\bar{q}_j = 0$.

(b) If either $\{m_k^2 : k = 1, 2, \cdots\} \subset \sigma(A_2)$ and the multiplicity of each eigenvalue m_k^2 is $d-1$ or $\{(m_k - \frac{1}{2})^2 : k = 1, 2, \cdots\} \subset \sigma(A_2)$ holds, then $\sum_{j=1}^{d}\bar{q}_j = 0$.

(c) If either $\{0\}\bigcup\{m_k^2 : k = 1, 2, \cdots\} \subset \sigma(A_2)$ and the multiplicity of each eigenvalue m_k^2 is $d-1$ or $\{0\}\bigcup\{(m_k - \frac{1}{2})^2 : k = 1, 2, \cdots\} \subset \sigma(A_2)$ holds, where 0 is the first eigenvalue of A_2, then $q_j(x) = 0$, $j = 1, 2, \cdots, d$.

3. The eigenvalue asymptotics

In this section, with the Gelfand-Levitan equation from [11, 30], we first derive the equation for eigenvalues of the operator A_1 or A_2, respectively. Then, with the help of the Rouché theorem we give the asymptotic expressions of large eigenvalues of the operators A_1 and A_2. The method used here is similar to the well-known techniques in the scalar case.

We first study the equation for eigenvalues of the operator A_1. Denote by $s_j(\lambda,x),\ j=1,2,\cdots,d$, the solutions of (2.1) satisfying the initial conditions

$$s_j(\lambda,0)=0,\ s_j'(\lambda,0)=1, \tag{3.1}$$

then the solutions of equations (2.1) satisfying the conditions (2.2) are

$$y_j(\lambda,x)=c_js_j(\lambda,x), \tag{3.2}$$

where c_j are arbitrary constants. Substituting (3.2) into (2.4) and (2.5), we obtain the following equation for eigenvalues of the operator A_1:

$$\varphi_1(\lambda)=\textstyle\sum_{j=1}^d s_j(\lambda,\pi)\prod_{l\neq j}s_l'(\lambda,\pi)=0. \tag{3.3}$$

Making use of the formulae in [11, 30], we have

$$\begin{aligned} s_j(\lambda,x)&=\tfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}-\tfrac{\cos(\sqrt{\lambda}x)}{\lambda}K_j(x,x)+\tfrac{1}{\lambda}\textstyle\int_0^x K_{j,t}'(x,t)\cos(\sqrt{\lambda}t)dt;\\ s_j'(\lambda,x)&=\cos(\sqrt{\lambda}x)+\tfrac{K_j(x,x)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}x)+\tfrac{1}{\sqrt{\lambda}}\textstyle\int_0^x K_{j,x}'(x,t)\sin(\sqrt{\lambda}t)dt,\end{aligned} \tag{3.4}$$

where both of the first partial derivatives $K_{j,x}'(x,t)$ and $K_{j,t}'(x,t)$ of $K_j(x,t),\ j=1,2,\cdots,d$, exist and $K_{j,x}'(x,\cdot)\in L^2[0,\pi]$ and $K_{j,t}'(x,\cdot)\in L^2[0,\pi]$.

If for brevity, we put

$$a_j=\int_0^\pi K_{j,x}'(\pi,t)\sin(\sqrt{\lambda}t)dt,\quad b_j=\int_0^\pi K_{j,t}'(\pi,t)\cos(\sqrt{\lambda}t)dt,$$

then by the Riemann-Lebesgue lemma,

$$a_j\to 0,\ b_j\to 0\ \text{ as real } \lambda\to\infty. \tag{3.5}$$

By (3.3) and (3.4), we have

$$\varphi_1(\lambda)=\sum_{j=1}^d[\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}+\frac{b_j-\cos(\sqrt{\lambda}\pi)K_j}{\lambda}]\times\prod_{j\neq l\in\{1,2,\cdots,d\}}[\cos(\sqrt{\lambda}\pi)+\frac{K_l}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi)+\frac{a_l}{\sqrt{\lambda}}], \tag{3.6}$$

where $K_j=K_j(\pi,\pi)=\frac{1}{2}\int_0^\pi q_j(x)dx$.

Now we try to get the equation for eigenvalues of the operator A_2. Denote by $\widetilde{s}_j(\lambda,x),\ j=1,2,\cdots,d$, the solutions of (2.1) satisfying the initial conditions

$$\widetilde{s}_j(\lambda,0)=1,\ \widetilde{s}_j'(\lambda,0)=0. \tag{3.7}$$

Then the solutions of equations (2.1) satisfying the conditions (2.3) are

$$y_j(\lambda,x)=\widetilde{c}_j\widetilde{s}_j(\lambda,x), \tag{3.8}$$

where $\widetilde{c}_j$ are arbitrary constants. Substituting (3.8) into (2.4) and (2.5), we obtain the following equation for eigenvalues of the operator A_2:

$$\varphi_2(\lambda) = \sum_{j=1}^{d} \widetilde{s}_j(\lambda,\pi) \prod_{l\neq j} \widetilde{s}'_l(\lambda,\pi) = 0. \tag{3.9}$$

Using the formulae in [11, 30], we have

$$\begin{aligned} \widetilde{s}_j(\lambda,x) &= \cos(\sqrt{\lambda}x) + \tfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \widetilde{K}_j(x,x) - \tfrac{1}{\sqrt{\lambda}} \textstyle\int_0^x \widetilde{K}'_{j,t}(x,t)\sin(\sqrt{\lambda}t)dt; \\ \widetilde{s}'_j(\lambda,x) &= -\sqrt{\lambda}\sin(\sqrt{\lambda}x) + \widetilde{K}_j(x,x)\cos(\sqrt{\lambda}x) + \textstyle\int_0^x \widetilde{K}'_{j,x}(x,t)\cos(\sqrt{\lambda}t)dt, \end{aligned} \tag{3.10}$$

where both of the first partial derivatives $\widetilde{K}'_{j,x}(x,t)$ and $\widetilde{K}'_{j,t}(x,t)$ of $\widetilde{K}_j(x,t)$, $j = 1,2,\cdots,d$, exist and $\widetilde{K}'_{j,x}(x,\cdot) \in L^2[0,\pi]$ and $\widetilde{K}'_{j,t}(x,\cdot) \in L^2[0,\pi]$.

If for brevity, we put

$$c_j = -\int_0^\pi \widetilde{K}'_{j,t}(\pi,t)\sin(\sqrt{\lambda}t)dt, \quad d_j = \int_0^\pi \widetilde{K}'_{j,x}(\pi,t)\cos(\sqrt{\lambda}t)dt,$$

then by the Riemann-Lebesgue lemma,

$$c_j \to 0,\ d_j \to 0 \ \text{ as real } \lambda \to \infty. \tag{3.11}$$

From (3.9) and (3.10), we obtain that

$$\varphi_2(\lambda) = \textstyle\sum_{j=1}^{d} [\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} K_j + \frac{c_j}{\sqrt{\lambda}}] \times \prod_{l\neq j} [-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + K_l\cos(\sqrt{\lambda}\pi) + d_l], \tag{3.12}$$

where $K_j = \frac{1}{2}\int_0^\pi q_j(x)dx$.

Furthermore, the kernels of the transformations $K_j(x,t), \widetilde{K}_j(x,t)$, $j = 1,2,\cdots,\ d$, satisfy the following partial differential equations [8, 11]

$$\begin{aligned} &K''_{j,xx} - q_j(x)K_j = K''_{j,tt},\ K_j(x,x) = \tfrac{1}{2}\textstyle\int_0^x q_j(x)dx,\ K_j(x,0) = 0; \\ &\widetilde{K}''_{j,xx} - q_j(x)\widetilde{K}_j = \widetilde{K}''_{j,tt},\ \widetilde{K}_j(x,x) = \tfrac{1}{2}\textstyle\int_0^x q_j(x)dx,\ \widetilde{K}'_{j,t}(x,0) = 0. \end{aligned} \tag{3.13}$$

When $q_j(x) \in C^1[0,\pi]$, (3.13) can be written as Volterra integral equations

$$\begin{aligned} K_j(x,t) &= \tfrac{1}{2}[\textstyle\int_0^{\frac{x+t}{2}} q_j(x)dx - \int_0^{\frac{x-t}{2}} q_j(x)dx] + \int_0^{\frac{x-t}{2}} d\tau \int_{\frac{x-t}{2}}^{\frac{x+t}{2}} q_j(\sigma+\tau)K_j(\sigma+\tau,\sigma-\tau)d\sigma, \\ \widetilde{K}_j(x,t) &= \tfrac{1}{2}[\textstyle\int_0^{\frac{x+t}{2}} q_j(x)dx + \int_0^{\frac{x-t}{2}} q_j(x)dx] + \int_0^{\frac{x-t}{2}} d\tau \int_{\tau}^{\frac{x+t}{2}} q_j(\sigma+\tau)\widetilde{K}_j(\sigma+\tau,\sigma-\tau)d\sigma, \end{aligned} \tag{3.14}$$

which are solvable. By (3.14) a direct calculation yields that

$$\begin{aligned} \tfrac{\partial K_j(x,x)}{\partial t} &= \tfrac{q_j(x)+q_j(0)}{4} - \tfrac{[\int_0^x q_j(x)dx]^2}{8}, \\ \tfrac{\partial K_j(x,x)}{\partial x} &= \tfrac{q_j(x)-q_j(0)}{4} + \tfrac{[\int_0^x q_j(x)dx]^2}{8}; \\ \tfrac{\partial \widetilde{K}_j(x,x)}{\partial t} &= \tfrac{q_j(x)-q_j(0)}{4} - \tfrac{[\int_0^x q_j(x)dx]^2}{8}, \\ \tfrac{\partial \widetilde{K}_j(x,x)}{\partial x} &= \tfrac{q_j(x)+q_j(0)}{4} + \tfrac{[\int_0^x q_j(x)dx]^2}{8}. \end{aligned} \tag{3.15}$$

When $q_j(x) \in C[0,\pi]$, by integration by parts we get

$$\begin{aligned} a_j &= -\tfrac{\cos(\sqrt{\lambda}\pi)K'_{j,x}(\pi,\pi)}{\sqrt{\lambda}} + \tfrac{1}{\sqrt{\lambda}}\int_0^\pi K''_{j,xt}(\pi,t)\cos(\sqrt{\lambda}t)dt, \\ b_j &= \tfrac{\sin(\sqrt{\lambda}\pi)K'_{j,t}(\pi,\pi)}{\sqrt{\lambda}} - \tfrac{1}{\sqrt{\lambda}}\int_0^\pi K''_{j,tt}(\pi,t)\sin(\sqrt{\lambda}t)dt \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} c_j &= \tfrac{\cos(\sqrt{\lambda}\pi)\widetilde{K}'_{j,t}(\pi,\pi)}{\sqrt{\lambda}} - \tfrac{1}{\sqrt{\lambda}}\int_0^\pi \widetilde{K}''_{j,tt}(\pi,t)\cos(\sqrt{\lambda}t)dt, \\ d_j &= \tfrac{\sin(\sqrt{\lambda}\pi)\widetilde{K}'_{j,x}(\pi,\pi)}{\sqrt{\lambda}} - \tfrac{1}{\sqrt{\lambda}}\int_0^\pi \widetilde{K}''_{j,xt}(\pi,t)\sin(\sqrt{\lambda}t)dt. \end{aligned} \tag{3.17}$$

Now we can prove the theorems in this paper.

Proof of Theorem 2.1

Write $\varphi_1(\lambda)$ as

$$\varphi_1(\lambda) = \varphi_1^{(0)}(\lambda) + \mathcal{E}_1(\lambda), \tag{3.18}$$

where

$$\varphi_1^{(0)}(\lambda) = \frac{d\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\cos^{d-1}(\sqrt{\lambda}\pi) \tag{3.19}$$

and $\mathcal{E}_1(\lambda)$ is the remainder.

It is easy to obtain zeros $\mu_{n,j}^D$ of the function $\varphi_1^{(0)}(\lambda)$, counting multiplicities of zero,

$$\sqrt{\mu_{n,d}^D} = n, \quad \sqrt{\mu_{n,j}^D} = n - \frac{1}{2}, \; j = 1,2,\cdots,d-1; n = 1,2,\cdots, \tag{3.20}$$

where $\{n^2\}_{n=1}^\infty$ are all simple zeros and $\{n-\frac{1}{2})^2\}_{n=1}^\infty$ are all zeros of order $d-1$.

Since the zeros of $\varphi_1(\lambda)$, the eigenvalues for the self-adjoint operator A_1, are real, we may suppose $|\mathrm{Im}\lambda| < \kappa$ for some fixed constant $\kappa > 0$.

Now it follows from (3.6), (3.18) and (3.19) that there exists a constant $c > 0$ such that

$$|\mathcal{E}_1(\lambda)| = |\varphi_1(\lambda) - \varphi_1^{(0)}(\lambda)| < \frac{c}{|\lambda|}$$

for all $|\mathrm{Im}\lambda| < \kappa$ and $|\lambda| \geq 1$. Since the function $d\sin(\sqrt{\lambda}\pi)\cos^{d-1}(\sqrt{\lambda}\pi)$ is a periodic function we can find $\Lambda > 0$ such that $|\varphi_1^{(0)}(\lambda)| > \frac{\Lambda}{|\sqrt{\lambda}|}$ for all $\lambda \in \mathbf{C}\backslash\bigcup_n C_n$, where C_n are circles of radii r with the centers at the points $\mu_{n,j}^D$, $j = 1,2,\cdots,d$. Thus, for all $\lambda \in \{\lambda | \lambda \in \mathbf{C}\backslash\bigcup_n C_n, |\sqrt{\lambda}| > \max\{\frac{c}{\Lambda}, 1\}\}$, we have

$$|\varphi_1(\lambda) - \varphi_1^{(0)}(\lambda)| < \frac{c}{|\lambda|} < \frac{\Lambda}{|\sqrt{\lambda}|} < |\varphi_1^{(0)}(\lambda)|. \tag{3.21}$$

Let $\lambda_{n,j}^D, j = 1,2,\cdots,d, n = 1,2,\cdots$, be the eigenvalues of the operator A_1, i.e., zeros of $\varphi_1(\lambda)$. By the Rouché theorem and taking sufficiently small r, we obtain the following results. For sufficiently large integer n, there lie exactly 1 and $d-1$ zeros of $\varphi_1(\lambda)$ in a suitable neighborhood of $\mu_{n,d}^D$ and $\mu_{n,j}^D (j \neq d)$, respectively, and denote

$$\sqrt{\lambda_{n,d}^D} = n + \alpha_n, \tag{3.22}$$

$$\sqrt{\lambda_{n,j}^D} = n - \frac{1}{2} + \beta_{n,j},\ j = 1, 2, \cdots, d-1, \tag{3.23}$$

where $\alpha_n = o(1)$ and $\beta_{n,j} = o(1)$ as $n \to \infty$. It is not difficult to see that $\alpha_n = O(1/n)$ and $\beta_{n,j} = O(1/(n-1/2))$. In fact, we can calculate $\lim_{n\to\infty} n\alpha_n$ and $\lim_{n\to\infty}(n-\frac{1}{2})\beta_{n,j}$.

Substituting $\lambda_{n,d}^D$ into $\varphi_1(\lambda) = 0$, then, from (3.6), (3.16) and (3.22), we have

$$\sin(\alpha_n \pi) = O(1/n).$$

Using Lagrange inversion formula, then we get

$$\alpha_n = \frac{c_0}{n} + \frac{\gamma_n}{n}, \tag{3.24}$$

where c_0 is a constant depending on $q_j(x)$, $j = 1, 2, \cdots, d$, and $\gamma_n \to 0$ as $n \to \infty$.

Similarly, we get

$$\beta_{n,i} = \frac{c_{i,0}}{n - \frac{1}{2}} + \frac{\gamma_{i,n}}{n}, \tag{3.25}$$

where $c_{i,0}, 1 \le i \le d-1$, are constants depending on $q_j(x)$, $j = 1, 2, \cdots, d$, and $\gamma_{i,n} \to 0$ as $n \to \infty$.

Substituting (3.22) and (3.24) into the equation $\varphi_1(\lambda) = 0$, we obtain

$$\begin{aligned} \textstyle\sum_{j=1}^d [(-1)^n \sin(\frac{c_0}{n} + o(1/n))\pi - \frac{(-1)^n K_j \cos(\frac{c_0}{n} + o(1/n))\pi}{n} + o(1/n)] \\ \times \textstyle\prod_{l \neq j} [(-1)^n \cos(\frac{c_0}{n} + o(1/n))\pi + O(1/n)] = 0, \end{aligned}$$

expanding the left-hand side of the resulting equation in power series, we have

$$\sum_{j=1}^d [c_0 \pi - K_j + o(1)] \prod_{l \neq j} [1 + o(1)] = 0,$$

and let $n \to \infty$, we obtain

$$c_0 = \frac{1}{\pi d} \sum_{j=1}^d K_j = \frac{1}{d} \sum_{j=1}^d \bar{q}_j. \tag{3.26}$$

Substituting (3.23) and (3.25) into the equation $\varphi_1(\lambda) = 0$, by (3.16), then it yields

$$\begin{aligned} 0 &= \textstyle\sum_{j=1}^d [\cos(\frac{c_{i,0}}{n-\frac{1}{2}} + o(1/n))\pi + o(1/n)] \times \prod_{l \neq j} [\sin(\frac{c_{i,0}}{n-\frac{1}{2}} + o(1/n))\pi - \frac{K_l \cos(\frac{c_{i,0}}{n-\frac{1}{2}} + o(1/n))\pi}{n - \frac{1}{2}} + o(1/n)] \\ &= \textstyle\sum_{j=1}^d [1 + o(1/n)] \times \prod_{l \neq j} [\frac{c_{i,0}\pi}{n-\frac{1}{2}} - \frac{K_l}{n-\frac{1}{2}} + o(1/n)]. \end{aligned}$$

Let $n \to \infty$, we have

$$\sum_{j=1}^{d} \prod_{l \neq j} (c_{i,0} - \bar{q}_l) = 0. \tag{3.27}$$

From (3.22)—(3.27), the theorem follows. □

Proof of Theorem 2.2

Its proof is similar to that of Theorem 2.1.

Write $\varphi_2(\lambda)$ as

$$\varphi_2(\lambda) = \varphi_2^{(0)}(\lambda) + \mathcal{E}_2(\lambda), \tag{3.28}$$

where

$$\varphi_2^{(0)}(\lambda) = d\cos(\sqrt{\lambda}\pi)[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)]^{d-1} \tag{3.29}$$

and $\mathcal{E}_2(\lambda)$ is the remainder. It is easy to obtain zeros $\mu_{n,j}^N$ of function $\varphi_2^{(0)}(\lambda)$:

$$\begin{aligned} \sqrt{\mu_{n,d}^N} &= n - \tfrac{1}{2},\ n = 1, 2, \cdots; \\ \sqrt{\mu_{n,j}^N} &= n,\quad j = 1, 2, \cdots, d-1;\ n = 0, 1, 2, \cdots, \end{aligned} \tag{3.30}$$

where $\{(n - \frac{1}{2})^2\}_{n=1}^{\infty}$ are all simple zeros and $\{n^2\}_{n=0}^{\infty}$ are all zeros of order $d-1$.

By the Rouch$\acute{e}$ theorem we have

$$\sqrt{\lambda_{n,d}^N} = n - \frac{1}{2} + \theta_n, \tag{3.31}$$

$$\sqrt{\lambda_{n,j}^N} = n + \nu_{n,j},\ j = 1, 2, \cdots, d-1, \tag{3.32}$$

where $\theta_n = o(1)$ and $\nu_{n,j} = o(1)$ as $n \to \infty$. It is not difficult to see that $\theta_n = O(1/(n - \frac{1}{2}))$ and $\nu_{n,j} = O(1/n)$.

From (3.28) and (3.31) we get

$$\theta_n = \frac{f_0}{n - \frac{1}{2}} + \frac{\widehat{\gamma}_n}{n}, \tag{3.33}$$

where f_0 is a constant depending on $q_j(x)$, $j = 1, 2, \cdots, d$, and $\widehat{\gamma}_n \to 0$ as $n \to \infty$.

Similarly,

$$\nu_{n,j} = \frac{g_{j,0}}{n} + \frac{\widehat{\gamma}_{j,n}}{n}, \tag{3.34}$$

where $g_{j,0}, 1 \le j \le d-1$, are constants depending on $q_j(x)$, and $\widehat{\gamma}_{j,n} \to 0$ as $n \to \infty$.

Moreover, substituting (3.31) and (3.33) into the equation $\varphi_2(\lambda) = 0$, we have

$$f_0 = \frac{1}{d} \sum_{j=1}^{d} \bar{q}_j, \tag{3.35}$$

and $g_{j,0},\ 1 \le j \le d-1$, are the solutions of the equation (2.13).

By (3.31), (3.32), (3.33), (3.34) and (3.35), the theorem follows. □

4. Trace formulae

Let Γ_{N_0} be the counterclockwise square contours $ABCD$, integer $N_0 = 0, 1, 2, \cdots \to \infty$, with

$$A = (N_0 + \tfrac{1}{4})(1 - i), \quad B = (N_0 + \tfrac{1}{4})(1 + i),$$
$$C = (N_0 + \tfrac{1}{4})(-1 + i), \quad D = (N_0 + \tfrac{1}{4})(-1 - i).$$

Obviously, $\mu_{n,j}^D$ and $\mu_{n,j}^N$ defined in (3.20) and (3.30), which are the zeros of the function $\varphi_k^{(0)}(\lambda), k = 1, 2$, don't lie on the contour Γ_{N_0}. To obtain trace formulae we need the following lemma in complex analysis.

Lemma 4.1 (refer to [1, 8]) *Suppose $\omega(\lambda), \omega_0(\lambda)$ are two entire functions, $\omega_0(\lambda)$ has no zeros on a closed contour Γ_{N_0} of λ-complex plane. If these functions satisfy the estimate*

$$\frac{\omega(\lambda)}{\omega_0(\lambda)} = 1 + \frac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\alpha_2(\sqrt{\lambda})}{\lambda} + O(1/\sqrt{\lambda^3}) \quad on\ \Gamma_{N_0},$$

where the functions $\frac{\alpha_k(\sqrt{\lambda})}{\sqrt{\lambda^k}}$, $k = 1, 2$, are single valued and analytic on Γ_{N_0} and $\alpha_k(\sqrt{\lambda})$ are uniformly bounded on Γ_{N_0}. Then, on Γ_{N_0},

$$\begin{aligned} \textstyle\sum_{\Gamma_{N_0}}(\lambda_n - \mu_n) &= -\tfrac{1}{2\pi i}\oint_{\Gamma_{N_0}} \log\tfrac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda \\ &= -\tfrac{1}{2\pi i}\oint_{\Gamma_{N_0}}[\tfrac{\alpha_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \tfrac{\alpha_2(\sqrt{\lambda}) - \alpha_1^2(\sqrt{\lambda})/2}{\lambda}]d\lambda + O(1/N_0), \end{aligned} \tag{4.1}$$

where λ_n, μ_n are the zeros of entire functions $\omega(\lambda), \omega_0(\lambda)$ inside the contour Γ_{N_0} listed with multiplicity, respectively.

Proof of Theorem 2.3

The computation of trace for the operator A_1 is based on Lemma 4.1 and asymptotic analysis method.

Step 1, we give the estimate for $\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)}$ on the contour Γ_{N_0}.

By (3.6) and (3.16), and integration by parts, on the contour Γ_{N_0}, we have

$$\begin{aligned} \tfrac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)} = \;& \tfrac{1}{d}\textstyle\sum_{j=1}^d[1 - \tfrac{K_j\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \tfrac{b_j}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}] \\ & \times\textstyle\prod_{l\neq j}[1 + \tfrac{a_l}{\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)} + \tfrac{K_l\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}] \\ = \;& 1 + \tfrac{1}{\sqrt{\lambda}}[\tfrac{d-1}{d}\textstyle\sum_{j=1}^d K_j\tan(\sqrt{\lambda}\pi) - \tfrac{\sum_{j=1}^d K_j\cot(\sqrt{\lambda}\pi)}{d}] + \tfrac{1}{\lambda}[-\tfrac{d-1}{d} \\ & \times\textstyle\sum_{j=1}^d K'_{j,x}(\pi,\pi) + \tfrac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\tan^2(\sqrt{\lambda}\pi) \\ & -\tfrac{2}{d}\textstyle\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2} + \tfrac{\sum_{j=1}^d K'_{j,t}(\pi,\pi)}{d}] + O(1/\sqrt{\lambda^3}). \end{aligned}$$

Next, the power series expansion tells us

$$\log \frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)} = \frac{1}{\sqrt{\lambda}}[\frac{d-1}{d}\sum_{j=1}^d K_j \tan(\sqrt{\lambda}\pi) - \frac{\sum_{j=1}^d K_j \cot(\sqrt{\lambda}\pi)}{d}] + \frac{1}{\lambda}[-\frac{d-1}{d}\times \sum_{j=1}^d K'_{j,x}(\pi,\pi) + \frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}} K_{i_1}K_{i_2}\tan^2(\sqrt{\lambda}\pi) - \frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}} K_{i_1}K_{i_2} + \frac{\sum_{j=1}^d K'_{j,t}(\pi,\pi)}{d} - \frac{(d-1)^2}{2d^2}(\sum_{j=1}^d K_j)^2\tan^2(\sqrt{\lambda}\pi) - \frac{1}{2d^2}(\sum_{j=1}^d K_j)^2\cot^2(\sqrt{\lambda}\pi) + \frac{d-1}{d^2}(\sum_{j=1}^d K_j)^2] + O(1/\sqrt{\lambda^3}). \tag{4.2}$$

From the above arguments it follows that the zeros $\lambda_{n,j}^D$ of $\varphi_1(\lambda)$ are the eigenvalues of the operator A_1, and the zeros $\mu_{n,j}^D$ of $\varphi_1^{(0)}(\lambda)$ are the eigenvalues of the problem (2.1), (2.2), (2.4) and (2.5) with $q_j=0, j=1,2,\cdots,d$. By Rouché's theorem, the number of zeros of $\varphi_1(\lambda)$ and $\varphi_1^{(0)}(\lambda)$ inside the contour Γ_{N_0} is just the same for sufficiently large N_0.

Finally, by (4.2) and Lemma 4.1, for sufficiently large N_0, it follows that

$$\sum_{n=1}^{N_0}[\lambda_{n,d}^D - n^2] + \sum_{n=1}^{N_0}\sum_{j=1}^{d-1}[\lambda_{n,j}^D - (n-\frac{1}{2})^2] = -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}} \log\frac{\varphi_1(\lambda)}{\varphi_1^{(0)}(\lambda)}d\lambda. \tag{4.3}$$

Using well-known formulae

$$\cot z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}, \quad \tan z = \sum_{n=0}^{\infty}\frac{8z}{(2n+1)^2\pi^2-4z^2}, \quad \csc^2 z = \sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^2}, \quad \sec^2 z = \sum_{n=-\infty}^{\infty}\frac{1}{[z+\{(2n+1)\pi/2\}]^2}, \tag{4.4}$$

we get

$$\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot\sqrt{\lambda}\pi}{\sqrt{\lambda}}d\lambda = \frac{2N_0+1}{\pi}, \quad \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\tan\sqrt{\lambda}\pi}{\sqrt{\lambda}}d\lambda = -\frac{2N_0}{\pi}, \quad \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot^2\sqrt{\lambda}\pi}{\lambda}d\lambda = -1+O(1/N_0), \quad \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\tan^2\sqrt{\lambda}\pi}{\lambda}d\lambda = -1+O(1/N_0). \tag{4.5}$$

Substituting (4.2) into (4.3), together with (3.15) and (4.5), we have

$$\sum_{n=1}^{N_0}\sum_{j=1}^{d}(\lambda_{n,j}^D - \mu_{n,j}^D) = \frac{1}{4}\sum_{j=1}^d[q_j(\pi)-q_j(0)] - \frac{1}{2d}\sum_{j=1}^d q_j(\pi) + \frac{2N_0d+1}{\pi d}\sum_{j=1}^d K_j + O(1/N_0),$$

i.e.

$$\sum_{n=1}^{N_0}[\sum_{j=1}^{d}(\lambda_{n,j}^D - \mu_{n,j}^D) - \frac{2}{\pi}\sum_{j=1}^d K_j] = \frac{1}{4}\sum_{j=1}^d[q_j(\pi)-q_j(0)] - \frac{1}{2d}\sum_{j=1}^d q_j(\pi) + \frac{1}{\pi d}\sum_{j=1}^d K_j + O(1/N_0). \tag{4.6}$$

Let $N_0\to\infty$ in (4.6), we have

$$\sum_{n=1}^{\infty}[\sum_{j=1}^{d}(\lambda_{n,j}^D - \mu_{n,j}^D) - 2\sum_{j=1}^d \bar{q}_j] = \frac{1}{4}\sum_{j=1}^d[q_j(\pi)-q_j(0)] - \frac{1}{2d}\sum_{j=1}^d q_j(\pi) + \frac{1}{d}\sum_{j=1}^d \bar{q}_j.$$

The proof of theorem is completed. □

Proof of Theorem 2.4

Its proof is similar to that of Theorem 2.3.

Step 1, we give the estimate for $\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}$ on the contour Γ_{N_0}.

By (3.12), (3.17) and (3.29), and integration by parts, on contour Γ_{N_0}, we obtain

$$\begin{aligned}\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)} = & \frac{1}{d}\sum_{j=1}^d[1+\frac{K_j\tan(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}+\frac{c_j}{\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)}] \\ & \times\prod_{l\neq j}[1-\frac{d_l}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}-\frac{K_l\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}] \\ & =1+\frac{1}{\sqrt{\lambda}}[-\frac{(d-1)}{d}\sum_{j=1}^d K_j\cot(\sqrt{\lambda}\pi)+\frac{\sum_{j=1}^d K_j\tan(\sqrt{\lambda}\pi)}{d}]+\frac{1}{\lambda}[-\frac{d-1}{d} \\ & \times\sum_{j=1}^d \widetilde{K'}_{j,x}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\cot^2(\sqrt{\lambda}\pi) \\ & -\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}+\frac{\sum_{j=1}^d\widetilde{K'}_{j,t}(\pi,\pi)}{d}]+O(1/\sqrt{\lambda^3}).\end{aligned}$$

Next, the power series expansion tells us

$$\begin{aligned}\log\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)} = & \frac{1}{\sqrt{\lambda}}[-\frac{d-1}{d}\sum_{j=1}^d K_j\cot(\sqrt{\lambda}\pi)+\frac{\sum_{j=1}^d K_j\tan(\sqrt{\lambda}\pi)}{d}]+\frac{1}{\lambda}[-\frac{d-1}{d}\times \\ & \sum_{j=1}^d\widetilde{K'}_{j,x}(\pi,\pi)+\frac{d-2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}\cot^2(\sqrt{\lambda}\pi) \\ & -\frac{2}{d}\sum_{i_1<i_2\in\{1,2,\cdots,d\}}K_{i_1}K_{i_2}+\frac{\sum_{j=1}^d\widetilde{K'}_{j,t}(\pi,\pi)}{d} \\ & -\frac{(d-1)^2}{2d^2}(\sum_{j=1}^d K_j)^2\cot^2(\sqrt{\lambda}\pi)-\frac{1}{2d^2}(\sum_{j=1}^d K_j)^2\tan^2(\sqrt{\lambda}\pi) \\ & +\frac{d-1}{d^2}(\sum_{j=1}^d K_j)^2]+O(1/\sqrt{\lambda^3}).\end{aligned} \tag{4.7}$$

By Lemma 4.1, we obtain

$$\sum_{n=1}^{N_0}[\lambda_{n,d}^N-(n-\frac{1}{2})^2]+\sum_{n=0}^{N_0}[\sum_{j=1}^{d-1}(\lambda_{n,j}^N-n^2)]=-\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\log\frac{\varphi_2(\lambda)}{\varphi_2^{(0)}(\lambda)}d\lambda. \tag{4.8}$$

Substituting (4.7) into (4.8), together with (3.15) and (4.5), we have

$$\begin{aligned}&\sum_{j=1}^{d-1}\lambda_{0,j}^N+\sum_{n=1}^{N_0}\sum_{j=1}^d(\lambda_{n,j}^N-\mu_{n,j}^N) \\ &=\frac{1}{4}\sum_{j=1}^d[q_j(\pi)+q_j(0)]-\frac{1}{2d}\sum_{j=1}^d q_j(\pi)+\frac{2N_0d+d-1}{\pi d}\sum_{j=1}^d K_j+O(1/N_0),\end{aligned}$$

i.e.

$$\begin{aligned}&\sum_{j=1}^{d-1}\lambda_{0,j}^N+\sum_{n=1}^{N_0}[\sum_{j=1}^d(\lambda_{n,j}^N-\mu_{n,j}^N)-2\sum_{j=1}^d\bar{q}_j] \\ &=\frac{1}{4}\sum_{j=1}^d[q_j(\pi)+q_j(0)]-\frac{1}{2d}\sum_{j=1}^d q_j(\pi)+\frac{d-1}{\pi d}\sum_{j=1}^d K_j+O(1/N_0).\end{aligned}$$

Let $N_0\to\infty$, we have

$$\begin{aligned}&\sum_{j=1}^{d-1}\lambda_{0,j}^N+\sum_{n=1}^{\infty}[\sum_{j=1}^d(\lambda_{n,j}^N-\mu_{n,j}^N)-2\sum_{j=1}^d\bar{q}_j] \\ &=\frac{1}{4}\sum_{j=1}^d[q_j(\pi)+q_j(0)]-\frac{1}{2d}\sum_{j=1}^d q_j(\pi)+\frac{d-1}{d}\sum_{j=1}^d\bar{q}_j.\end{aligned}$$

The proof of theorem is finished. □

5. The inverse problems

From a historical viewpoint, the paper [2] of Ambarzumyan may be thought to be the starting point of the inverse spectral theory aiming to reconstruct the potential from the spectrum (or spectra), Ambarzumyan proved the following theorem:

If $q \in C[0,\pi]$, *and* $\{n^2 : n = 0,1,2,\cdots\}$ *is the spectra set of the boundary value problem*

$$-y''(x) + q(x)y(x) = \lambda y(x),\ \ y'(0) = y'(\pi) = 0,$$

then $q(x) \equiv 0$ *in* $[0,\pi]$.

Proof of Theorem 2.5

(a) If $\{m_k^2 : k = 1,2,\cdots\} \subset \sigma(A_1)$ and $m_k, k = 1,2,\cdots$, be a strictly ascending infinite sequence of positive integers, by the estimate (2.11) of large eigenvalue, it follows $\sum_{j=1}^d \bar{q}_j = 0$.

If $\{(m_k - \frac{1}{2})^2 : k = 1,2,\cdots\} \subset \sigma(A_1)$ and the multiplicity of each eigenvalue $(m_k - \frac{1}{2})^2$ is $d-1$, then, by the estimate (2.12) of large eigenvalue, we have $c_{j,0} = 0,\ j = 1,2,\cdots,d-1$. Since $c_{j,0},\ 1 \le j \le d-1$, are the solutions of the equation (2.13), it follows $\sum_{j=1}^d \bar{q}_j = 0$.

(b) Similarly, applying estimates (2.14) and (2.15) of large eigenvalue, we obtain $\sum_{j=1}^d \bar{q}_j = 0$.

(c) From (b), we first obtain

$$\sum_{j=1}^d \bar{q}_j = 0. \tag{5.1}$$

Next, we show that $Y_{k,0} = (y_1(x), y_2(x), \cdots, y_d(x))^T = \frac{1}{\sqrt{d(d-1)\pi}}[e_1 + e_2 + \cdots + e_{k-1} - (d-1)e_k + e_{k+1} + \cdots + e_d]$, which satisfy boundary conditions (2.3), (2.4) and (2.5), is an eigenfunction corresponding to the first eigenvalue 0 of the operator A_2, where e_k is the unit vector whose k-th component is 1 $(k = 1,2,\cdots,d)$. By the variational principle, we obtain

$$0 = \inf_{Y \in D(A_2), ||Y||=1} (A_2 Y, Y) = \inf_{Y \in D(A_2), \sum_{j=1}^d ||y_j||^2 = 1} \left(-\int_0^\pi \sum_{j=1}^d y_j'' \overline{y_j} dx + \int_0^\pi \sum_{j=1}^d q_j(x)|y_j|^2 dx\right),$$

where $Y = (y_1, y_2, \cdots, y_d)^T, ||y_j||^2 = \int_0^\pi |y_j|^2 dx$. Now $||Y_{k,0}|| = 1$ and $Y_{k,0} \in D(A_2)$ are obvious, and so, for $1 \le k \le d$, it follows

$$0 \le (A_2 Y_{k,0}, Y_{k,0}) = \frac{1}{d(d-1)\pi}\left[\sum_{j=1, j\ne k}^d \int_0^\pi q_j(x)dx + (d-1)^2 \int_0^\pi q_k(x)dx\right] \triangleq \alpha_k. \tag{5.2}$$

Together with (5.1), we get

$$\begin{aligned}\sum_{k=1}^d \alpha_k &= \tfrac{1}{d(d-1)\pi} \sum_{k=1}^d \left[\sum_{j=1, j\ne k}^d \int_0^\pi q_j(x)dx + (d-1)^2 \int_0^\pi q_k(x)dx\right] \\ &= \tfrac{1}{\pi} \sum_{j=1}^d \int_0^\pi q_j(x)dx = 2\sum_{j=1}^d \bar{q}_j = 0.\end{aligned}$$

Thus, the right hand side of (5.2) is exactly 0, the test function $Y_{k,0}$ makes the functional $(A_2Y, Y)/||Y||^2$ achieve its minimum value and is thus the first eigenfunction. Substituting $Y_{k,0}$ into the equation (2.1), we obtain $q_j(x) = 0,\ j = 1, 2, \cdots, d$. The proof is finished. □

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