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Direct and inverse theorems for the Bézier variant of certain summation-integral type operators

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Abstract

Recently, the Bézier variant of some well known operators were introduced (cf. [8]–[9])and their rates of convergence for bounded variation functions have been investigated (cf. [2], [10]). In this paper we establish direct and inverse theorems for the Bézier variant of the operators M_n introduced in [5] in terms of Ditzian-Totik modulus of smoothness $\omega_{\varphi^\lambda}(f,t)(0 \leqslant \lambda \leqslant 1)$. These operators include the well known Baskakov-Durrmeyer and Szász-Durrmeyer type operators as special cases.

Key Words: Degree of approximation, Ditzian-Totik modulus of continuity.

1. Introduction

In order to approximate Lebesgue integrable functions on the interval $[0, \infty)$, Gupta and Mohapatra [5] considered the operators

$$M_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^{\infty} b_{n,k}(t,c) f(t)\, dt, \tag{1.1}$$

where $p_{n,k}(x,c) = (-1)^k \frac{x^k}{k!} \varphi_{n,c}^{(k)}(x)$, $b_{n,k}(t,c) = (-1)^{k+1} \frac{t^k}{k!} \varphi_{n,c}^{(k+1)}(t)$ and

(i) for $c > 0, \varphi_{n,c}(x) = (1+cx)^{-n/c}$ and $x \in [0, \infty)$;

(ii) for $c = 0, \varphi_{n,c}(x) = e^{-nx}$ and $x \in [0, \infty)$.

Here we observe that, for the case $c > 0$, the operators M_n reduce to Baskakov-Durrmeyer operators; and when $c = 0$ these become Szász-Durrmeyer type operators. Some approximation properties of these operators were studied in [6]. The rate of convergence by the operators M_n for the particular value $c = 1$ was studied in [4].

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For $\alpha \geqslant 1$, and $f \in L_B[0,\infty)$, the class of all bounded Lebesgue integrable functions on the positive real line, the Bézier variant $M_{n,\alpha}$ of the operators M_n is defined by

$$M_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x,c) \int_0^{\infty} b_{n,k}(t,c) f(t)\, dt, \tag{1.2}$$

where $Q_{n,k}^{\alpha}(x,c) = J_{n,k}^{\alpha}(x,c) - J_{n,k+1}^{\alpha}(x,c)$ with $J_{n,k}(x,c) = \sum_{\nu=k}^{\infty} p_{n,\nu}(x,c)$.

For $\alpha = 1$, the operators $M_{n,\alpha}$ reduce to the operators M_n.

In order to make the paper self contained we recall the definitions of the unified K-functional and the Ditzian-Totik modulus of smoothness (cf. [3]).

Let $\varphi(x) = \sqrt{x(1+cx)}, 0 \leqslant \lambda \leqslant 1$, then

$$\begin{aligned} \omega_{\varphi^\lambda}(f,t) &= \sup_{0<h\leqslant t} \sup_{x-h\varphi^\lambda(x)/2\geqslant 0} \left|\tilde{\Delta}_{h\varphi^\lambda(x)} f(x)\right| \\ &= \sup_{0<h\leqslant t} \sup_{x-h\varphi^\lambda(x)/2\geqslant 0} \left| f\left(x + \frac{h\varphi^\lambda(x)}{2}\right) - f\left(x - \frac{h\varphi^\lambda(x)}{2}\right)\right|, \end{aligned}$$

where $0 \leqslant \lambda \leqslant 1$, $\varphi(x)$ is an admissible weight function of Ditzian-Totik modulus of smoothness and the corresponding K-functional is defined as

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda} \left\{\|f-g\| + t\|\varphi^\lambda g'\|\right\},\ t \in (0,\infty),$$

where $W_\lambda = \left\{g : g \in AC_{\text{loc}}, \|\varphi^\lambda g'\| < \infty\right\}$.

It is well known that (cf. [3]) there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\varphi^\lambda}(f,t) \leqslant K_{\varphi^\lambda}(f,t) \leqslant C\,\omega_{\varphi^\lambda}(f,t). \tag{1.3}$$

Our main result is the following theorem.

Theorem 1 *Let $f \in L_B[0,\infty), \varphi(x) = \sqrt{x(1+cx)}, 0 \leqslant \lambda \leqslant 1, c \geqslant 0$ and $0 < \beta < 1$. Then, there holds the implication (i) $\Leftrightarrow$ (ii) in the following statements:*

(i) $\left|M_{n,\alpha}(f,x) - f(x)\right| = O\left(\frac{\alpha^{1/2}\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)$

(ii) $\omega_{\varphi^\lambda}(f,x) = O(x^\beta)$.

Corollary 1 *For $\alpha = 1, \lambda = 0,$ and $c = 0$ we get, in particular, the following error estimate for the Szász-Durrmeyer type operators, obtained in [7]:*

$$\left|M_n(f,x) - f(x)\right| \leqslant \omega\left(f, \sqrt{\frac{x}{n}}\right).$$

Corollary 2 *For $\alpha = 1, \lambda = 0,$ and $c = 1,$ the following error estimate for the Baskakov Durrmeyer operators is obtained as in [4]:*

$$\left|M_n(f,x) - f(x)\right| \leqslant C\omega\left(f, \sqrt{\frac{x(1+x)}{n}}\right).$$

Section 2 of this paper contains some definitions and auxiliary results. In Section 3 we establish our main theorem. Further, the constant C is not the same at each occurrence.

2. Preliminaries

In this section we give some Lemmas and their corollaries which will be used in our main theorem.

Lemma 1 *[5] For $m \in N \cup \{0\},$ if we define the m-th order moment for the operators M_n by*

$$\mu_{n,m}(x,c) = \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^{\infty} b_{n,k}(t,c)(t-x)^m \, dt$$

then

$$\mu_{n,0}(x,c) = 1 \;\; \mu_{n,1}(x,c) = \frac{1+cx}{n-c}$$

and

$$\mu_{n,2}(x,c) = \frac{2cx^2(n+c) + 2x(n+2c) + 2}{(n-c)(n-2c)}.$$

Also, there holds the following recurrence relation

$$\begin{aligned} [n - c(m+1)]\mu_{n,m+1}(x,c) &= x(1+cx)[\mu^{(1)}_{n,m}(x,c) + 2m\mu_{n,m-1}(x,c)] \\ &+ [(1+2cx)(m+1) - cx]\mu_{n,m}(x,c), \;\; n > c(m+1). \end{aligned}$$

Corollary 3 *If $c \geqslant 0$ and $K > 2,$ then for sufficiently large $n,$ we have*

$$\mu_{n,2}(x,c) \leqslant \frac{K\varphi^2(x)}{n}. \tag{2.1}$$

Lemma 2 *For the functions $J_{n,k}(x,c)$ and $Q^{\alpha}_{n,k}(x,c)$, we have*

$$1 = J_{n,0}(x,c) > J_{n,1}(x,c) > ... > J_{n,k}(x,c) > J_{n,k+1}(x) > ... \tag{2.2}$$

$$0 < Q^{\alpha}_{n,k}(x,c) < \alpha\, p_{n,k}(x,c),\ \ \alpha \geqslant 1, \tag{2.3}$$

$$M'_{n,\alpha}(1,x) = 0 \tag{2.4}$$

$$\begin{aligned} \left|M'_{n,\alpha}(f,x)\right| &\leqslant \alpha \Bigg| \sum_{k=0}^{\infty} \big(J^{\alpha-1}_{n,k}(x,c) - J^{\alpha-1}_{n,k+1}(x)\big) J'_{n,k+1}(x) \times \\ &\quad \times \int_0^{\infty} f(t) b_{n,k}(t,c)\,dt + M'_n(f,x) \Bigg|. \end{aligned} \tag{2.5}$$

Proof. (2.2-2.4) are easy to prove therefore we leave their proofs. Now, from definition of $Q^{\alpha}_{n,k}(x,c)$ and in view of the inequality

$$|a^{\alpha} - b^{\alpha}| \leqslant \alpha |a-b| \text{ with } 0 \leqslant a, b \leqslant 1 \text{ and } \alpha \geqslant 1 \tag{2.6}$$

(cf. [9], Lemma 3) we have

$$\begin{aligned} Q^{\alpha}_{n,k}(x,c) &= J^{\alpha}_{n,k}(x,c) - J^{\alpha}_{n,k+1}(x,c) \\ &\leqslant \alpha\left(J_{n,k}(x,c) - J_{n,k+1}(x,c)\right) = \alpha\, p_{n,k}(x,c). \end{aligned}$$

Again,

$$\begin{aligned} M'_{n,\alpha}(f,x) &= \sum_{k=0}^{\infty} Q'^{\alpha}_{n,k}(x,c) \int_0^{\infty} b_{n,k}(t,c) f(t)\,dt \\ &= \alpha \sum_{k=0}^{\infty} \left\{ J^{\alpha-1}_{n,k}(x,c) J'_{n,k}(x,c) - J^{\alpha-1}_{n,k+1}(x,c) J'_{n,k+1}(x,c) \right\} \times \\ &\qquad \times \int_0^{\infty} b_{n,k}(t,c) f(t)\,dt \\ &= \alpha \sum_{k=0}^{\infty} \Bigg[\left\{ J^{\alpha-1}_{n,k}(x,c) - J^{\alpha-1}_{n,k+1}(x,c) \right\} J'_{n,k+1}(x,c) \times \\ &\qquad \times \int_0^{\infty} b_{n,k}(t,c) f(t)\,dt \\ &+ \left\{ J'_{n,k}(x,c) - J'_{n,k+1}(x,c) \right\} J^{\alpha-1}_{n,k}(x,c) \int_0^{\infty} b_{n,k}(t,c) f(t)\,dt \Bigg]. \end{aligned}$$

□

Now, (2.5) follows in view of (2.2).

Corollary 4 *From (2.1) and (2.3), it follows that*

$$M_{n,\alpha}\big((t-x)^2, x\big) \leqslant \alpha \frac{C\varphi^2(x)}{n},\ \ C > 2.$$

Lemma 3 *For the functions $A_{m,n}(x)$ given by*

$$A_{m,n}(x) \equiv n^m \sum_{\nu=0}^{\infty} \left(\frac{\nu}{n} - x\right)^m p_{n,\nu}(x,c),$$

we have $A_{0,n}(x) = 1,\ A_{1,n}(x) = 0$ and there holds the recurrence relation

$$A_{m+1,n}(x) = \varphi^2(x) \left[A'_{m,n}(x) + n\, m\, A_{m-1,n}(x)\right], \tag{2.7}$$

where $m \geqslant 1, x \in [0, \infty)$ and $\varphi^2(x) = x(1 + cx)$.

Corollary 5 *From the recurrence relation (2.7) there holds*

$$A_{2m,n}(x) \leqslant C_m\, n^m \varphi^{2m}(x), \forall\, m \in N^0,$$

where C_m is a constant that depends on m.

Using induction on m in the recurrence relation (2.7) this result follows easily hence details are omitted.

Lemma 4 *For $f \in W_\lambda, \varphi(x) = \sqrt{x(1 + cx)}, 0 \leqslant \lambda,\ t, x > 0,$ we have*

$$\left|\int_x^t f'(u)\, du\right| \leqslant 2\Big(x^{-\lambda/2}(1 + ct)^{-\lambda/2} + \varphi^{-\lambda}(x)\Big)|t - x|\big\|\varphi^\lambda f'\big\|.$$

Proof. In view of Hölder's inequality, we have

$$\begin{aligned} \left|\int_x^t f'(u)\, du\right| &\leqslant \big\|\varphi^\lambda f'\big\| \left|\int_x^t \frac{du}{\varphi^\lambda(u)}\right| \\ &\leqslant \big\|\varphi^\lambda f'\big\| |t - x|^{1-\lambda} \left|\int_x^t \frac{du}{\varphi(u)}\right|^\lambda. \end{aligned}$$

Since,

$$\left|\int_x^t \frac{du}{\varphi(u)}\right| \leqslant \left|\int_x^t \frac{du}{\sqrt{u}}\right| \left(\frac{1}{\sqrt{1 + cx}} + \frac{1}{\sqrt{1 + ct}}\right)$$

and

$$\left|\int_x^t \frac{du}{\sqrt{u}}\right| \leqslant \frac{2|t - x|}{\sqrt{x}},$$

using the inequality $|a + b|^p \leqslant |a|^p + |b|^p,\ 0 \leqslant p \leqslant 1,$ we get

$$\begin{aligned} \left|\int_x^t f'(u)\, du\right| &\leqslant \big\|\varphi^\lambda f'\big\| |t - x| \frac{2^\lambda}{x^{\lambda/2}} \left|\frac{1}{\sqrt{1 + cx}} + \frac{1}{\sqrt{1 + ct}}\right|^\lambda \\ &\leqslant \big\|\varphi^\lambda f'\big\| |t - x| \frac{2^\lambda}{x^{\lambda/2}} \big((1 + ct)^{-\lambda/2} + (1 + cx)^{-\lambda/2}\big). \end{aligned}$$

Hence the lemma follows. □

Lemma 5 *For any non negative real number m, there holds the ineguality*

$$M_{n,\alpha}\big((1+c\,t)^{-m},x\big)\leqslant K_m(1+c\,x)^{-m}, \tag{2.8}$$

where K_m is a constant depending on m only.

Proof. For $c=0$, there is nothing to prove. Hence we assume $c>0$. From the definition of $M_{n,\alpha}$, we get

$$\begin{aligned} M_{n,\alpha}\big((1+c\,t)^{-m},x\big) &= \sum_{k=0}^{\infty}\frac{Q_{n,k}^{\alpha}(x,c)\prod_{i=0}^{k}(n+ci)}{k!}\int_0^{\infty}\frac{t^k}{(1+ct)^{\frac{n}{c}+k+1+m}}\,dt,\\ &= \sum_{k=0}^{\infty}Q_{n,k}^{\alpha}(x)\,\frac{\Gamma(\frac{n}{c}+m)}{\Gamma(\frac{n}{c}+m+k+1)}\frac{\prod_{i=0}^{k}(n+ci)}{c^{k+1}}. \end{aligned}$$

Now,

$$\begin{aligned} Q_{n,k}^{\alpha}(x) &= J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)\\ &= \left(\sum_{\nu=k}^{\infty}\frac{x^{\nu}}{\nu!}n(n+c)...(n+c(\nu-1))(1+c\,x)^{\frac{-n}{c}-\nu}\right)^{\alpha}\\ &\quad-\left(\sum_{\nu=k+1}^{\infty}\frac{x^{\nu}}{\nu!}n(n+c)...(n+c(\nu-1))(1+c\,x)^{\frac{-n}{c}-\nu}\right)^{\alpha}\\ &\leqslant \frac{\alpha(1+c\,x)^{-\frac{n}{c}-k}}{k!}n(n+c)...(n+c(k-1))\quad(\text{using}(2.6))\\ &\leqslant \alpha(1+cx)^{-m}\left(\frac{\alpha(1+c\,x)^{m-\frac{n}{c}-k}}{k!}n(n+c)...(n+c(k-1))\right). \end{aligned}$$

Hence, we have the estimate

$$\begin{aligned} &M_{n,\alpha}\big((1+c\,t)^{-m},x\big)\\ &\leqslant \alpha(1+c\,x)^{-m}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{c}+m)\prod_{i=0}^{k-1}(n+ci)^2(n+ck)}{\Gamma(\frac{n}{c}+m+k+1)c^{k+1}}\times\\ &\times\frac{(1+c\,x)^{m-\frac{n}{c}-k}}{k!}. \end{aligned} \tag{2.9}$$

The series in the right hand side of (2.9) is convergent.

This follows the lemma. □

Lemma 6 *For the functions* $J_{n,k}(x,c)$ *and* $p_{n,k}(x,c)$, *there hold the relations:*

(i) $\varphi^2(x)\sum_{\nu=k}^{\infty} p'_{n,\nu}(x) = \sum_{\nu=k}^{\infty}(\nu - nx)p_{n,\nu}(x,c)$;

(ii) $(1+cx)J'_{n,k}(x,c) + nJ_{n,k}(x,c) = nJ_{n,k-1}(x,c) + cxJ'_{n,k-1}(x,c)$,

where $c \geqslant 0, \varphi^2(x) = x(1+cx)$.

Proof. The relation (i) is easy to prove, hence the proof is omitted.

We consider the case $c > 0$ as (ii) is true for $c = 0$. We have

$$\begin{aligned} I &= \sum_{\nu=k}^{\infty} \nu p_{n,\nu}(x,c) \\ &= \sum_{\nu=k}^{\infty} \frac{(-1)^{\nu}(x^{\nu})}{(\nu-1)!}\frac{\partial^{\nu}}{\partial x^{\nu}}(1+cx)^{-n/c} \\ &= -\frac{cx}{1+cx}\sum_{\nu=k}^{\infty}\frac{(-1)^{\nu-1}(x^{\nu-1})}{(\nu-1)!}\frac{\partial^{\nu-1}}{\partial x^{\nu-1}}(1+cx)^{-n/c} \\ &= \frac{cx}{1+cx}\sum_{\nu=k-1}^{\infty} p_{n,\nu}(x,c)\left(\frac{n}{c}+m\right) \\ &= \frac{cx}{1+cx}\left[\frac{n}{c}J_{n,k-1}(x,c) + x(1+cx)J'_{n,k-1}(x,c) + nxJ_{n,k-1}(x,c)\right] \\ &= nxJ_{n,k-1}(x,c) + cx^2J'_{n,k-1}(x,c). \end{aligned}$$

This together (i) gives (ii). □

Corollary 6 *From (i), we get*

$$\begin{aligned} x(1+cx)J'_{n,k}(x,c) &= \sum_{\nu=k}^{\infty}(\nu-nx)p_{n,\nu}(x,c) \\ &= \sum_{\nu=k+1}^{\infty}(\nu-nx)p_{n,\nu}(x,c) + kp_{n,k}(x,c) - nxp_{n,\nu}(x,c) \\ &= cx^2J'_{n,k}(x,c) + kp_{n,k}(x,c) \ \ (\text{Using}(ii)). \end{aligned}$$

Hence, we have

$$J'_{n,k}(x,c) = \frac{k}{x}p_{n,k}(x,c).$$

We now establish a Bernstein type lemma for the operators $M_{n,\alpha}$ which is useful while establishing the inverse theorem.

Lemma 7 *For the operators* $M_{n,\alpha}$ *there hold the estimates:*

(i) $|\varphi^\lambda(x) M'_{n,\alpha}(f,x)| \leqslant C\,\alpha \|\varphi^\lambda f'\|$;

(ii) $|\varphi^\lambda(x) M'_{n,\alpha}(f,x)| \leqslant C\,\alpha \varphi^{\lambda-1}(x)\sqrt{n}\|f\|$,

where $\lambda \geqslant 1$.

Proof. In view of (2.4) and (2.5), we can write

$$\begin{aligned}
M'_{n,\alpha}(f,x) &= M'_{n,\alpha}(f,x) - f(x)M'_{n,\alpha}(1,x)\\
&= M'_{n,\alpha}\Big(\int_x^t f'(u)\,du, x\Big)\\
&\leqslant \alpha\Big|\sum_{k=0}^{\infty}\big(J_{n,k}^{\alpha-1}(x,c) - J_{n,k+1}^{\alpha-1}(x)\big)J'_{n,k+1}(x)\times\\
&\quad \times\int_0^\infty\Big(\int_x^t f'(u)\,du\Big)b_{n,k}(t,c)\,dt\Big| + \Big|M'_n\Big(\Big(\int_x^t f'(u)\,du\Big),x\Big)\Big|\\
&:= E_1 + E_2,\ \text{say.}
\end{aligned} \tag{2.10}$$

Now, we find estimates for E_1 and E_2 separately as follows. In view of the inequality (2.6) and Corallary 6, we have

$$\begin{aligned}
E_1 &= \alpha\Big|\sum_{k=0}^{\infty}\big(J_{n,k}^{\alpha-1}(x,c) - J_{n,k+1}^{\alpha-1}(x)\big)J'_{n,k+1}(x)\int_0^\infty\Big(\int_x^t f'(u)\,du\Big)b_{n,k}(t,c)\,dt\\
&\leqslant \frac{\alpha}{x}\sum_{k=0}^{\infty} k p_{n,k}(x,c)\int_0^\infty\Big|\int_x^t f'(u)\,du\Big|b_{n,k}(t,c)\,dt\\
&\leqslant \frac{2\alpha}{x}\sum_{k=0}^{\infty} k p_{n,k}(x,c)\times\\
&\quad\times\int_0^\infty\Big(x^{-\lambda/2}(1+ct)^{-\lambda/2} + \varphi^{-\lambda}(x)\Big)|t-x|\big\|\varphi^\lambda f'\big\|b_{n,k}(t,c)\,dt
\end{aligned}$$

Using Lemma 4, we get

$$\begin{aligned} E_1 \quad & \leqslant \frac{2\alpha\|\varphi^\lambda f'\|}{x^{1+\lambda/2}} \sum_{k=0}^{\infty} kp_{n,k}(x,c) \times \\ & \times \int_0^\infty (1+ct)^{-\lambda/2}|t-x| b_{n,k}(t,c)\,dt + \\ & + \frac{2\alpha\|\varphi^\lambda f'\|}{x\varphi^\lambda(x)} \sum_{k=0}^{\infty} kp_{n,k}(x,c) \int_0^\infty |t-x| b_{n,k}(t,c)\,dt \\ & = E_{11} + E_{12}, \text{ say.} \end{aligned} \tag{2.11}$$

Applying Hölder's inequality, we get

$$\begin{aligned} E_{11} \quad \leqslant \quad & \frac{2\alpha\|\varphi^\lambda f'\|}{x^{1+\lambda/2}} \sum_{k=0}^{\infty} kp_{n,k}(x,c) \times \\ & \times \Big(\int_0^\infty (1+ct)^{-\lambda} b_{n,k}(t,c)\,dt \Big)^{1/2} \Big(\int_0^\infty (t-x)^2 b_{n,k}(t,c)\,dt \Big)^{1/2}. \end{aligned}$$

Now, it can be easily shown that $\int_0^\infty (t-x)^2 b_{n,k}(t,c)\,dt = O(x^2)$. Therefore, we get the following estimate for E_{11}:

$$\begin{aligned} E_{11} \quad & \leqslant \quad \frac{2\alpha\|\varphi^\lambda f'\|}{x^{\lambda/2}} \sum_{k=0}^{\infty} kp_{n,k}(x,c) \Big(\int_0^\infty (1+ct)^{-\lambda} b_{n,k}(t,c)\,dt \Big)^{1/2} \\ & \leqslant \quad \frac{2\alpha\|\varphi^\lambda f'\|}{x^{\lambda/2}} \Big(\sum_{k=0}^{\infty} k^2 p_{n,k}(x,c) \Big)^{1/2} \Big(\sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty (1+ct)^{-\lambda} b_{n,k}(t,c)\,dt \Big)^{1/2} \\ & \leqslant \quad \frac{2\alpha\|\varphi^\lambda f'\|}{x^{\lambda/2}} \frac{1}{(1+cx)^{\lambda/2}} \quad \text{(in view of Lemma 5).} \end{aligned}$$

Next, using Corallary 4, and Hölder's inequality for summation, we get

$$\begin{aligned} E_{12} \quad & \leqslant \frac{2\alpha\|\varphi^\lambda f'\|}{x\varphi^\lambda(x)} \Big(\sum_{k=0}^{\infty} k^2 p_{n,k}(x,c) \Big)^{1/2} \Big(\sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty (t-x)^2 b_{n,k}(t,c)\,dt \Big)^{1/2} \\ & \leqslant \frac{2\alpha\|\varphi^\lambda f'\|}{\varphi^\lambda(x)}. \end{aligned}$$

Combining, the estimates for E_{11} and E_{12}, we obtain

$$E_1 \leqslant \frac{C\alpha\|\varphi^\lambda f'\|}{\varphi^\lambda(x)}. \tag{2.12}$$

Again, the estimate for E_2 are obtained along the lines of E_1 for $\alpha = 1$. Hence, we get (i).

Now, we have as in the proof of part (i)

$$\begin{aligned} M'_{n,\alpha}(f,x) &= \leqslant \alpha\left|\sum_{k=0}^{\infty}\left(J_{n,k}^{\alpha-1}(x,c) - J_{n,k+1}^{\alpha-1}(x)\right)J'_{n,k+1}(x)\times\right. \\ &\quad \left.\times\int_0^{\infty} f(t)b_{n,k}(t,c)\,dt\right| + \left|M'_n(f(t),x)\right| \\ &:= F_1 + F_2,\ \text{say.} \end{aligned} \tag{2.13}$$

For F_1, we get the estimate

$$\begin{aligned} F_1 &\leqslant \alpha\|f\|\left|\sum_{k=0}^{\infty}\left(J_{n,k}^{\alpha-1}(x,c) - J_{n,k+1}^{\alpha-1}(x)\right)J'_{n,k+1}(x)\right. \\ &\leqslant \alpha\|f\|\frac{n}{\varphi^2(x)}\sum_{\nu=k}^{\infty}\left|\frac{\nu}{n} - x\right| p_{n,\nu}(x,c) \\ &\leqslant \alpha\|f\|\frac{\sqrt{n}}{\varphi(x)}\ \text{Using Corrollary 3.} \end{aligned} \tag{2.14}$$

Similar estimate is established for F_2 as it is obtained by putting $\alpha = 1$ in the estimate of F_1.

Hence the Lemma is established from (2.10) to (2.14). □

3. Proof of the main theorem

Proof. By the definition of $K_{\varphi^\lambda}(f,t)$ for fixed n, x, λ, we can choose $g = g_{n,x,\lambda} \in W_\lambda$ such that

$$\|f - g\| + \frac{\alpha^{1/2}\varphi^{1-\lambda}(x)}{\sqrt{n}}\|\varphi^\lambda g'\| \leqslant 2K_{\varphi^\lambda}\left(f, \frac{\alpha^{1/2}\varphi^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{3.1}$$

Since, $M_{n,\alpha}$ is constant preserving, we can write

$$|M_{n,\alpha}(f,x) - f(x)| \quad \leqslant \quad 2\|f - g\| + |M_{n,\alpha}(g,x) - g(x)|. \tag{3.2}$$

Using the representation $g(t) = g(x) + \int\limits_x^t g'(u)\,du$, and in view of Lemma 4, we have

$$\begin{aligned}
|M_{n,\alpha}(g,x) - g(x)| &= \Big|M_{n,\alpha}\Big(\int\limits_x^t g'(u)\,du\Big)\Big| \\
&\leqslant 2\big\|\varphi^\lambda g'\big\|\Big[\varphi^{-\lambda}(x)M_{n,\alpha}(|t-x|,x) \\
&\quad + x^{-\lambda/2}M_{n,\alpha}\left(\frac{|t-x|}{(1+c\,t)^{\lambda/2}},x\right)\Big] \\
&:= 2\big\|\varphi^\lambda g'\big\|[J_1 + J_2]. \qquad (3.3)
\end{aligned}$$

Now, in view of Schwarz's inequality and Corollary 4, we get

$$\begin{aligned}
J_1 &= \varphi^{-\lambda}(x)\sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x,c)\int\limits_0^\infty b_{n,k}(t,c)|t-x|\,dt \\
&\leqslant \alpha^{1/2}\varphi^{-\lambda}(x)\left(\sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x,c)\int\limits_0^\infty b_{n,k}(t,c)\,dt\right)^{1/2} \times \\
&\quad \times\left(\sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x,c)\int\limits_0^\infty b_{n,k}(t,c)(t-x)^2\,dt\right)^{1/2} \\
&\leqslant \alpha^{1/2}\frac{K\varphi^{1-\lambda}(x)}{\sqrt{n}}. \qquad (3.4)
\end{aligned}$$

Next, using Schwarz's inequality, Corollary 4 and Lemma 5 we get

$$\begin{aligned}
J_2 &= x^{-\lambda/2}M_{n,\alpha}\left(\frac{|t-x|}{(1+c\,t)^{\lambda/2}},x\right) \\
&\leqslant \big(M_{n,\alpha}\left((t-x)^2,x\right)\big)^{1/2}\ \big(M_{n,\alpha}\left((1+c\,t)^{-\lambda},x\right)\big)^{1/2} \\
&\leqslant x^{-\lambda/2}\alpha^{1/2}(1+c\,x)^{-\lambda/2}\frac{\varphi(x)}{\sqrt{n}} \\
&= C\alpha^{1/2}\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}. \qquad (3.5)
\end{aligned}$$

Hence, from (3.4) and (3.5) we have the estimate

$$|M_{n,\alpha}(g,x) - g(x)| \leqslant C\alpha^{1/2}\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\big\|\varphi^\lambda g'\big\|. \qquad (3.6)$$

Thus, from (3.3)–(3.6) the implication (ii) $\Rightarrow$ (i) follows.

Proof of the implication (i) $\Rightarrow$ (ii).

We have

$$\begin{aligned}\left|\tilde{\Delta}_{h\varphi^{\lambda}(x)}f(x)\right| &\leqslant \left|M_{n,\alpha}(z)-f(z)\right|+\left|M_{n,\alpha}(x)-f(x)\right|+\left|\tilde{\Delta}_{h\varphi^{\lambda}(x)}M_{n,\alpha}(f,x)\right| \\ &\leqslant 2C\delta_{x,h}^{\beta}+\left|\tilde{\Delta}_{h\varphi^{\lambda}(x)}M_{n,\alpha}(f,x)\right|,\end{aligned}$$

where $\delta_{x,h}=\max\{y,z\},\ y=x-h\varphi^{\lambda}(x)/2$ and $z=x+h\varphi^{\lambda}(x)/2$.

We define a weighted Steklov type average function g as

$$g(x):=\frac{1}{\delta\varphi^{\lambda}(x)}\int_{\frac{-\delta}{2}\varphi^{\lambda}(x)}^{\frac{\delta}{2}\varphi^{\lambda}(x)} f(x+u)\,du \quad \lambda\geqslant 0.$$

Then, we obtain

$$\begin{aligned}(g-f)(x) &= \frac{1}{\delta\varphi^{\lambda}(x)}\int_{\frac{-\delta}{2}\varphi^{\lambda}(x)}^{\frac{\delta}{2}\varphi^{\lambda}(x)}[f(x+u)-f(x)]\,du \\ &\leqslant C\omega_{\varphi^{\lambda}(x)}(f,\delta).\end{aligned}$$

Also, it follows that

$$\begin{aligned}|g'(x)| &= \left|\frac{1}{\delta\varphi^{\lambda}(x)}\Big(f\big(x+\frac{\delta}{2}\varphi^{\lambda}(x)\big)-f\big(x-\frac{\delta}{2}\varphi^{\lambda}(x)\big)\Big)\right| \\ &\leqslant \frac{1}{\delta\varphi^{\lambda}(x)}\omega_{\varphi^{\lambda}(x)}(f,\delta). \qquad (3.7)\end{aligned}$$

In view of Lemma 7(i), it follws that

$$\left|M'_{n,\alpha}(g,x)\right| \leqslant C\frac{\alpha}{\delta\varphi^{\lambda}(x)}\omega_{\varphi^{\lambda}(x)}(f,\delta). \qquad (3.8)$$

Using Bernstein type inequalities, and then using (3.7) and (3.8), we get the estimate

$$\begin{aligned}\left|\tilde{\Delta}_{h\varphi^{\lambda}(x)}M_{n,\alpha}(f,x)\right| &\leqslant h\varphi^{\lambda}(x)\left(|M'_{n,\alpha}(f-g,x)|+|M'_{n,\alpha}(g,x)|\right) \\ &\leqslant C\alpha h\varphi^{\lambda}(x)\left(\varphi^{-1}(x)\sqrt{n}\|f-g\|+\|g'\|\right) \\ &\leqslant C\,\alpha h\left(\varphi^{\lambda-1}\sqrt{n}+\frac{1}{\delta}\right)\omega_{\varphi^{\lambda}(x)}(f,\delta).\end{aligned}$$

Consequently, we obtain

$$\omega_{h\varphi^{\lambda}(x)}(f,x) \leqslant 2C\delta_{x,h}^{\beta}+C\,\alpha h\left(\varphi^{\lambda-1}\sqrt{n}+\frac{1}{\delta}\right)\omega_{\varphi^{\lambda}(x)}(f,\delta).$$

Choosing $\delta_{x,h}=\delta$ and following the argument of [1] the implication (i) $\Rightarrow$ (ii) follows. $\square$

Remark 1 *From (3.1)–(3.6), we get*

$$|M_{n,\alpha}(f,x)-f(x)| \leqslant CK_{\varphi^\lambda}\left(f, \frac{\alpha^{1/2}\varphi^{1-\lambda}(x)}{\sqrt{n}}\right).$$

In view of (1.3), this further gives

$$|M_{n,\alpha}(f,x)-f(x)| \leqslant \omega_{\varphi^\lambda}\left(f, \frac{\alpha^{1/2}\varphi^{1-\lambda}(x)}{\sqrt{n}}\right).$$

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