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The principal eigencurves for a nonselfadjoint elliptic operator

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Abstract

In this paper we study the existence of the principal eigencurves for a nonselfadjoint elliptic operator. We obtain their variational formulation. We establish also the continuity and the differentiability of the principal eigencurves.

Key Words: Nonsefadjoint elliptic operator , principl eigenvalue, principl eigencurve, Holland's formula.

1. Introduction

In this paper we consider the following problem

$$(P_\mu)\begin{cases}\text{To find } (\lambda, u) \in \mathbb{R} \times H^1(\Omega) \setminus \{0\} \text{ such that} & \\ Lu - \mu m_1(x)u = \lambda m_2(x)u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^N (N \geq 1)$ with boundary $\partial\Omega$, L is a second order elliptic operator of the form

$$Lu := -div(A(x) < u) + < a(x), \nabla u > +a_0(x)U,$$

and B is a first order boundary operator of Neumann or Robin type:

$$Bu := \langle b(x), \nabla u\rangle + b_0(x)u,$$

where $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^N$, the coefficient of L and B satisfy the condition where $A(x) = (a_{i,j}(x))$ is a symmetric, uniformly positive definite $N \times N$ matrix, with $a_{i,j} \in C^{0,1}(\overline{\Omega})$, a and $a_0 \in \mathbb{L}^\infty(\Omega)$, b and $b_0 \in C^{0,1}(\overline{\Omega})$, with $\langle b, \nu\rangle > 0$ (where ν is the unit exterior normal) and $b_0 \geq 0$ on $\partial\Omega$, μ is a real parameter; and m_1 and $m_2 \in \mathbb{L}^\infty(\Omega)$ are possibly indefinite weights, with m_1 and $m_2 \not\equiv 0$.
The selfadjoint case ($a \equiv 0$) was considered by several authors, in particular P.A. Binding and Y. X. Huang in [1], A. Dakkake and M. Hadda in [2]. For $\mu = 0$, the problem (P_μ) was studied by T. Godoy, J. P. Gossez and

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S. Paczka in [3]. They gave a formula of minimax type (called Holland's formula (cf., e.g., [6])) for the principal eigenvalues of this problem. They gave also an application of this formula of minimax to the antimaximum principle.

In this paper we study the existence of the principal eigencurves for the problem (P_μ). We obtain their variational formulation. We also establish the continuity and the differentiability of the principal eigencurves. The remainder of this paper is organized as follows: In section 2, which has a preliminary character, we collect some results on the existence of principal eigencurves. In section 3, we establish a formula of the minmax type for the principal eigencurves of the problem (P_μ) (cf. Theorem 3.1). In section 4, we establish the continuity (cf. Proposition 4.1) and the differentiability (cf. Proposition 4.3) of the principal eigencurve.

2. Existence of the principal eigencurves

Our purpose in this preliminary section is to collect some results relative to the principal eigenvalues $\lambda^\star(\mu)$ of problem (P_μ).

Definition. We say that a principal eigenvalue of problem (P_μ), an eigenvalue $\lambda^\star(\mu) \in \mathbb{R}$ such that (P_μ) admits a solution $(\lambda^\star(\mu), u)$ with $u \geq 0$. The graph of $\mu \to \lambda^\star(\mu)$ is called the principal eigencurve of (P_μ).

Unless otherwise stated solutions of (P_μ) are understood in the strong sense, i.e., $u \in W^{2,p}(\Omega)$ for some $1 < p < \infty$, with the equation in Ω satisfied a.e., and the boundary condition satisfied in the sense of traces. We will denote by $W(\Omega)$ the intersection of all $W^{2,p}(\Omega)$ spaces for $1 < p < \infty$.

The following proposition concerns the maximum principle.

Proposition 2.1 [3] *Assume $a_0 \geq 0$ and let $u \in W^{2,p}(\Omega)$ with $p \geq N$ satisfy $Lu \geq 0$ in Ω, $Bu \geq 0$ on $\partial\Omega$, with either $Lu \not\equiv 0$ or $Bu \not\equiv 0$. Then $u > 0$ on $\overline{\Omega}$.*

Another basic tool is the following existence, unicity and regularity result, which follows e.g. from Theorem 2.4.2.7 in [5].

Proposition 2.2 *Let $1 < p < \infty$. If $l \in \mathbb{R}$ is sufficiently large, then the problem*

$$(L + l)u = f \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega, \tag{2.1}$$

has for any $f \in L^p(\Omega)$ a unique solution $u \in W^{2,p}(\Omega)$. Moreover, the solution operator $(L + l)^{-1} : f \to u$ is continuous from $L^p(\Omega)$ onto $W^{2,p}(\Omega)$. This operator will be looked as an operator from $C(\overline{\Omega})$ (equipped with its usual norm) into itself (and then sometime denoted by S_{lC}).

Proposition 2.3 [3] *Assume l sufficiently large.*

1. *S_{lC} is compact and strongly positive.*
2. *The spectral radius ρ_l of S_{lC} is > 0 and ρ_l is an algebraically simple eigenvalue of S_{lC}, having an eigenfunction which is > 0 on $\overline{\Omega}$. In addition, there is no other eigenvalue having a nonnegative eigenfunction.*

3. *ρ_l is also an algebraically simple eigenvalue of the adjoint $S^{\star}_{lC}$, having a nonnegative eigenvector $\psi \in L^p(\Omega)$ for any $p < \infty$. In addition ψ does not depend on l.*

4. *For every $f \in C(\overline{\Omega})$ with $f \geq 0$, $f \not\equiv 0$, the equation $\lambda u - S_{lC} u = f$ has exactly one solution u, which is > 0 on $\overline{\Omega}$, if $\lambda > \rho_l$, and no solution $u \geq 0$ if $\lambda \leq \rho_l$.*

Remark 2.4 The above propositions apply in particular to the operator $L - \mu m_1$.

Theorem 2.5 *Assume first $a_0 - \mu m_1 \geq 0$ with either $a_0 - \mu m_1 \not\equiv 0$ or $b_0 \not\equiv 0$. Then*

1. *If m_2 changes sign, then (P_μ) admits exactly two principal eigencurves, one is positive and the other is negative.*

2. *If $m_2 \geq 0$, then (P_μ) admits exactly one principal eigenvalues, which is positive.*

3. *If $m_2 \leq 0$, then (P_μ) admits exactly one principal eigencurves, which is negative.*

   Assume now $a_0 - \mu m_1 \equiv 0$ and $b_0 \equiv 0$. Then

4. *Let ψ be the function provided by Proposition 2.3. If m_2 changes sign and $\int_\Omega \psi m_2 < 0$, then (P_μ) admits exactly two principal eigencurves, 0 and another one which is positive.*

5. *If m_2 changes sign and $\int_\Omega \psi m_2 > 0$, then (P_μ) admits exactly two principal eigencurves, 0 and another one which is negative.*

6. *If m_2 changes sign and $\int_\Omega \psi m_2 = 0$, then (P_μ) admits only 0 as principal eigencurve.*

7. *If $m_2 \geq 0$, then (P_μ) admits only 0 as principal eigencurve.*

8. *If $m_2 \leq 0$, then (P_μ) admits only 0 as principal eigencurve.*

Moreover all the principal eigenvalues associated to the principal eigencurves above are simple.

Proof. [Proof of Theorem 2.5] The proof of this theorem follows directly from Theorem 2.6 in [3] by taking $L - \mu m_1$ instead of L. □

3. Minimax formula of principal eigencurves

The weights and the operators L and B in this section are assumed to satisfy the conditions indicated at the beginning of introduction, with in addition $a \in C^{0,1}(\bar{\Omega})$, $a_0 - \mu m_1 \geq 0$ and $b = A\nu$ (i.e., the derivation in B is taken in the conormal direction). Our purpose is to give a formula of minimax type for the principal eigencurves of (P_μ).

Theorem 3.1 *Let $\mu \to \lambda^\star(\mu) \geq 0$ be the largest principal eigencurve of (P_μ) corresponding to any of the case 1, 2, 4, 5, 7 of Theorem 2.5. Then*

$$\lambda^\star(\mu) = \min_{u\in U} \max_{v\in H^1(\Omega)} \frac{\Lambda(u) - Q_u(v) + \int_{\partial\Omega} b_0(x)u^2 - \mu\int_\Omega m_1(x)u^2}{\int_\Omega m_2(x)u^2}, \tag{3.1}$$

where

$$U := \{u \in H^1(\Omega)\cap L^\infty(\Omega) : ess\inf u > 0 \text{ and } \int_\Omega m_2u^2 > 0\},$$

$$\Lambda(u) := \int_\Omega \langle A\nabla u, \nabla u\rangle + \langle a, \nabla u\rangle u + a_0u^2,$$

$$Q_u(v) := \int_\Omega u^2(\langle A\nabla v, \nabla v\rangle - \langle a, \nabla v\rangle).$$

Moreover, the minimum in (3.1) *is achieved at some $u \in W(\Omega)$ and the corresponding maximum in* (3.1) *is then achieved at some $v \in W(\Omega)$.*

Remark 3.2 Formula (3.1) can be stated equivalently as

$$\lambda^\star(\mu) = \min_{u\in U} \frac{\Lambda(u) - Q_u(W_u) + \int_{\partial\Omega} b_0(x)u^2 - \mu\int_\Omega m_1u^2}{\int_\Omega m_2u^2}, \tag{3.2}$$

where W_u is the function provided by the following lemma.

Lemma 3.3 [3] *For any $u \in H^1(\Omega)\cap L^\infty(\Omega)$ with $ess\inf u > 0$, the minimum of Q_u on $H^1(\Omega)$ is achieved at some W_u. This W_u is unique up to an additive constant and can be characterized at the weak solution (also unique up to an additive constant) of*

$$-div(u^2(2A\nabla W_u - a)) = 0 \text{ in } \Omega, \langle u^2(2A\nabla W_u - a), \nu\rangle = 0 \text{ on } \partial\Omega. \tag{3.3}$$

Moreover,

$$Q_u(W_u) = -\int_\Omega u^2\langle A\nabla W_u, \nabla W_u\rangle = -\frac{1}{2}\int_\Omega u^2\langle a, \nabla W_u\rangle.$$

Finally, if $u \in C^{0,1}(\overline{\Omega})$, then $W_u \in W(\Omega)$ and satisfies (3.3) *in the strong sense.*

Remark 3.4 The remaining cases of Theorem 2.5 (i.e., the negative eigencurve in 1 and in 3, the zero eigencurve in 4, the negative eigencurve in 5 and the zero eigencurve in 8 can be handled by Theorem 3.1, after changing m_2 to $-m_2$.

Remark 3.5 The zero eigencurve in case 6 of Theorem 2.5 is given by the following theorem

Theorem 3.6 *Assume $a_0 - \mu m_1 \equiv 0$ and $b_0 \equiv 0$. Assume also that m_2 changes sign with $\int_\Omega m_2\psi \equiv 0$. Then*

$$0 = \inf_{u\in U} \frac{\int_\Omega\langle A\nabla u, \nabla u\rangle + \int_\Omega\langle a, \nabla u\rangle u - Q_u(W_u)}{\int_\Omega m_2u^2}. \tag{3.4}$$

Formula (3.4) in the selfadjoint case was established in [4]. It is also proved in [4], in that case, that the infimum is never achieved.

The proofs of Theorem 3.1 and Theorem 3.6 follow respectively from Theorem 3.1 and Theorem 3.6 in [3] by taking $L - \mu m_1$ instead of L.

Remark 3.7 In the selfadjoint case, the minimum of the Rayleigh quotient is achieved at the eigenfunction. In the nonselfadjoint case, the minimum in (3.1) is achieved after multiplying the eigenfunction by a suitable function $\sqrt{G}$. This function G is introduced in the following Lemma 3.8.

Lemma 3.8 *[3] Let $u \in C^{0,1}(\Omega)$ with $u > 0$ on $\overline{\Omega}$. Then the problem*

$$-div(u^2(A\nabla G + aG)) = 0 \text{ on } \Omega, \langle u^2(A\nabla G + aG), \nu\rangle = 0 \text{ on } \partial\Omega, \tag{3.5}$$

has a solution $G \in W(\Omega)$, which is unique up to a multiplicative constant and which satisfies $G > 0$ on $\overline{\Omega}$.

4. Continuity and differentiability of the principal eigencurves

The assumptions on the weights and on the operators L and B in this section are those of Section 3, with in addition $a_0 - \mu m_1 > 0$ a.e. We will denote by $\mathcal{D} := \{\mu \in \mathbb{R}; a_0 - \mu m_1 > 0\}$. Our purpose in this section is to study the continuity and the differentiability of the principal eigencurve $\mu \to \lambda^\star(\mu)$ in $\mathcal{D}$. In the particular case where $m_2 \equiv 1$, some of the results of this section are contained in Lemma 2.5 of [3].

Proposition 4.1 *The function $\mu \to \lambda^\star(\mu)$ is concave and continuous in $\mathcal{D}$.*

Proof. For μ_1, $\mu_2 \in \mathcal{D}$ and $t \in (0,1)$, we have

$$\begin{aligned}
\lambda^\star(t\mu_1 + (1-t)\mu_2) &= \min_{u\in U} \frac{\Lambda(u) - Q_u(W_u) + \int_{\partial\Omega} b_0 u^2 - (t\mu_1 + (1-t)\mu_2)\int_\Omega m_1 u^2}{\int_\Omega m_2 u^2} \\
&\geq t \min_{u\in U} \frac{\Lambda(u) - Q_u(W_u) + \int_{\partial\Omega} b_0 u^2 - \mu_1 \int_\Omega m_1 u^2}{\int_\Omega m_2 u^2} \\
&+ (1-t) \min_{u\in U} \frac{\Lambda(u) - Q_u(W_u) + \int_{\partial\Omega} b_0 u^2 - \mu_2 \int_\Omega m_1 u^2}{\int_\Omega m_2 u^2} \\
&= t\lambda^\star(\mu_1) + (1-t)\lambda^\star(\mu_2).
\end{aligned}$$

Thus $\mu \to \lambda^\star(\mu)$ is concave in $\mathcal{D}$, then the continuity of $\mu \to \lambda^\star(\mu)$ in $\mathcal{D}$ follows from the concavity. □

Next we study the continuity of $\mu \to u^\star(\mu)$ in $\mathcal{D}$, where $u^\star(\mu)$ is a principal eigenfunction associated to $\lambda^\star(\mu)$.

Theorem 4.2 *Let $\mu, \mu_0 \in \mathcal{D}$. And let $u^\star(\mu)$ be a principal eigenfunction associated to $\lambda^\star(\mu)$ such that $u^\star(\mu) > 0$ and $\|u^\star(\mu)\|_{H^1(\Omega)} := \left[\int_\Omega \langle A\nabla u^\star(\mu), \nabla u^\star(\mu)\rangle + \int_\Omega (u^\star(\mu))^2\right]^{1/2} = 1$. Then $u^\star(\mu) \to u^\star(\mu_0)$ in $H^1(\Omega)$ as $\mu \to \mu_0$, where $u^\star(\mu_0)$ is a principal eigenfunction associated to $\lambda^\star(\mu_0)$.*

Proof. $(\lambda^\star(\mu), u^\star(\mu))$ is a solution of problem (P_μ). Thus for all $v \in H^1(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \langle A\nabla u^\star(\mu), \nabla v\rangle + \int_\Omega \langle a, \nabla u^\star(\mu)\rangle v + \int_{\partial\Omega} b_0 u^\star(\mu) v + \int_\Omega (a_0 - \mu m_1) u^\star(\mu) v \\ &= \lambda^\star(\mu) \int_\Omega m_2 u^\star(\mu) v.\end{aligned} \tag{4.1}$$

$u^\star(\mu)$ is bounded in $H^1(\Omega)$. So, for a subsequence, $u^\star(\mu) \to u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Passing to the limit in (4.1), we obtain

$$\int_\Omega \langle A\nabla u, \nabla v\rangle + \int_\Omega \langle a, \nabla u\rangle v + \int_{\partial\Omega} b_0 uv + \int_\Omega (a_0 - \mu_0 m_1) uv = \lambda^\star(\mu_0) \int_\Omega m_2 uv. \tag{4.2}$$

Taking $v = u^\star(\mu)$ in (4.1), we obtain

$$\begin{aligned}&\int_\Omega \langle A\nabla u^\star(\mu), \nabla u^\star(\mu)\rangle + \int_\Omega \langle a, \nabla u^\star(\mu)\rangle u^\star(\mu) + \int_{\partial\Omega} b_0 (u^\star(\mu))^2 + \int_\Omega (a_0 - \mu m_1)(u^\star(\mu))^2 \\ &= \lambda^\star(\mu) \int_\Omega m_2 (u^\star(\mu))^2,\end{aligned}$$

which implies that

$$1 + \int_\Omega \langle a, \nabla u^\star(\mu)\rangle u^\star(\mu) + \int_{\partial\Omega} b_0 (u^\star(\mu))^2 + \int_\Omega (a_0 - 1 - \mu m_1)(u^\star(\mu))^2 = \lambda^\star(\mu) \int_\Omega m_2 (u^\star(\mu))^2. \tag{4.3}$$

Passing to the limit in (4.3), we obtain

$$1 + \int_\Omega \langle a, \nabla u\rangle u + \int_{\partial\Omega} b_0 u^2 + \int_\Omega (a_0 - 1 - \mu m_1) u^2 = \lambda^\star(\mu_0) \int_\Omega m_2 u^2. \tag{4.4}$$

For $v = u$ in (4.2), we have

$$\begin{aligned}&\int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega u^2 + \int_\Omega \langle a, \nabla u\rangle u + \int_{\partial\Omega} b_0 u^2 + \int_\Omega (a_0 - 1 - \mu_0 m_1) u^2 \\ &= \lambda^\star(\mu_0) \int_\Omega m_2 u^2.\end{aligned} \tag{4.5}$$

By (4.4) and (4.5), we have

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega u^2 = 1.$$

Thus $u \not\equiv 0$ and by (4.2), u is a principal eigenfunction associated to $\lambda^\star(\mu_0)$. By the simplicity of $\lambda^\star(\mu_0)$, we have $u = u^\star(\mu_0)$, where $\|u^\star(\mu_0)\|_{H^1(\Omega)} = 1$. Finally by the uniform convexity of $H^1(\Omega)$, one concludes that

$u^\star(\mu) \to u^\star(\mu_0)$ in $H^1(\Omega)$. □

Now we study the differentiability of $\mu \to \lambda^\star(\mu)$.

Proposition 4.3 *The function $\mu \to \lambda^\star(\mu)$ is differentiable in $\mathcal{D}$ with*

$$(\lambda^\star)'(\mu) = -\frac{\int_\Omega m_1(\tilde{u}(\mu))^2}{\int_\Omega m_2(\tilde{u}(\mu))^2},$$

where $\tilde{u}(\mu)$ satisfies the conditions of the following lemma

Lemma 4.4 *Let $\mu, \mu_0 \in \mathcal{D}$, and let $u^\star(\mu)$ be an eigenfunction associated to $\lambda^\star(\mu)$ such that $u^\star(\mu) > 0$, with $\|u^\star(\mu)\|_{H^1(\Omega)} = 1$. And let $G^\star(\mu)$ be the function provided by Lemma 3.8 for $u = u^\star(\mu)$. If $\|G^\star(\mu)\|_{H^1(\Omega)} = 1$, then $G^\star(\mu) \to G^\star(\mu_0)$ strongly in $L^2(\Omega)$ and $\tilde{u}(\mu) := u^\star(\mu)\sqrt{G^\star(\mu)} \to \tilde{u}(\mu_0) := u^\star(\mu_0)\sqrt{G^\star(\mu_0)}$ strongly in $L^2(\Omega)$ as $\mu \to \mu_0$.*

Proof. [Proof of Lemma 4.4.] Assume that $p > \max(2, N)$. By Theorem 4.2, we have $u^\star(\mu) \to u^\star(\mu_0)$ strongly in $H^1(\Omega)$ as $\mu \to \mu_0$. The equation

$$Lu^\star(\mu) - \mu m_1(x)u^\star(\mu) = \lambda^\star(\mu)m_2(x)u^\star(\mu) \text{ in } \Omega$$

implies that

$$|div(A\nabla u^\star(\mu))|^2 + |u^\star(\mu)|^2 = |(\lambda^\star(\mu)m_2(x) + \mu m_1(x) - a_0)u^\star(\mu) - \langle a, \nabla u^\star(\mu)\rangle|^2 + |u^\star(\mu)|^2 \text{ in } \Omega.$$

Integrating and using the Hölder inequality, we obtain $\|u^\star(\mu)\|_{W^{2,p}(\Omega)} := (|div(A\nabla u^\star(\mu))|^2 + |u^\star(\mu)|^2)^{\frac{1}{2}} \leq c_\mu$, where $\mu \to c_\mu$ is a continuous function in $\mathcal{D}$. Thus $u^\star(\mu)$ is bounded in $W^{2,p}(\Omega)$. So, for a subsequence, $u^\star(\mu) \to u^\star$ weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ as $\mu \to \mu_0$. Moreover $u^\star(\mu) \to u^\star$ strongly in $H^1(\Omega)$ as $\mu \to \mu_0$. By unicity of the limit, we have $u^\star = u^\star(\mu_0)$. Consequently $u^\star(\mu) \to u^\star(\mu_0)$ strongly in $C(\bar{\Omega})$. On the other hand, $G^\star(\mu)$ is bounded in $H^1(\Omega)$. So, for a subsequence, $G^\star(\mu) \to G^\star$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. By (3.5), we have for all $v \in H^1(\Omega)$

$$\int_\Omega (u^\star(\mu))^2\langle A\nabla G^\star(\mu), \nabla v\rangle + \int_\Omega (u^\star(\mu))^2\langle aG^\star(\mu), \nabla v\rangle = 0.$$

Passing to the limit, we obtain

$$\int_\Omega (u^\star(\mu_0))^2\langle A\nabla G^\star, \nabla v\rangle + \int_\Omega (u^\star(\mu_0))^2\langle aG^\star, \nabla v\rangle = 0.$$

One deduces by Lemma 3.8 that $G^\star = G^\star(\mu_0)$ and $G^\star(\mu) \to G^\star(\mu_0)$ strongly in $L^2(\Omega)$. Thus $\tilde{u}(\mu) \to \tilde{u}(\mu_0)$ strongly in $L^2(\Omega)$. □

Proof. [Proof of Proposition 4.3.] Let $\tilde{u}(\mu) := u^\star(\mu)\sqrt{G^\star(\mu)}$, where $u^\star(\mu)$ is an eigenfunction associated to $\lambda^\star(\mu)$ such that $u^\star(\mu) > 0$ and $\|u^\star(\mu)\|_{H^1(\Omega)} = 1$. And $G^\star(\mu)$ is the function provided by Lemma 3.8 for $u = u^\star(\mu)$ with $\|G^\star(\mu)\|_{H^1(\Omega)} = 1$. By Remark 3.7, we have

$$\begin{aligned}
\lambda^\star(\mu) &= \frac{\Lambda(\tilde{u}(\mu)) - Q_{\tilde{u}(\mu)}(W_{\tilde{u}(\mu)}) + \int_{\partial\Omega} b_0(x)(\tilde{u}(\mu))^2 - \mu\int_\Omega m_1(\tilde{u}(\mu))^2}{\int_\Omega m_2(\tilde{u}(\mu))^2} \\
&\leq \frac{\Lambda(\tilde{u}(\mu_0)) - Q_{\tilde{u}(\mu_0)}(W_{\tilde{u}(\mu_0)}) + \int_{\partial\Omega} b_0(x)(\tilde{u}(\mu_0))^2 - \mu\int_\Omega m_1(\tilde{u}(\mu_0))^2}{\int_\Omega m_2(\tilde{u}(\mu_0))^2} \\
&= \frac{\Lambda(\tilde{u}(\mu_0)) - Q_{\tilde{u}(\mu_0)}(W_{\tilde{u}(\mu_0)}) + \int_{\partial\Omega} b_0(x)(\tilde{u}(\mu_0))^2 - \mu_0\int_\Omega m_1(\tilde{u}(\mu_0))^2}{\int_\Omega m_2(\tilde{u}(\mu_0))^2} \\
&+ (\mu_0 - \mu)\frac{\int_\Omega m_1(\tilde{u}(\mu_0))^2}{\int_\Omega m_2(\tilde{u}(\mu_0))^2} \\
&= \lambda^\star(\mu_0) + (\mu_0 - \mu)\frac{\int_\Omega m_1(\tilde{u}(\mu_0))^2}{\int_\Omega m_2(\tilde{u}(\mu_0))^2}.
\end{aligned}$$

In the same way, we have

$$\lambda^\star(\mu_0) \leq \lambda^\star(\mu) + (\mu - \mu_0)\frac{\int_\Omega m_1(\tilde{u}(\mu))^2}{\int_\Omega m_2(\tilde{u}(\mu))^2}.$$

Thus

$$(\mu_0 - \mu)\frac{\int_\Omega m_1(\tilde{u}(\mu))^2}{\int_\Omega m_2(\tilde{u}(\mu))^2} \leq \lambda^\star(\mu) - \lambda^\star(\mu_0) \leq (\mu_0 - \mu)\frac{\int_\Omega m_1(\tilde{u}(\mu_0))^2}{\int_\Omega m_2(\tilde{u}(\mu_0))^2}.$$

Dividing by $\mu_0 - \mu$. Letting $\mu \to \mu_0$ and using the Lemma 4.4, we obtain the result. □

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