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Structural properties of Bilateral Grand Lebesque Spaces

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Abstract

In this paper we study the multiplicative, tensor, Sobolev and convolution inequalities in certain Banach spaces, the so-called Bilateral Grand Lebesque Spaces. We also give examples to show the sharpness of these inequalities when possible.

Key word and phrases: Grand Lebesgue and rearrangement invariant spaces, Sobolev embedding theorem, convolution operator.

1. Introduction

Let (X, Σ, μ) be a σ-finite measure space. We suppose the measure μ to be non-trivial and diffuse. The latter means that, for all $A \in \Sigma$ such that $\mu(A) \in (0, \infty)$, there exists $B \subset A$ with $\mu(B) = \mu(A)/2$.

For a and b constants, $1 \le a < b \le \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous log-convex positive function such that $\psi(a+0)$ and $\psi(b-0)$ exist, with $\max\{\psi(a+0), \psi(b-0)\} = \infty$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$.

The Bilateral Grand Lebesgue Space (in notation, BGLS) $G_X(\mu; \psi; a, b) = G_X(\psi; a, b) = G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $h : X \to \mathbb{R}$ endowed with the norm

$$||h||_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a,b)} |h|_p / \psi(p), \quad |h|_p = \left[\int_X |h(x)|^p \, d\mu(x) \right]^{1/p}.$$

The $G(\psi)$ spaces with $\mu(X) = 1$ appeared in [12]; it was proved that in this case each $G(\psi)$ space coincides with certain exponential Orlicz space, up to norm equivalence. Partial cases of these spaces have been intensively studied, in particular, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.; see, e.g., [1],[4]–[9],[11], [13], [16], and a recent paper [3]. These spaces are also Banach function spaces and, moreover, rearrangement invariant (r.i.) (see [1, Ch. 1, §1]).

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [5]–[8], probability in Banach spaces [14], in the modern non-parametrical statistics, for example, in the so-called regression problem [15, 16].

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We shall be concerned not with all functions ψ described above but with an essential subset of these functions satisfying certain natural conditions.

Let again $a \ge 1,\ b \in (a,\infty]$, and let $\psi = \psi(p)$ be a positive continuous function on the *open* interval (a,b) such that there exists a measurable function $f : X \to \mathbb{R}$ for which

$$f(\cdot) \in \cap_{p\in(a,b)} L_p(X,\mu) = \cap_{p\in(a,b)} L_p,\ \psi(p) = |f|_p,\ p \in (a,b). \tag{1.1}$$

We say that the equality (1.1) and the function $f(\cdot)$ from (1.1) is the *representation* of the function ψ. The existence of representation implies, by the way, the log-convexity of ψ.

We denote the subset of all the functions ψ having representation by $\Psi = \Psi(a,b)$. For complete description of these functions see, for example, [15, p.p. 21–27], [16].

We also denote by $C_+ := C_+(a,b)$ the class of all continuous positive functions $v : (a,b) \to \mathbb{R}^1_+$ separated from zero, that is, $\inf_{p\in(a,b)} v(p) > 0$.

At the endpoints we need more in the case when $\psi(p) \to \infty$. This may occur when either $p \to a+$ or $p \to b-$ or in both cases. In detail, this means that, in the case, when $\psi(a+0) = \infty$ while $\psi(b-0) < \infty$, there holds

$$\lim_{p\to a+0} \psi(p)/\nu(p) = 1;$$

in the case when $\psi(b-0) = \infty$ while $\psi(a+0) < \infty$, there holds

$$\lim_{p\to b-0} \psi(p)/\nu(p) = 1;$$

and when in both cases $\psi(a+0) = \psi(b-0) = \infty$, there holds

$$\lim_{p\to a+0} \psi(p)/\nu(p) = \lim_{p\to b-0} \psi(p)/\nu(p) = 1.$$

Remark 1 *Observe that if $\psi_1 \in \Psi(a_1,b_1)$ and $\psi_2 \in \Psi(a_2,b_2)$, with $(a_1,b_1)\cap(a_2,b_2) = (a_3,b_3) \ne \emptyset$, then $\psi_1\psi_2 \in \Psi(a_3,b_3)$. Indeed, let $\psi_1(p) = |f_1|_p$ and $\psi_2(p) = |f_2|_p$. Among various representations one can choose such that functions f_1 and f_2 are independent in the probabilistic sense, that is, for all Borel sets A, B on the real axis $\mathbb{R}$*

$$\mu\{x:\ f_1(x) \in A,\ f_2(x) \in B\} = \mu\{x : f_1(x) \in A\}\ \mu\{x:\ f_2(x) \in B\}.$$

By this $\psi_1(p)\psi_2(p) = |f_1 f_2|_p,\ p \in (a_3,b_3)$.

We note that the $G(\psi)$ spaces are also interpolation spaces (the so-called Σ-spaces), see [1],[4]-[9],[13, 16], etc. However, we hope that our direct representation of these spaces is of certain convenience in both theory and applications. A natural question arises what happens if the spaces other than L_p are used in the definition.

Indeed, this is possible and might be of interest, but, for example, using Lorenz spaces in this capacity leads to the same object (see [16]).

The main goal of this paper is to prove new (and extend known) results on the Boyd indices, tensor, Sobolev embedding, multiplicative and convolution operators in BGLS spaces.

The paper is organized as follows. In the next section we start with an exemplary case just to give a feeling of what happens and then present the main results of above-mentioned type, each of these in a separate subsection. Further, the section follows where we prove the statements. In the last section we discuss sharpness of the obtained results.

We use symbols $C(X,Y),\ C(p,q;\psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X,Y)$ and $C_2(X,Y)$. The relation $g(\cdot) \asymp h(\cdot),\ p \in (A,B),$ where $g = g(p),\ h = h(p),\ g,h : (A,B) \to \mathbb{R}_+,$ denotes as usual

$$0 < \inf_{p \in (A,B)} h(p)/g(p) \le \sup_{p \in (A,B)} h(p)/g(p) < \infty.$$

The symbol $\sim$ will denote the usual equivalence in the limit sense.

2. Results

To get a flavor of the setting we work in, let, for instance, $X = \mathbb{R}^n,\ \sigma$ be a constant, and $\mu = \mu_\sigma$ be the measure on Borel subsets of X with density $d\mu_\sigma/dx = |x|^\sigma$. As usual, $x = (x_1, x_2, \ldots, x_n) \in X$ with $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. Let $L = L(z),\ z \in (0,\infty)$ be a slowly varying, as $z \to \infty$, positive continuous function, and $\chi_A = \chi_A(x) = 1$ for $x \in A$ and 0 otherwise be the indicator of the set A. Let

$$\begin{aligned}
f(x) = f_L(x) &= f(x; L, a, \alpha) = \chi_{\{|x|>1\}}(x)\ |x|^{-1/a}\ [\log(|x|)]^\alpha\ L(\log|x|),\\
g(x) = g_L(x) &= g(x; L, b, \beta) = \chi_{(-1,1)}(x)\ |x|^{-1/b}\ [|\log(|x|)|]^\beta\ L(|\log|x||),\\
A = A(a,n,\sigma) &= a(n+\sigma) \ge 1,\ B = B(b,n,\sigma) = b(n+\sigma) \in (A,\infty),\\
\gamma &= \alpha + 1/A,\ \delta = \beta + 1/B,\ p \in (A,B),
\end{aligned}$$

and

$$\psi_L(p) = \psi_L(p; A, B; \gamma, \delta) \stackrel{def}{=} (p-A)^{-\gamma}\ (B-p)^{-\delta}\ \max\{L(A/(p-A)), L(B/(B-p))\}.$$

The function $h_L(x) = h_L(x; a, b; \alpha, \beta) = f_L(x) + g_L(x)$ belongs to the space $G(\psi_L)$:

$$h_L(\cdot) \in G(A, B; \gamma, \delta) \stackrel{def}{=} G(\psi_L),$$

and this belonging is exact in the sense that for $p \in (A,B)$ there holds $|h_L|_p \asymp \psi_L(p)$.

Denoting now $\omega(n) = \pi^{n/2}/\Gamma(n/2+1)$ and $\Omega(n) = n\omega(n) = 2\pi^{n/2}/\Gamma(n/2)$, we set

$$R = R(\sigma, n) = [(\sigma+n)/\Omega(n)]^{1/(\sigma+n)}, \ \sigma + n > 0,$$

and let $h = h(|x|)$ be a non-negative measurable function vanishing for $|x| \geq R(\sigma, n)$. For $u \geq e^2$ let

$$\mu_\sigma\{x : h(|x|) > u)\} = \min\{1, \exp(-W(\log u))\},$$

where $W = W(z)$ is a twice differentiable strictly convex for $z \in [2, \infty)$ and strictly increasing function. Denoting by

$$W^*(p) = \sup_{z>2}(pz - W(z))$$

the Young-Fenchel transform of the function $W(\cdot)$, we define the function

$$\psi(p) = \exp(W^*(p)/p)\,.$$

It follows from the theory of Orlicz spaces ([15, p.p. 22–27]) that if for $p \in [a, \infty)$ we have $|h|_p \asymp \psi(p)$, then $h(\cdot) \in G(\psi; a, \infty)$, where, in particular, $G(\psi; 1, \infty)$ coincides with some exponential Orlicz space.

We will restrict ourselves to the case $b(n+\sigma) < \infty$, with $p \to a(n+\sigma) + 0$. Let

$$f(x) = f(x; L, a, \gamma) = \chi_{\{|x|>1\}}(x) \ |x|^{-1/a} \ (\log|x|)^\gamma \ L(\log|x|).$$

Using then multidimensional polar coordinates and well-known properties of slowly varying functions (see [18, Ch. 1, Sect. 1.4–1.5]), we obtain

$$\begin{aligned}
\Omega(n) \ ||f||_p^p &= \int_1^\infty r^{-p/a+n+\sigma-1} \ (\log r)^{\gamma \ p} \ L^p(\log r) \ dr \\
&= (p/a - n - \sigma)^{-\gamma p - 1} \int_0^\infty e^{-z} \ z^{\gamma p} \ L^p(az/(p-A)) \, dz \\
&\sim (p/a - n - \sigma)^{-\gamma p - 1} \ L^p(a/(p-A)) \int_0^\infty e^{-z} \ z^{\gamma p} \, dz \\
&= (p/a - n - \sigma)^{-\gamma p - 1} \ L^p(a/(p-A)) \ \Gamma(1 + \gamma \ p),
\end{aligned}$$

where the equivalence follows by the Lebesgue dominated convergence theorem. This yields for $p \in (A, B)$

$$|f_L|_p \asymp (p-A)^{-\gamma - 1/A} \ L(a/(p-A)).$$

2.1. Indices

In this subsection we give an expression for the so-called Boyd indices of $G(\psi; a, b)$ spaces. There are 2 such indices which can be defined only in special cases; for example in the case of $X = [0, \infty)$ with usual Lebesque measure. These indices play important role in the theory of interpolation of operators, in Fourier Analysis on r.i. spaces, etc. (see, e.g., [1, p.p. 22–31, 192–204]).

We recall the definitions. Given a family of (linear) operators $\{\sigma_s\}$ taking a r.i. space G into itself in accordance with $\sigma_s f(x) = f(x/s)$, for $s > 0$. Denoting $||\sigma_s|| = ||\sigma_s||_{G\to G}$, we define (see, e.g., [1, Ch. 3]) for $X = [0, \infty)$

$$\gamma_1(G) = \lim_{s\to 0+} \log ||\sigma_s||/\log s \quad \text{and} \quad \gamma_2(G) = \lim_{s\to\infty} \log ||\sigma_s||/\log s.$$

Theorem 2 *There holds for* $\psi \in \Psi(a, b)$

$$\gamma_1(G(\psi; a, b)) = 1/b, \ \gamma_2(G(\psi; a, b)) = 1/a.$$

In a more general case of $X = \mathbb{R}^n$ with $\mu = \mu_\sigma, \ \sigma \geq 0$, we analogously have

$$\gamma_1(G(\psi; a, b)) = (n + \sigma)/b \quad \text{and} \quad \gamma_2(G(\psi; a, b)) = (n + \sigma)/a.$$

2.2. Tensor and multiplicative inequalities

Let (X, Σ_1, μ) and (Y, Σ_2, ν) be two measure spaces with σ-finite measures μ and ν, respectively. Let $f = f(x) \in G_X(\psi_1; a_1, b_1)$ and $g = g(y) \in G_Y(\psi_2; a_2, b_2)$, where $x \in X, \ y \in Y, \ \psi_1 \in C_+(a_1, b_1), \ \psi_2 \in C_+(a_2, b_2)$, and let $a = \max(a_1, a_2) < \min(b_1, b_2) = b$. We set $\psi(p) \stackrel{def}{=} \psi_1(p)\,\psi_2(p)$ for $p \in (a, b)$. Obviously, if $\psi_1 \in C_+(a_1, b_1)$ and $\psi_2 \in C_+(a_2, b_2)$, then $\psi \in C_+(a, b)$; while if $\psi_1 \in \Psi(a_1, b_1)$ and $\psi_2 \in \Psi(a_2, b_2)$, then $\psi \in \Psi(a, b)$.

Let us consider the so-called *tensor product* of f, g : $z(x, y) \stackrel{def}{=} f(x)\ g(y)$. Since both functions f and g are independent on the space $(X \times Y, \ \Sigma_1 \times \Sigma_2, \ \zeta)$, with $\zeta = \mu \times \nu$, we have:

Lemma 1 *The following tensor inequality holds:*

$$||z||_{G(\psi;a,b)} \leq ||f||_{G(\psi_1;a_1,b_1)}||g||_{G(\psi_2;a_2,b_2)}.$$

This inequality is sharp, for example, when $\psi_1(p) = |f|_p = |f|_p(X, \Sigma_1, \mu)$ and $\psi_2(p) = |g|_p = |g|_p(Y, \Sigma_2, \nu)$. By the way, in this case $\psi_1 \in \Psi(a_1, b_1)$ and $\psi_2 \in \Psi(a_2, b_2)$.

We now go on to the so-called *multiplicative* inequality. Let again $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$, and let $a_1, a_2 \geq 1$ and $1/b_1 + 1/b_2 > 1$. We denote

$$A_1 = \max(1, a_1a_2/(a_1 + a_2)), \quad B_1 = b_1b_2/(b_1 + b_2),$$

and for $r \in (A_1, B_1)$, $\psi_1 \in C_+(a_1, b_1)$ and $\psi_2 \in C_+(a_2, b_2)$, define

$$\psi_3(r) = \inf_{p,q:\, p,q>1, 1/p+1/q=1} \{\psi_1(pr)\, \psi_2(qr)\}.$$

Observe that $\psi_3 \in C_+(A_1, B_1)$.

Theorem 3 *There holds*

$$||f\, g||_{G(\psi_3; A_1, B_1)} \le ||f||_{G(\psi_1; a_1, b_1)}\ ||g||_{G(\psi_2; a_2, b_2)}. \tag{2.2}$$

We mention that the sharpness of (2.2) up to multiplicative constant can be seen from letting $f = f_L \in G(\psi_1)$ and $g = g_L \in G(\psi_2)$ with $\psi_1 = \psi_L(p; a_1, b_1; \alpha_1, \beta_1)$ and $\psi_2 = \psi_L(p, a_2, b_2; \alpha_2, \beta_2)$. More precisely, in the considered case

$$||f_L\, g_L||_{G(\psi_3; A_1, B_1)} \ge C(L, a_1, a_2, b_1, b_2)\ ||f_L||_{G(\psi_1; a_1, b_1)}\ ||g_L||_{G(\psi_2; a_2, b_2)}. \tag{2.3}$$

We note, in addition that, if $f \in G(\psi; a, b)$ and for $\gamma = const \in [a, b]$ we have $\psi_\gamma(p) = \psi^\gamma(\gamma p)$, then $g(x) = f^\gamma(x) \in G(\psi_\gamma; 1, b/\gamma)$ and

$$||f^\gamma||_{G(\psi_\gamma)} = ||f||^\gamma_{G(\psi)}.$$

2.3. Sobolev embedding and convolution operators

Let X be a convex domain in $\mathbb{R}^n, n \ge 2$, with smooth boundary, B be the projection operator on the m-dimensional smooth (piece-wise C^∞) convex sub-manifold Y of X, $m \le n$,

$$Bu(x) = u(x), \qquad x \in Y,$$

endowed with the corresponding surface measure θ, and let $\psi = \psi(p; a, b) \in C_+(a, b)$, $1 \le a < b < n$. We denote $A_2 = \max(1, am/(n-a))$ and $B_2 = bm/(n-b)$, $\eta(q) = q^{1-1/n}\psi(qn/(q+m))$, with $q \in (A_2, B_2)$. By this $\eta \in C_+(A_2, B_2)$. Let $u = u(x) \in C_0^1(X)$, i.e., $u(\cdot)$ is continuously differentiable and $\lim_{|x|\to\infty} u(x) = 0$.

Theorem 4 *The following Sobolev type inequality holds:*

$$||Bu||_{G_Y(\theta, \eta; A_2, B_2)} \le C(X, Y; \psi)\, ||\, |\mathrm{grad}\, u|\, ||_{G_X(\mu, \psi; a, b)}. \tag{2.4}$$

This result is supplied with the following interesting assertion, contrary to that for the classical L_p spaces.

Theorem 5 *Let $\psi \in \Psi(a, b)$ and either $\lim_{p\to a+0} \psi(p) = \infty$ or $\lim_{p\to b-0} \psi(p) = \infty$. The corresponding embedding Sobolev operator B is not compact as an operator taking $G_X(\mu, \psi; a, b)$ into $G_Y(\theta, \eta; A_2, B_2)$.*

We now consider the (bilinear) generalized convolution operator of the form

$$v(x) = (f * g)(x) = \int_X g(xy^{-1}) f(y)\, d\mu(y),$$

where X is a unimodular Lie group, μ is the associated Haar measure. The unimodularity means, in particular, that μ is bilaterally invariant. For the commutative group X, with standard notation $y^{-1} = -y,\ xy^{-1} = x - y$, this definition coincides with the classical definition of convolution.

Let $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$, with $\psi_1 \in \Psi(a_1, b_1)$ and $\psi_2 \in \Psi(a_2, b_2)$, provided $1/a_1 + 1/a_2 > 1$ and $1/b_1 + 1/b_2 > 1$. We denote

$$A_3 = a_1 a_2/(a_1 + a_2 - a_1 a_2) \quad \text{and} \quad B_3 = b_1 b_2/(b_1 + b_2 - b_1 b_2),$$

and define for the values $r \in (A_3, B_3)$ and $p, q > 1$ with $1/p + 1/q = 1 + 1/r$

$$\tau(r) = \inf_{p,q} \{\psi_1(p)\, \psi_2(q)\}.$$

It is evident that $\tau \in \Psi(A_3, B_3)$.

Theorem 6 *There holds*

$$||f * g||_{G(\tau; A_3, B_3)} \le ||f||_{G(\psi_1; a_1, b_1)}\ ||g||_{G(\psi_2; a_2, b_2)}. \tag{2.5}$$

3. Proofs

Proof of Theorem 2.

The assertion of the theorem follows immediately from the identity

$$||\sigma_s|| = \max\left(s^{1/a}, s^{1/b}\right), \quad s > 0, \quad \psi \in \Psi, \tag{3.6}$$

where

$$||\sigma_s|| \stackrel{def}{=} ||\sigma_s||_{G(\psi) \to G(\psi)}.$$

It remains to prove (3.6). The upper bound is obtained as follows. Let $f : f \in G(\psi),\ ||f||_{G(\Psi)} \ne 0$. We have

$$|\sigma_s f|_p^p = \int_0^\infty |f(x/s)|^p\ dx = s \int_0^\infty |f(y)|^p\ dy.$$

It follows from

$$\begin{aligned} |\sigma_s f|_p &= s^{1/p}|f|_p \le \max\left(s^{1/a}, s^{1/b}\right)\ |f|_p \\ &\le \max\left(s^{1/a}, s^{1/b}\right)\ \psi(p)\,||f||_{G(\psi)} \end{aligned}$$

that

$$||\sigma_s|| \le \max\left(s^{1/a}, s^{1/b}\right).$$

For the lower bound, let $g(\cdot)$ be a representation of $\psi:\ |g|_p = \psi(p),\ p \in (a,b)$; then $||g||_{G(\psi)} = 1$ and

$$\begin{aligned} ||\sigma_s|| \ge ||\sigma_s\ g||_{G(\psi)} &= \sup_{p\in(a,b)} [\ |\sigma_s\ g|_p/\psi(p)\ ] \\ = \sup_{p\in(a,b)} [s^{1/p}|g|_p/\psi(p)] &= \sup_{p\in(a,b)} s^{1/p} = \max\left(s^{1/a}, s^{1/b}\right). \end{aligned}$$

The proof is complete. □

The proofs Theorems 3, 4 and 6 go along similar lines and are strongly based on definitions and preliminary matter given above.

Proof of Theorem 3. Let $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$. By definition of these spaces,

$$|f|_p \le \psi_1(p)\,||f||_{G(\psi_1)}, \quad p \in (a_1, b_1); \tag{3.7}$$

$$|g|_q \le \psi_2(q)\,||g||_{G(\psi_1)}, \quad q \in (a_2, b_2). \tag{3.8}$$

It follows from Hölder's inequality that for $r \in (A_1, A_2)$

$$|f\,g|_r \le |f|_{pr}\ |g|_{qr} \le \psi_1(pr)\ \psi_2(qr)\,||f||_{G(\psi_1)}\ ||g||_{G(\psi_2)}.$$

Minimizing the right-side over p and q provided $p, q > 1$ and $1/p + 1/q = 1$, we obtain the desired assertion. □

Proof of Theorem 4. Here we will use the known Sobolev inequality in the L_p spaces (see, e.g., [10, Part 2, Ch. 11, Sect. 4] or [19] and [17] for newer and more extended versions) which can be rewritten as

$$|Bu|_{L_q(Y,\theta)} \le C_1(X,Y)\,q^{1-1/n}\,|\,|\mathrm{grad}\,u|\,|_{L_p(X,\mu)}, \quad p = qn/(q+m).$$

for the values $q \in (A_2, B_2)$. Let $|\mathrm{grad}\ u| \in G(\psi; a, b)$ with $b < n$ (the case $b = n$ can be treated analogously); we then have

$$\begin{aligned}|Bu|_q &\le C_2(X,Y)\, ||\,|\mathrm{grad}\, u|\,||_{G(\psi)}\, q^{1-1/n}\, \psi(qn/(q+m))\\ &= C_2(X,Y)\, ||\,|\mathrm{grad}\, u|\,||_{G(\psi)}\, \eta(q);\end{aligned}$$

and

$$||Bu||_{G(\eta)} \le C_2(X,Y)\, ||\,|\mathrm{grad}\, u|\,||_{G(\psi)}.$$

This completes the proof. □

Proof of Theorem 6. With (3.7) and (3.8) in hand, using classical Young's inequality (see, e.g., [2] where additional details are given), we obtain

$$|f * g|_r \le C(p,q)\; |f|_p\, |g|_q, \quad C(p,q) \le 1,\; 1 + 1/r = 1/p + 1/q. \tag{3.9}$$

We just mention that in the case $X = \mathbb{R}^n$, with $s = p/(p-1)$, $t = q/(q-1)$, and $z = r/(r-1)$, the constant $C(p,q)$ is explicitly indicated in [2] as

$$C(p,q) = \left[p^{1/p} s^{-1/s} q^{1/q} t^{-1/t} r^{1/r} z^{-1/z}\right]^{n/2}.$$

It follows from (3.9) that

$$|f * g|_r \le \psi_1(p)\, \psi_2(q)\, ||f||_{G(\psi_1)} ||g||_{G(\psi_2)}.$$

The proof can now be completed as above, by minimizing over p and q, where $p, q > 1$ and $1/p + 1/q = 1 + 1/r$. □

Proof of Theorem 5. Let $\psi \in \Psi(a,b)$ and either $\lim_{p \to a+0} \psi(p) = \infty$ or $\lim_{p \to b-0} \psi(p) = \infty$. We introduce the Sobolev-Grand Lebesque spaces $W_1(\psi)$ with the (finite) norm of a function $u = u(x)$, defined on X,

$$||u||_{W_1(\psi)} = ||\,|\mathrm{grad}\, u|\,||_{G(\psi)} + ||u||_{G(\psi)}.$$

We are going to prove that the classical Sobolev embedding operator $S:\; W_1(\psi) \to G(\eta)$, $Su = Bu$, is not compact, unlike in the case of classical L_p spaces.

This fact follows from the assertion that for the considered function $u = u(|x|)$ the family of "small" shifts $\{T_\varepsilon\, u(|x|) = u(\varepsilon + |x|)\}$, with $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, has a positive distance in both $G(\psi)$ and $W_1(\psi)$ spaces: there exists $C > 0$ such that

$$||T_\epsilon u - T_\delta u||_{G(\psi)} \ge C(\varepsilon_0)$$

and

$$||T_\varepsilon u - T_\delta u||_{W_1(\psi)} \geq C,$$

where $\varepsilon, \delta \in (0, \epsilon_0),\ \epsilon \neq \delta.$

In order to prove the first of these inequalities (the second one can be proved analogously), we introduce certain subspaces of the $G(\psi; a, b)$ space. Let

$$G_t(\psi) = G_t(\psi; a, b) = \{f : \lim_{\delta \to 0+} \sup_{A:\mu(A) \leq \delta} ||f\,\chi_A||_{G(\psi)} = 0\}.$$

Let $GB(\psi) = GB(\psi; a, b)$ be the set of all $f,\ f : X \to \mathbb{R}$, such that for all $\varepsilon > 0$ there exist $B \in (0, \infty)$ and $A \in \Sigma,\ \mu(A) \leq B$, and measurable $g,\ g : X \to \mathbb{R}$ vanishing off A, for which $\sup_x |g(x)| < B$ and $||f - g|| < \varepsilon$. Let also

$$G^0(\psi; a, b) = G^0(\psi) = \{f : \lim_{\psi(p) \to \infty} |f|_p / \psi(p) = 0\}.$$

The spaces $G_t(\psi),\ GB(\psi),\ G^0(\psi)$ are closed subspaces of $G(\psi)$. We denote here by h a function $h : X \to \mathbb{R}$ such that $|h|_p = \psi(p),\ p \in (a, b)$.

It follows from the theory of r.i. spaces ([1, Ch. 1, p.p. 22–28]) that if $\psi \in \Psi$, then

$$G_t(\psi) = GB(\psi) = G^0(\psi) \neq G(\psi).$$

Without loss of generality we can assume $X = (0, 2\pi],\ \sigma = 0$, and define also $x \pm y = x \pm y (\mathrm{mod}\, 2\pi)$ for $x, y \in X$.

There exists a function $u = u(x)$ such that $u(\cdot) \in G(\psi) \setminus G^o(\psi),\ ||u||_{G(\Psi)} = 1$. Then (see [1, Ch.3, p.p. 192–198])

$$\inf_{\varepsilon \neq \delta} ||T_\varepsilon u - T_\delta u||_{G(\psi)} = \inf_{\varepsilon \neq \delta} ||T_{\varepsilon - \delta} u - u||_{G(\psi)} > 0,$$

which completes the proof. □

4. Sharpness

We will discuss either the sharpness or lack of that for obtained results. Since the sharpness of Theorems 2, 3 and 4 was obtained readily, we did not postpone it and gave immediately after the formulations.

Let us demonstrate (briefly) the sharpness of Theorem 6 by considering only the case $n = 1$ and $\sigma = 0$. With $\gamma_1, \gamma_2 \geq 0$ and $b_1, b_2 > 1$, let us denote $h(t) = (f * g)(t)$ and define

$$f(x) = \chi_{(0,1)}(x)\; x^{-1/b_1}\; |\log x|^{\gamma_1},$$
$$g(x) = \chi_{(0,1)}(x)\; x^{-1/b_2}\; |\log x|^{\gamma_2}.$$

It suffices to consider the case $t \in (0,1/4)$, since on $[1/4,2]$ the function $h = h(t)$ is bounded and for $t \in (-\infty,0)\cup(2,\infty)$ this function vanishes.

For $t \to 0+$ we get

$$\begin{aligned} h(t) &= \int_0^t x^{-1/b_1}|\log x|^{\gamma_1}\; (t-x)^{-1/b_2}\; |\log(t-x)|^{\gamma_2}\; dx \\ &= t^{1-1/b_1-1/b_2}\int_0^1 y^{-1/b_1}\; (1-y)^{-1/b_2}\, |\log t + \log y|^{\gamma_1}\; |\log t + \log(1-y)|^{\gamma_2}\, dy \\ &\sim t^{1-1/b_1-1/b_2}\; |\log t|^{\gamma_1+\gamma_2}\int_0^1 y^{-1/b_1}\; (1-y)^{-1/b_2}\, dy \\ &= B(1-1/b_1, 1-1/b_2)\; t^{1-1/b_1-1/b_2}\; |\log t|^{\gamma_1+\gamma_2}, \end{aligned}$$

where the equivalence follows from the Lebesgue dominated convergence theorem and $B(\cdot,\cdot)$ stands for the beta-function. Taking then the $|\cdot|_p$ norm, we obtain

$$|f*g|_p \sim C(B_3-p)^{-\gamma_1-\gamma_2-1/b_1-1/b_2+1},\; p\in[1,B_3),\; p\to B_3-0.$$

The situation is more complicated with (2.4) and (2.5). Roughly speaking, we can present examples when the inequalities are achieved but for the same examples the actual bounds are better. In other words, the sharpness of these inequalities is an open problem.

We first analyze (2.4). Let $\psi(p) = (p-a)^{-\alpha}\; (b-p)^{-\beta},\; \sigma = 0,\; 1 \le a < b < n.$ Consider the function $u = u(|x|)$, with $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, for which $|\text{grad } u| \in G(a,b;\alpha,\beta)$, i.e., for $p \to a+0$ and $p \to b-0$ we have $|\; |\text{grad } u|\; |_p \sim (p-a)^{-\alpha}\; (b-p)^{-\beta}$, then it follows from the inequality (2.4) that

$$|Bu|_p \le C(p-A_2)^{-\alpha}\; (B_2-p)^{-\beta},\; p\in(A_2,B_2).$$

However, in the considered case for the same range of p

$$|Bu|_p \sim C(p-A_2)^{-\alpha+1/n}\,(B_2-p)^{-\beta+1/n}.$$

In the same way, analogous examples may be constructed in the cases when either $b_1 = \infty$ or $b_2 = \infty$.

Therefore, the *bounds* A_2 and B_2 are, in general, exact, but between the *exponents* obtained $-\alpha$ and $-\beta$ on one side and $-\alpha+1/n$ and $-\beta+1/n$ in the example on the other side there is the $1/n$ “gap”.

However, for all dimensions n at once (2.4) is sharp.

Almost the same example works for (2.5). Let $X = R,\ \sigma = 0,\ \gamma_1, \gamma_2 \geq 0,\ 1 \leq a_1, a_2 < b_1, b_2 < \infty,$ and $1/b_1 + 1/b_1 > 1$. Considering the same functions f and g but with argument $|x|$ in place of x and, correspondingly, $\chi_{(-1,1)}$, we then have

$$f \in G(1, b_1; 0, \gamma_1 + 1/b_1),\ g \in G(1, b_2; 0, \gamma_2 + 1/b_2).$$

It follows from (2.5) that for $p \in [1, B_3)$

$$|h|_p \leq C(B_3 - p)^{-\gamma_1 - \gamma_2 - 1/b_1 - 1/b_2}.$$

In fact

$$|h|_p \sim C(B_3 - p)^{-\gamma_1 - \gamma_2 - 1/b_1 - 1/b_2 + 1}.$$

Analogously, if $h = f * g,$ where

$$\begin{aligned} f(x) &= \chi_{\{|x|>1\}}(x)\ |x|^{-1/a_1}\ |\log|x|\ |^{\gamma_1}, \\ g(x) &= \chi_{(-1,1)}(x)\ |x|^{-1/a_2}\ |\log|x|\ |^{\gamma_2}, \end{aligned}$$

then

$$f \in G(a_1, b_1; \gamma_1 + 1/a_1, 0),\ g \in G(a_2, b_2; \gamma_2 + 1/b_2, 0).$$

It follows from (2.5) that

$$|h|_p \leq C(p - A_3)^{-\gamma_1 - \gamma_2 - 1/a_1 - 1/a_2},$$

but in fact

$$|h|_p \sim C(p - A_3)^{-\gamma_1 - \gamma_2 - 1/a_1 - 1/a_2 + 1}.$$

Therefore, like above, the *bounds* A_3 and B_3 are in this case exact, but between the *exponents* there is a 1 “gap”.

Note that in the considered case the ”gap” does not tends to zero as $n \to \infty$, contrary to the previous (convolution) case.

Finding sharp estimates in these two cases is an interesting open problem.

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